UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 2 June, 2022 9:00am to 12:00pm

PAPER 1

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Vectors and Matrices

(a) Consider the equation

$$|z - a| + |z - b| = c\,,$$

for $z \in \mathbb{C}$, where $a, b \in \mathbb{C}$, $a \neq b$, and $c \in \mathbb{R}$, $c > 0$.

For each of the following cases, either show the equation has no solutions for z or give a rough sketch of the set of solutions:

(i) $c < |a - b|$,

(ii) $c = |a - b|$,

(iii) $c > |a - b|$.

(b) Let ω be the solution to $\omega^3 = 1$ with $\mathrm{Im}(\omega) > 0$. Calculate the following:

(i) $(1+\omega)^{10^6}$,

(ii) all values of $(1+\omega)^{1+\omega}$.

2B Vectors and Matrices

Consider the equation

$$M^T J M = J\,, \qquad (*)$$

where M is a 2×2 real matrix and

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) (i) What are the possible values of $\det(M)$ if M satisfies $(*)$? Justify your answer.

(ii) Suppose that $(*)$ holds when $M = M_1$ and when $M = M_2$. Show that $(*)$ also holds when $M = M_1 M_2$ and when $M = (M_1)^{-1}$.

(b) Show that if M satisfies $(*)$ and its first entry satisfies $M_{11} > 0$ then M takes one of the forms

$$\begin{pmatrix} a(u) & b(u) \\ c(u) & d(u) \end{pmatrix}, \qquad \begin{pmatrix} a(u) & -b(u) \\ c(u) & -d(u) \end{pmatrix},$$

where $u \in \mathbb{R}$ and $a(u), b(u), c(u), d(u)$ are hyperbolic functions whose form you should determine.

3D Analysis I

State the alternating series test. Deduce that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges. Is this series absolutely convergent? Justify your answer.

Find a divergent series which has the same terms $\frac{(-1)^n}{\sqrt{n}}$ taken in a different order. You should justify the divergence.

[*You may use the comparison test, provided that you accurately state it.*]

4D Analysis I

Let $a \in \mathbb{R}$ and let f and g be continuous real-valued functions defined on $\mathbb{R}$ which are not identically zero on any interval containing a.

Must the function $F(x) = f(x) + g(x)$ be non-differentiable at $a \in \mathbb{R}$ if (a) f is differentiable at a and g is not differentiable at a; (b) both f and g are not differentiable at a?

Must the function $G(x) = f(x)g(x)$ be non-differentiable at $a \in \mathbb{R}$ if (a) f is differentiable at a and g is not differentiable at a; (b) both f and g are not differentiable at a?

Justify your answers.

SECTION II

5B Vectors and Matrices

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be vectors in $\mathbb{R}^3$.

(a) (i) Define the scalar product $\mathbf{A} \cdot \mathbf{B}$ and the vector product $\mathbf{A} \times \mathbf{B}$, expressing the products in terms of vector components.

(ii) Obtain expressions for $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ as linear combinations of $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.

(iii) Now suppose that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent. By considering the expression $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})$, obtain an expression for $\mathbf{D}$ as a linear combination of $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

(b) Again suppose that the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent, and that they are position vectors of points on a sphere S that passes through the origin $\mathbf{O}$. By writing the position vector of the centre of S in the form

$$\mathbf{P} = \alpha \mathbf{A} + \beta \mathbf{B} + \gamma \mathbf{C},$$

obtain three linear equations for α, β, γ in terms of $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Hence find $\mathbf{P}$ when $\mathbf{A} = (1, 0, 0), \mathbf{B} = (1, 1, 0), \mathbf{C} = (0, 1, 2)$ in Cartesian coordinates.

6B Vectors and Matrices

(a) (i) Find, with brief justification, the 2×2 matrix R representing an anticlockwise rotation through angle θ in the xy-plane and the 2×2 matrix M representing reflection in the x-axis in the xy-plane.

(ii) Show that $MRM = R^a$, where a is an integer that you should determine.

(iii) Can $R^a = MR^b$ for some integers a, b? Justify your answer.

(b) Now let $n \geqslant 3$ be an integer and $\theta = \frac{2\pi}{n}$. Consider matrices of the form

$$M^{m_1} R^{n_1} M^{m_2} R^{n_2} \ldots M^{m_k} R^{n_k},$$

where $k \geqslant 1$ is an integer and $m_i \geqslant 0, n_i \geqslant 0$ are integers for $i = 1, \ldots, k$.

Show that there are precisely $2n$ distinct matrices of this form, and give explicit expressions for them as 2×2 matrices.

7B Vectors and Matrices

An $n \times n$ complex matrix P is called an *orthogonal projection matrix* if $P^2 = P = P^\dagger$, where † denotes the Hermitian conjugate. An $n \times n$ complex matrix A is *positive semidefinite* if $(\mathbf{x}, A\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in \mathbb{C}^n$. [Recall that the standard inner product on $\mathbb{C}^n$ is defined by $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\dagger \mathbf{y}$.]

(a) Show that the eigenvalues of any $n \times n$ Hermitian matrix are real.

(b) Show that every $n \times n$ orthogonal projection matrix is positive semidefinite.

(c) If P is an $n \times n$ orthogonal projection matrix, show that every vector $\mathbf{v} \in \mathbb{C}^n$ can be written in the form $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$, where $\mathbf{v}_0$ is in the kernel of P, $P\mathbf{v}_1 = \mathbf{v}_1$ and $(\mathbf{v}_0, \mathbf{v}_1) = 0$.

(d) If A and B are distinct $n \times n$ Hermitian matrices, show that there is an orthogonal projection matrix P such that $\mathrm{Tr}(PA) \neq \mathrm{Tr}(PB)$.

(e) If P and Q are $n \times n$ orthogonal projection matrices, is PQ necessarily a positive semidefinite matrix? Justify your answer.

8B Vectors and Matrices

Let M be an $n \times n$ complex matrix with columns $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{C}^n$. Verify that $\mathbf{c}_1 = M\mathbf{e}_1, \ldots, \mathbf{c}_n = M\mathbf{e}_n$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors for $\mathbb{C}^n$. If P is also an $n \times n$ complex matrix, show that PM has columns $P\mathbf{c}_1, \ldots, P\mathbf{c}_n$.

For an $n \times n$ complex matrix A with characteristic polynomial

$$\chi_A(x) = (-1)^n (x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0)\,,$$

consider the matrix

$$C = \begin{pmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{pmatrix}.$$

Show that there exists an invertible matrix S such that

$$S^{-1}AS = C$$

if and only if there is a vector $\mathbf{v} \in \mathbb{C}^n$ such that $\mathbf{v}, A\mathbf{v}, \ldots, A^{n-1}\mathbf{v}$ are linearly independent. You may assume that $\chi_A(A) = 0$ (the Cayley-Hamilton theorem).

[*Hint: consider the columns of* S.]

9D Analysis I

(a) Let a_n be a sequence of real numbers. Show that if a_n converges, the sequence $\frac{1}{n}\sum_{k=1}^{n} a_k$ also converges and $\lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = \lim\limits_{n\to\infty} a_n$.

If $\frac{1}{n}\sum_{k=1}^{n} a_k$ converges, must a_n converge too? Justify your answer.

(b) Let x_n be a sequence of real numbers with $x_n > 0$ for all n. By considering the sequence $\log x_n$, or otherwise, show that if x_n converges then $\lim\limits_{n\to\infty} \sqrt[n]{x_1 x_2 \ldots x_n} = \lim\limits_{n\to\infty} x_n$. You may assume that exp and log are continuous functions.

Deduce that if the sequence $\dfrac{x_n}{x_{n-1}}$ converges, then $\lim\limits_{n\to\infty} \sqrt[n]{x_n} = \lim\limits_{n\to\infty} \dfrac{x_n}{x_{n-1}}$.

(c) What is a *Cauchy sequence*? State the general principle of convergence for real sequences.

Let a_n be a decreasing sequence of positive real numbers and suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim\limits_{n\to\infty} n a_n = 0$.

10D Analysis I

Prove that every continuous real-valued function on a closed bounded interval is bounded and attains its bounds. [The Bolzano–Weierstrass theorem can be assumed provided it is accurately stated.]

Give an example of a continuous function $\phi : (0,1) \to \mathbb{R}$ that is bounded but does not attain its bounds and an example of a function $\psi : [0,1] \to \mathbb{R}$ that is not bounded on any interval $[a,b]$ such that $0 \leqslant a < b \leqslant 1$. Justify your examples.

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Prove that the functions

$$m(x) = \inf_{a \leqslant \xi \leqslant x} f(\xi) \qquad \text{and} \qquad M(x) = \sup_{a \leqslant \xi \leqslant x} f(\xi)$$

are also continuous on $[a,b]$.

Let a function $g : (0,\infty) \to \mathbb{R}$ be continuous and bounded. Show that for every $T > 0$ there exists a sequence x_n such that $x_n \to \infty$ and

$$\lim_{n\to\infty} \left(g(x_n + T) - g(x_n)\right) = 0.$$

[The intermediate value theorem can be assumed.]

11D Analysis I

In this question $a < b$ are real numbers.

(a) State and prove Rolle's theorem. State and prove the mean value theorem.

(b) Prove that if a continuous function $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and is not a linear function, then $f'(\xi) > \dfrac{f(b) - f(a)}{b - a}$ for some ξ with $a < \xi < b$.

(c) Let $f : [a, b] \to \mathbb{R}$ be a continuous function and let f be differentiable on (a, b). Must there exist, for every $\xi \in (a, b)$, two points x_1, x_2 with $a \leqslant x_1 < \xi < x_2 \leqslant b$ such that $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi)$? Give a proof or counterexample as appropriate.

(d) Let functions f and g be continuous on $[a, b]$ and differentiable on (a, b) with $g(a) \neq g(b)$ and suppose that $f'(x)$ and $g'(x)$ never vanish for the same value of x. By considering $\lambda f + \mu g + \nu$ for suitable real constants λ, μ, ν, or otherwise, prove that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)} \qquad \text{for some } \xi \text{ with } a < \xi < b.$$

Give an example to show that the condition that $f'(x)$ and $g'(x)$ never vanish for the same x cannot be omitted.

12D Analysis I

Let $f : [0, 1] \to \mathbb{R}$ be a monotone function.

Show that for all dissections $\mathcal{D}$ and $\mathcal{D}'$ of $[0, 1]$ one has $L_{\mathcal{D}}(f) \leqslant U_{\mathcal{D}'}(f)$, where $L_{\mathcal{D}}(f)$ and $U_{\mathcal{D}'}(f)$ are the lower and upper sums of f for the respective dissections. Show further that for each $\varepsilon > 0$ there is a dissection $\mathcal{D}$ such that $U_{\mathcal{D}}(f) - L_{\mathcal{D}}(f) < \varepsilon$. Deduce that f is integrable.

Show that

$$\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{k=1}^{n} f\big(\frac{k}{n}\big) \right| < \frac{|f(1) - f(0)|}{n}$$

for all positive integers n.

Let a function F be continuous on some open interval containing $[0, 1]$ and have a continuous derivative F' on $[0, 1]$. Denote

$$\Delta_n = \int_0^1 F(x)dx - \frac{1}{n} \sum_{k=1}^{n} F\big(\frac{k}{n}\big).$$

Stating clearly any results from the course that you require, show that

$$\lim_{n \to \infty} n\Delta_n = (F(0) - F(1))/2.$$

[*Hint: it might be helpful to consider* $\int_{(k-1)/n}^{k/n} \big(F(x) - F(\frac{k}{n})\big) dx$.]

END OF PAPER