UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II 2022

List of Courses

Riemann Surfaces
Statistical Modelling
Statistical Physics
Stochastic Financial Models
Topics in Analysis
Waves

Paper 1, Section II
25H Algebraic Geometry

Define the *local ring* at a point p of an irreducible algebraic variety V. Define the *Zariski tangent space* to V at p.

Let $V \subset \mathbb{A}^2 \times \mathbb{P}^1$ be defined by the equation

$$XZ - WY = 0\,,$$

where X and Y are the coordinates on $\mathbb{A}^2$ and W and Z are the homogeneous coordinates on $\mathbb{P}^1$. Determine whether V is smooth.

Consider the projection morphism

$$\pi : V \to \mathbb{A}^2$$

obtained by restricting the projection from $\mathbb{A}^2 \times \mathbb{P}^1$ onto the first factor. Prove that π is birational but not an isomorphism. Use this to calculate the function field of V.

Let V' be an affine variety and $\varphi : V \to V'$ a morphism. Prove that φ is not injective. Deduce that V is not affine.

Assume the ground field is $\mathbb{C}$. Prove that if V is equipped with the Euclidean topology, then it is not homeomorphic to any projective variety.

Paper 2, Section II
25H Algebraic Geometry

State the *Riemann–Hurwitz theorem*. Show that, if C and C' are smooth projective connected curves over a characteristic zero field with $g(C) < g(C')$, then any morphism

$$C \to C'$$

is constant.

Let $C_d \subset \mathbb{P}^2$ be a smooth plane curve of degree d. Construct a morphism

$$\varphi : C_d \to \mathbb{P}^1$$

of degree $d-1$. Let $B \subset \mathbb{P}^1$ be the set of branch points for φ. Give an upper bound for the cardinality of B in terms of d.

Now let D be the divisor on C_d associated to a hyperplane section of C_d. Prove that if $d \geqslant 5$ then D is not linearly equivalent to the canonical divisor of C_d.

The *gonality* of a curve C is the minimum degree of a non-constant morphism $C \to \mathbb{P}^1$. Prove that a smooth plane curve of degree 4 has gonality equal to 3. What is the gonality of a smooth projective curve of genus 1?

Paper 3, Section II
24H Algebraic Geometry

What is a *singular point* on an irreducible algebraic variety? Let X be an irreducible affine variety. Prove that the set of nonsingular points in X is dense in the Zariski topology.

Find the set of singular points on the projective variety

$$\mathbb{V}(X_0^2 + \cdots + X_{n-1}^2) \subset \mathbb{P}^n,$$

where $X_0, \ldots, X_n$ are the homogeneous coordinates on $\mathbb{P}^n$.

Let X be an irreducible variety of dimension n and let $Z \subset X$ be the closed subvariety consisting of all singular points of X. Suppose the dimension of Z is k. If Y is smooth of dimension m, what is the dimension of the set of singular points of $X \times Y$? Justify your answer.

Given integers $n > k \geqslant 0$, give an example of an n-dimensional irreducible subvariety of projective space whose subvariety of singular points is nonempty and has dimension k.

Let C be an irreducible curve in $\mathbb{P}^2$. If C is birational to a smooth projective curve of genus 2, show that C contains a singular point.

Paper 4, Section II
24H Algebraic Geometry

What is the *degree* of a divisor on a smooth projective algebraic curve? What is a *principal divisor* on a smooth projective algebraic curve?

Let $D = \sum a_i p_i$ be a divisor of degree 0 on $\mathbb{P}^1$. Construct a rational function f such that $\mathrm{div}(f)$ is D. Deduce that if E and E' are divisors of the same degree on $\mathbb{P}^1$ then E is linearly equivalent to E'.

Let X_0, X_1 be the usual homogenous coordinates on $\mathbb{P}^1$, and let t be the rational function X_0/X_1. Calculate the divisor associated to the rational differential dt on $\mathbb{P}^1$.

Fix an integer m and let D be a divisor equivalent to $mK_{\mathbb{P}^1}$, where $K_{\mathbb{P}^1}$ is the canonical divisor computed above. Without appealing to the Riemann–Roch theorem, calculate the dimension of the vector space $L(D)$ of rational functions with poles bounded by D.

Let C be a smooth projective curve of genus at least 1. Prove that for distinct points p and q in C, the divisor $p - q$ is not principal.

Paper 1, Section II
21I Algebraic Topology

Suppose $f, g : C_* \to C'_*$ are chain maps. Define what it means for f and g to be *chain homotopic*. Show that if f and g are chain homotopic then $f_* = g_*$.

Let $C_* = \widetilde{C}_*(\Delta^n)$ be the reduced chain complex of the n-dimensional simplex. Show that id_{C_*} is chain homotopic to 0_{C_*}. Hence compute $H_*(\Delta^n)$.

Now let $K = \Delta^6_2$ be the 2-skeleton of Δ^6. Compute $H_*(K)$. Let $f : K \to K$ be the simplicial map given by $f(e_i) = e_{\sigma(i)}$, where σ is the permutation given in cycle notation by $(0123)(456)$. Compute the trace of the linear map $f_* : H_2(K;\mathbb{Q}) \to H_2(K;\mathbb{Q})$.

Paper 2, Section II
21I Algebraic Topology

State the *snake lemma* and derive the exactness of the Mayer–Vietoris sequence from it.

Suppose that K is a simplicial complex of dimension $n \geqslant 1$, that every $(n-1)$-simplex of K is a face of precisely two n-simplices, and that if σ and σ' are n-simplices of K then there is a sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ of n-simplices in K such that for all i, σ_i and σ_{i+1} have an $(n-1)$-simplex in common. Show that $H_n(K)$ is either trivial or isomorphic to $\mathbb{Z}$.

Now suppose that K is as above and that $H_n(K) \cong \mathbb{Z}$ is generated by $x \in H_n(K)$. If K is the union of subcomplexes L_1 and L_2 such that $L_1 \cap L_2$ has dimension less than n, describe ∂x, where ∂ is the boundary map in the Mayer–Vietoris sequence associated to the decomposition $K = L_1 \cup L_2$. Justify your answer. When is $\partial x \neq 0$?

Finally, suppose that K, L_1 and L_2 are as in the previous paragraph, that K is homeomorphic to S^3, that L_1 is homeomorphic to $S^1 \times D^2$, and that the image of $L_1 \cap L_2$ under this homeomorphism is $S^1 \times S^1 \subset S^1 \times D^2$. Compute $H_*(L_2)$.

Paper 3, Section II
20I Algebraic Topology

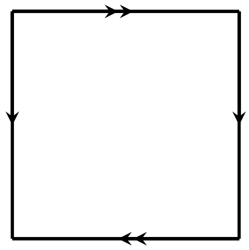
Suppose $f : S^{n-1} \to X$ is a continuous map. Show that f extends to a continuous map $F : D^n \to X$ if and only if f is homotopic to a constant map.

Let X be a path-connected and locally path-connected topological space. Define what it means for a space $\widetilde{X}$ to be a *universal covering space* of X. State a suitable lifting property and use it to prove that any two universal covering spaces of X are homeomorphic.

Now suppose that $\widetilde{X}$ is a universal covering space of X, and that $\widetilde{X}$ is contractible. Let K be a path-connected simplicial complex with 1-skeleton K_1, and let $i : K_1 \to K$ be the inclusion. Given a continuous map $f : |K_1| \to X$, prove that f extends to a continuous map $F : |K| \to X$ if and only if there is a homomorphism $\Phi : \pi_1(|K|, v) \to \pi_1(X, f(v))$ with $f_* = \Phi \circ i_*$, where v is any vertex of K. [*Hint: Induct on the number of simplices in* $K \setminus K_1$.]

Paper 4, Section II
21I Algebraic Topology

Let K be the Klein bottle obtained by identifying the sides of the unit square as shown in the figure, and let $k_0 \in K$ be the image of the corners of the square.



Show that K is the union of two Möbius bands with their boundaries identified. Deduce that $\pi_1(K, k_0)$ has a presentation

$$\pi_1(K, k_0) = \langle a, b \,|\, a^2 b^{-2} \rangle.$$

Show that there is a degree two covering map $p : (T^2, x_0) \to (K, k_0)$. Describe generators α, β for $\pi_1(T^2, x_0)$ and express $p_*(\alpha)$ and $p_*(\beta)$ in terms of a and b.

Let $Y = T^2 \times [0,1)/\sim$, where $\sim$ is the smallest equivalence relation with $(x, 0) \sim (x', 0)$ whenever $p(x) = p(x')$. What is $\pi_1(Y, y_0)$, where y_0 is the image of $(x_0, 0)$ in Y?

Suppose X is a path-connected Hausdorff space, that $U \subset X$ is an open subset, and that U is homeomorphic to Y. Can X be simply connected? Justify your answer.

Paper 1, Section II
23G Analysis of Functions

In this question, $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets and λ is Lebesgue measure on $\mathbb{R}^n$.

State *Lebesgue's differentiation theorem* and the *Radon–Nikodym theorem*. For a set $A \in \mathcal{M}$, and a measure μ defined on $\mathcal{M}$, let the *μ-density* of A at $x \in \mathbb{R}^n$ be

$$\rho_{\mu,A}(x) = \lim_{r \searrow 0} \frac{\mu(A \cap B_r(x))}{\mu(B_r(x))},$$

whenever the limit exists, where $B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$ is the open ball of radius r centred at x.

For each $t \in [0,1]$, give an example of a set $B \subset \mathbb{R}^2$ and point $z \in \mathbb{R}^2$ for which $\rho_{\lambda,B}(z)$ exists and is equal to t.

Show that for λ-almost every $x \in \mathbb{R}^n$, $\rho_{\lambda,A}(x)$ exists and takes the value 0 or 1. Show that $\rho_{\lambda,A}$ vanishes λ-almost everywhere if and only if A has Lebesgue measure zero.

Let ν be a measure on $\mathcal{M}$ such that $\nu \ll \lambda$ and $\lambda \ll \nu$. Show that $\rho_{\nu,A}(x)$ exists and takes the value 0 or 1 at λ-almost every $x \in \mathbb{R}^n$.

Paper 2, Section II
23G Analysis of Functions

Let X be a real vector space. State what it means for a functional $p : X \to \mathbb{R}$ to be *sublinear*.

Let $M \subsetneq X$ be a proper subspace. Suppose that $p : X \to \mathbb{R}$ is sublinear and the linear map $\ell : M \to \mathbb{R}$ satisfies $\ell(y) \leqslant p(y)$ for all $y \in M$. Fix $x \in X \setminus M$ and let $\widetilde{M} = \operatorname{span}\{M, x\}$. Show that there exists a linear map $\tilde{\ell} : \widetilde{M} \to \mathbb{R}$ such that $\tilde{\ell}(z) \leqslant p(z)$ for all $z \in \widetilde{M}$ and $\tilde{\ell}(y) = \ell(y)$ for all $y \in M$.

State the *Hahn–Banach theorem.*

Let $\{z_1, \ldots, z_n\}$ be a set of linearly independent elements of a real Banach space Z. Show that for each $j = 1, \ldots, n$ there exists $\ell_j \in Z'$ with $\ell_j(z_k) = \delta_{jk}$ for all $k = 1, \ldots, n$. Suppose $M \subset Z$ is a finite dimensional subspace. Show that there exists a closed subspace N such that $Z = M \oplus N$.

Paper 3, Section II
22G Analysis of Functions

State and prove the *Riemann–Lebesgue lemma*. State *Parseval's identity*, including any assumptions you make on the functions involved.

Suppose that $f : \mathbb{R}^n \to \mathbb{C}$ is given by

$$f(x) = \frac{|x|^a}{(1+|x|^2)^{\frac{b+a}{2}}}\,.$$

Show that if $2a > -n$ and $b > n$ then $\hat{f} \in L^p(\mathbb{R}^n)$ for all $2 \leqslant p \leqslant \infty$, where $\hat{f}$ is the Fourier transform of f.

Paper 4, Section II
23G Analysis of Functions

For $s \in \mathbb{R}$, define the *Sobolev space* $H^s(\mathbb{R}^n)$. Show that for any multi-index α, the map $u \mapsto D^\alpha u$ is a bounded linear map from $H^s(\mathbb{R}^n)$ to $H^{s-|\alpha|}(\mathbb{R}^n)$.

Given $f \in H^s(\mathbb{R}^n)$, show that the PDE

$$-\Delta u + u = f$$

admits a unique solution with $u \in H^{s+2}(\mathbb{R}^n)$. Show that the map taking f to u is a linear isomorphism of $H^s(\mathbb{R}^n)$ onto $H^{s+2}(\mathbb{R}^n)$.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider a sequence of functions $(u_j)_{j=1}^\infty$ with $u_j \in C^\infty(\mathbb{R}^n)$, supported in Ω, such that

$$\|\Delta u_j\|_{L^2(\Omega)} + \|u_j\|_{L^2(\Omega)} \leqslant K\,,$$

for some constant K independent of j. Show that there exists a subsequence $(u_{j_k})_{k=1}^\infty$ which converges strongly in $H^1(\mathbb{R}^n)$.

Paper 1, Section II
35D Applications of Quantum Mechanics

A particle of mass m and energy $E = \hbar^2 k^2/2m$, moving in one dimension, is incident on a localised potential barrier.

(a) Define *reflection* and *transmission coefficients*, r and t, for a right-moving particle incident from $x = -\infty$. Define corresponding coefficients r' and t' for a left-moving particle incident from $x = +\infty$. Prove that the S-matrix

$$\mathcal{S} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix}$$

is unitary. [You may use without proof the conservation of the probability current.]

(b) Explain what is meant by the *parity* of a wavefunction. Under what circumstances do energy eigenstates of the system described above have definite parity?

(c) Consider the potential barrier

$$V(x) = \begin{cases} V_0 & \text{for } |x| < a/2 \\ 0 & \text{for } |x| > a/2, \end{cases}$$

where $V_0 > 0$. Find an even parity wavefunction satisfying the Schrödinger equation for a particle of energy $E = \hbar^2 k^2/2m$ with $E < V_0$. Hence compute $r + t$.

Paper 2, Section II
36D Applications of Quantum Mechanics

A particle of mass m moves in one dimension in the periodic potential

$$V(x) = \sum_{n \in \mathbb{Z}} V_n \exp\left(\frac{2\pi i n x}{a}\right),$$

where $V_{-n} = (V_n)^*$. Treating the Hamiltonian $\hat{H} = \hat{H}_0 + V(x)$ as a small perturbation of the free Hamiltonian $\hat{H}_0$, show that the energy spectrum consists of continuous bands separated by gaps of width $2|V_n|$ that occur for each positive integer n.

What is meant by the *dispersion relation* of the particle? Determine an explicit form of the dispersion relation near each band gap.

Work out the locations and widths of the gaps in the energy spectrum for the potential

$$V(x) = \frac{8}{3} V_0 \cos^4\left(\frac{2\pi x}{a}\right).$$

Sketch the dispersion relation of a particle moving in this potential.

Paper 3, Section II
34D Applications of Quantum Mechanics

A two-dimensional Bravais lattice Λ has primitive basis vectors $\{\mathbf{a}_1, \mathbf{a}_2\}$, where

$$\mathbf{a}_1 = \hat{\mathbf{x}} \;, \qquad\qquad \mathbf{a}_2 = -\frac{1}{2}\hat{\mathbf{x}} + \frac{\sqrt{3}}{2}\hat{\mathbf{y}} \;,$$

and $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ is the standard Cartesian basis. Express a general primitive basis $\{\mathbf{a}'_1, \mathbf{a}'_2\}$ for Λ in terms of $\{\mathbf{a}_1, \mathbf{a}_2\}$.

Find the lattice Λ^* which is dual to Λ, giving a basis of primitive vectors dual to $\{\mathbf{a}_1, \mathbf{a}_2\}$. Sketch the region of the lattice Λ^* containing the origin, indicating all those points which are nearest neighbours of the origin. Determine the *Wigner-Seitz unit cell* of Λ^* as polygonal region of the plane, giving the coordinates of all vertices of this polygon. Determine the area of this unit cell.

A particle of mass m moves in a potential $V(\mathbf{x})$ which is invariant under shifts by vectors in Λ,

$$V(\mathbf{x} + \mathbf{l}) = V(\mathbf{x}) \qquad\qquad \forall\ \mathbf{l} \in \Lambda\,.$$

Define the n^{th} *Brillouin zone* of this system and briefly describe its physical significance. Draw a sketch showing the first and second Brillouin zones.

Paper 4, Section II

34D Applications of Quantum Mechanics

A particle of mass m and charge e moves in a constant homogeneous magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ with vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{B}{2}(-y, x, 0)\,,$$

where $\mathbf{x} = (x, y, z)$ are Cartesian coordinates on $\mathbb{R}^3$.

(a) Write down the Hamiltonian $\hat{H}$ for the particle as a differential operator in Cartesian coordinates. Find a corresponding expression for $\hat{H}$ in cylindrical polar coordinates (r, θ, z), where $x = r\cos\theta$ and $y = r\sin\theta$.

[*You may use without proof the relations*

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \qquad \text{and} \qquad x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}\,. \quad]$$

(b) Consider wavefunctions of the form

$$\psi_{k_z,n}(r, \theta, z) = \exp(ik_z z)\,\exp(in\theta)\,\phi_n(r)\,.$$

What is the physical interpretation of the quantum numbers $k_z \in \mathbb{R}$ and $n \in \mathbb{Z}$? For $n \geqslant 0$, show that $\psi_{k_z,n}$ is an eigenstate of $\hat{H}$ provided that

$$\phi_n(r) = r^\alpha \exp\left(-\beta\frac{r^2}{2}\right),$$

where α and β are (possibly n-dependent) constants which you should determine. Find the corresponding energy eigenvalue E.

(c) By noting that $\phi_n(r)$ is sharply peaked at a particular value of r, work out the total degeneracy of this energy level when the particle is confined to lie inside a large circle of radius R. Determine the number of states per unit area.

Paper 1, Section II
28J Applied Probability

(a) Define what it means for a matrix Q to be a *Q-matrix* on a finite or countably infinite state space S.

Suppose S is a finite state space. Express the generator Q of a continuous-time Markov chain $X = (X_t)$ on S in terms of its transition semigroup $(P(t))_{t\geqslant 0}$, and conversely express the semigroup in terms of the generator. You do not need to prove the expressions you give.

Write down the forward and backward *Kolmogorov equations* for a chain X as above.

(b) Let $X = (X_t)$ be a continuous-time Markov chain on the state space $S = \{1, 2\}$, with generator

$$Q = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix},$$

where $\lambda\mu > 0$.

(i) Compute the transition probabilities $p_{ij}(t)$, $i, j \in S$, $t > 0$.

(ii) Find Q^n for $n \geqslant 1$, and compute $\sum_{n=0}^{\infty} \frac{t^n}{n!} Q^n$ for $t > 0$. Compare the result with your answer in part (i).

(iii) Solve the equation $\pi Q = 0$ for a probability distribution π and identify the invariant distribution of X. Use your result in part (i) to verify that, indeed, the semigroup converges to the invariant distribution as $t \to \infty$.

(iv) Compute the probability $\mathbb{P}(X(t) = 2 | X(0) = 1, X(3t) = 1)$.

Paper 2, Section II
28J Applied Probability

(a) Let $X = (X_t)$ be a right-continuous process with values in a finite state space S, and let Q be a Q-matrix on S. State two different conditions that are equivalent to the statement that X is a continuous-time Markov chain with generator Q. Prove that these two conditions are equivalent.

(b) Let G be a finite connected graph and let A be a connected subgraph of G. Let X be a continuous time Markov chain that takes values in the vertices of A and evolves as follows: when at x it stays there for an exponential time of parameter 1 and then chooses a neighbour of x in G uniformly at random. If the neighbour is in A, then X jumps there, otherwise it waits for another independent exponential time of parameter 1 and proceeds as before. This continues until the first time that X chooses a neighbour of x in A and then jumps there. Find the Q-matrix and the invariant distribution of X. Justify your answer.

[*You may use the fact that, if N is a geometric random variable of parameter p and $(E_i)_{i\geqslant 1}$ is an i.i.d. sequence of exponential random variables of parameter* 1 *independent of N, then $\sum_{i=1}^{N} E_i$ is exponentially distributed with parameter p.*]

Paper 3, Section II
27J Applied Probability

(a) Define what we mean by a *renewal process* associated with the independent and identically distributed sequence of nonnegative random variables $\{\xi_n\}$.

(b) Define the *size-biased* distribution corresponding to ξ_1.

(c) Define the *excess process* $E = (E(t))$ and state a result regarding its asymptotic behaviour, giving the required conditions carefully.

(d) Let $X = (X_t)$ be a Poisson process and $N = (N_t)$ be a renewal process with non-arithmetic inter-renewal times, independent of X. Suppose that $Y = (Y_t)$ defined by $Y_t = X_t + N_t$, $t \geqslant 0$ is also a renewal process. Show that the first event time of Y has an exponential distribution by deriving an integral equation for its distribution function that is satisfied by the exponential.

Paper 4, Section II
27J Applied Probability

(a) Let $X = (X_t)$ be the queue length process of an $M/M/1$ queue with arrival rate $\lambda > 0$ and service rate $\mu > 0$. Suppose $\rho = \lambda/\mu < 1$. Show that X is positive recurrent and derive its invariant distribution π.

(b) Now suppose that each arriving customer observes the current queue length $X_t = n$, and either decides to join the queue with probability $p(n)$ or to leave the system with probability $1 - p(n)$, independently of all other customers.

(i) Find the invariant distribution π of X if $p(n) = 1/(n+1)$, $n \geqslant 0$.

(ii) Find the invariant distribution π of X if $p(n) = 2^{-n}$, $n \geqslant 0$, and show that, in equilibrium, an arriving customer joins the queue with probability $\mu(1-\pi_0)/\lambda$.

Paper 2, Section II
32E Asymptotic Methods

(a) Let $n = 1, 2, \dots$. Which of the following sequences are asymptotic and why?

(i) $\phi_n(x) = \ln(\cos(x^n)) \qquad \text{as } x \to 0\,.$

(ii) $\psi_n(x) = n^{1/x} \qquad \text{as } x \to \infty\,.$

(iii) $\chi_n(x) = \sin(x^n) \qquad \text{as } x \to \infty\,.$

(b) Let $\phi_n(x)$ and $\psi_n(x)$, for $n = 0, 1, 2, \dots$, be two sequences of real positive functions defined on $\{x \in \mathbb{R} : 0 < |x - x_0| < 1\}$ which are asymptotic sequences as $x \to x_0$.

For $n = 0, 1, 2, \dots$, show that the sequence

$$\chi_n(x) = \sum_{k=0}^{n} \phi_k(x)\psi_{n-k}(x)\,,$$

is an asymptotic sequence as $x \to x_0$.

Paper 3, Section II
30E Asymptotic Methods

(a) Derive the leading order term of the asymptotic expansion, as $x \to \infty$, for the integral

$$I(x) = \int_0^2 \ln t \; e^{x(t^3 - 2t^2 + t)}\, dt\,.$$

Justify your steps.

(b) The derivative of the Gamma function has the following integral representation

$$\Gamma'(z) = \int_0^\infty \frac{\ln t}{t} e^{z \ln t - t}\, dt \qquad \text{for} \quad \operatorname{Re} z > 0.$$

In what follows we assume $z \in \mathbb{R}$ and $z > 0$.

(i) Justify briefly why the integral converges. Explain why Laplace's method cannot be used directly to find the leading order behaviour of $\Gamma'(z)$ as $z \to \infty$.

(ii) Now perform the change of variables $t = zs$, then apply Laplace's method to show that

$$\Gamma'(z) \sim \sqrt{\frac{a}{z}} e^{z \ln z - z} \ln z \qquad \text{as} \quad z \to \infty\,,$$

for a real number a, which you should determine.

Paper 4, Section II

31E Asymptotic Methods

Consider the differential equation

$$x^2 y'' + xy' - \frac{1}{x^2} y = 0 \,. \qquad (*)$$

(i) What type of regular or singular point does equation $(*)$ have at $x = 0$?

(ii) For $x > 0$, find a transformation that maps equation $(*)$ to an equation of the form

$$u'' + q(x)u = 0 \qquad (\dagger)$$

and compute $q(x)$.

(iii) Determine the leading asymptotic behaviour of the solution u of equation $(\dagger)$, as $x \to 0^+$, using the Liouville-Green method and justifying your assumptions at each stage.

(iv) Conclude from the above an asymptotic expansion of two linearly independent solutions of equation $(*)$, as $x \to 0^+$.

Paper 1, Section I

4I Automata and Formal Languages

What are the nth *register machine* P_n and the nth *recursively enumerable set* W_n?

Given subsets $A, B \subseteq \mathbb{N}$, define a *many–one reduction* $A \leqslant_m B$ of A to B.

State *Rice's theorem.*

Is there a total algorithm that, on input n in register 1 and m in register 2, terminates with 0 if $W_m = W_n$ and 1 if $W_m \neq W_n$? Is there a partial algorithm that, with the same inputs as above, terminates with 0 if $W_m = W_n$ and never halts if $W_m \neq W_n$? Justify your answers.

[*You may assume without proof that the halting set* $\mathbb{K}$ *is not recursive.*]

Paper 2, Section I

4I Automata and Formal Languages

State and prove the *pumping lemma for regular languages.*

Are the following languages over the alphabet $\Sigma = \{0, 1\}$ regular? Justify your answers.

(i) $\{0^n 1 \mid n \geqslant 0\}$.

(ii) $\{0^n 1^{n^2} \mid n \geqslant 0\}$.

(iii) The set of all words in Σ^* containing the same number of 0s and 1s.

Paper 3, Section I

4I Automata and Formal Languages

Define a *context-free grammar* (CFG) and a *context-free language* (CFL).

State the *pumping lemma for CFLs.*

Which of the following languages over the alphabet $\{a, b, c\}$ are CFLs? Justify your answers.

(i) $\{a^n b^{2n} c^n \mid n \geqslant 0\}$.

(ii) $\{a^n b^{2i} c^n \mid n, i \geqslant 0\}$.

Paper 4, Section I

4I Automata and Formal Languages

Define what it means for a context-free grammar (CFG) to be in *Chomsky normal form*.

What are an *ϵ–production* and a *unit production*?

Let G_1 be the CFG

$$\begin{aligned} S &\rightarrow \epsilon \,|\, aTa \,|\, bTa \\ T &\rightarrow Ta \,|\, Tb \,|\, c \end{aligned}$$

and let G_2 be the CFG

$$\begin{aligned} S &\rightarrow XZ \,|\, YZ \\ T &\rightarrow TX \,|\, TY \,|\, c \\ X &\rightarrow a,\, Y \rightarrow b,\, Z \rightarrow TX. \end{aligned}$$

What is the relationship between the language of G_1 and the language of G_2? Justify your answer carefully.

Paper 1, Section II

12I Automata and Formal Languages

Give the definition of a *primitive recursive function* $f : \mathbb{N}^k \rightarrow \mathbb{N}$.

Show directly from the definition that, when $k = 2$, the functions

$$P(m,n) = m + n \text{ and } T(m,n) = mn$$

are both primitive recursive.

Show further that for $k \geqslant 2$ the function

$$T_k(n_1, \ldots, n_k) = n_1 \cdots n_k$$

is primitive recursive, as is $E_a : \mathbb{N} \rightarrow \mathbb{N}$ given by $E_a(n) = a^n$, where $a \geqslant 1$ is a fixed integer.

Suppose $F : \mathbb{N}^k \rightarrow \mathbb{N}^k$, where $F = (f_0, \ldots, f_{k-1})$ with each coordinate function f_i primitive recursive. Describe how F can be encoded as a primitive recursive function $\overline{F} : \mathbb{N} \rightarrow \mathbb{N}$.

Let the Fibonacci function $B : \mathbb{N} \rightarrow \mathbb{N}$ be defined by $B(0) = 0, B(1) = 1$ and $B(n+2) = B(n+1) + B(n)$ for $n \geqslant 0$. Is B primitive recursive? Justify your answer.

If $f : \mathbb{N} \rightarrow \mathbb{N}$ is a primitive recursive function, must there exist some $R > 0$ such that $f(n) \leqslant R^n$ for all $n \geqslant 1$? Justify your answer.

[*You may use without proof that for fixed $j \geqslant 2$ the maxpower function M_j is primitive recursive, where $M_j(n)$ is the exponent of the highest power of j that divides n. If you use any other results from the course, you should prove them.*]

Paper 3, Section II

12I Automata and Formal Languages

Let $D = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite-state automaton (DFA).

What does it mean to say that $q \in Q$ is an *accessible* state? What does it mean to say that $p, q \in Q$ are *equivalent* states?

Explain the construction of the *minimal DFA* $D/\!\sim$ and show that the languages of D and of $D/\!\sim$ are the same. Show also that no two distinct states of $D/\!\sim$ are equivalent.

Now let Σ be the single-letter alphabet $\{1\}$. Suppose that D is a DFA with no inaccessible states and exactly one accept state. Justifying your answer, describe the corresponding minimal DFA $D/\!\sim$ in the form of a transition diagram or otherwise. [Remember that you need only consider accessible states.]

Paper 1, Section I
8B Classical Dynamics

(a) Show that the canonical transformation $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{Q}, \mathbf{P})$ associated with a generating function $F_2(\mathbf{q}, \mathbf{P})$ of type 2 satisfies

$$\mathbf{p} = \frac{\partial F_2}{\partial \mathbf{q}}, \qquad \mathbf{Q} = \frac{\partial F_2}{\partial \mathbf{P}}.$$

(b) A physical system with two degrees of freedom is described by the Hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = H_0(p_1, p_2) + H_1(p_1, p_2) \cos\theta,$$

where

$$\theta = n_1 q_1 + n_2 q_2$$

and n_1 and n_2 are non-zero integers.

Show that a certain linear combination of p_1 and p_2 is conserved, and that there is a (linear) canonical transformation $(\mathbf{q}, \mathbf{p}) \mapsto (\mathbf{Q}, \mathbf{P})$ such that $Q_1 = \theta$ and the transformed Hamiltonian does not depend on Q_2.

Explain why the system is integrable.

Paper 2, Section I
8B Classical Dynamics

Show that Hamilton's equations for a system with n degrees of freedom can be written in the form

$$\dot{x}_a = \Omega_{ab} \frac{\partial H}{\partial x_b},$$

where $a, b \in \{1, 2, \ldots, 2n\}$ and Ω is a matrix that you should define.

Using a similar notation, define the Poisson bracket $\{f, g\}$ of two functions $f(\mathbf{x}, t)$ and $g(\mathbf{x}, t)$. Evaluate the Poisson bracket $\{x_a, x_b\}$.

Show that the transformation $\mathbf{x} \mapsto \mathbf{X}(\mathbf{x})$ preserves the form of Hamilton's equations if and only if the Jacobian matrix

$$J_{ab} = \frac{\partial X_a}{\partial x_b}$$

satisfies

$$J \Omega J^T = \Omega.$$

Deduce that such a canonical transformation leaves the phase-space volume invariant.

Paper 3, Section I
8B Classical Dynamics

The Lagrangian of the Lagrange top can be written as

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos \theta \right)^2 - Mgl \cos \theta \,.$$

Define the *generalized momenta* p_ϕ and p_ψ, and describe how they evolve in time.

Show that the nutation of the top is governed by the equation

$$\frac{1}{2} I_1 \dot{\theta}^2 + V_{\text{eff}}(\theta) = \text{constant} \,,$$

where $V_{\text{eff}}(\theta)$ is an effective potential energy that you should define.

Explain why p_ϕ and p_ψ must be equal in order for the top to reach the vertical position $\theta = 0$. In this case, show that $\theta = 0$ is a stable equilibrium provided that the top spins sufficiently fast.

Paper 4, Section I
8B Classical Dynamics

A particle of mass $m_1 = 3m$ is connected to a fixed point by a massless spring of natural length l and spring constant k. A second particle of mass $m_2 = 2m$ is connected to the first particle by an identical spring. The masses move along a vertical line in a uniform gravitational field g, such that mass m_i is a distance $z_i(t)$ below the fixed point and $z_2 > z_1 > 0$.

[*You may assume that the potential energy of a spring of length $l + x$ is $\frac{1}{2} k x^2$, where k is the spring constant and l is the natural length.*]

Write down the Lagrangian of the system.

Determine the equilibrium values of z_i.

Let q_i be the departure of z_i from its equilibrium value. Show that the Lagrangian can be written as

$$L = \frac{1}{2} T_{ij} \dot{q}_i \dot{q}_j - \frac{1}{2} V_{ij} q_i q_j + \text{constant} \,,$$

and determine the matrices T and V.

Calculate the angular frequencies and eigenvectors of the normal modes of the system.

In what sense are the eigenvectors orthogonal?

Paper 2, Section II
14B Classical Dynamics

(a) A homogeneous, solid ellipsoid of mass M occupies the region

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}<1\,,$$

where a, b and c are positive constants. Calculate the inertia tensor of the ellipsoid.

(b) According to Poinsot's construction, the evolution of the angular velocity vector $\boldsymbol{\omega}(t)$ of a rigid body undergoing free rotational motion corresponds to the movement of an inertia ellipsoid on an invariable plane. Derive this construction, explaining why the inertia ellipsoid is tangent to the invariable plane and rolls on it.

(c) Describe qualitatively the general free rotational motion of the body considered in part (a) in an inertial frame of reference, in the special case $a=b<c$.

Paper 4, Section II
15B Classical Dynamics

An isolated three-body system consists of particles with masses m_1, m_2 and m_3 and position vectors $\mathbf{r}_1(t)$, $\mathbf{r}_2(t)$ and $\mathbf{r}_3(t)$. The particles move under the action of their mutual gravitational attraction. Write down the Lagrangian L of the system.

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be defined by

$$\mathbf{a}=\mathbf{r}_1-\mathbf{r}_2\,,\qquad \mathbf{b}=\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2}-\mathbf{r}_3\,,\qquad \mathbf{c}=\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2+m_3\mathbf{r}_3}{m_1+m_2+m_3}\,.$$

By expressing $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, or otherwise, show that the total kinetic energy can be written as

$$\frac{1}{2}\alpha|\dot{\mathbf{a}}|^2+\frac{1}{2}\beta|\dot{\mathbf{b}}|^2+\frac{1}{2}\gamma|\dot{\mathbf{c}}|^2\,,$$

and obtain expressions for α, β and γ.

Show that the total potential energy can be expressed as a function of $\mathbf{a}$ and $\mathbf{b}$ only. What does this imply for the evolution of $\mathbf{c}$? Give a physical interpretation of this result.

Show also that the total angular momentum of the system about the origin is

$$\alpha\,\mathbf{a}\times\dot{\mathbf{a}}+\beta\,\mathbf{b}\times\dot{\mathbf{b}}+\gamma\,\mathbf{c}\times\dot{\mathbf{c}}\,.$$

Paper 1, Section I
3K Coding and Cryptography

(a) State *Kraft's inequality.*

Show that Kraft's inequality gives a necessary condition for the existence of a prefix-free code with given codeword lengths.

(b) A *comma code* is one where a special letter—the comma—occurs at the end of each codeword and nowhere else. Show that a comma code is prefix-free and give a direct argument to show that comma codes must satisfy Kraft's inequality.

Give an example of a non-decipherable code satisfying Kraft's inequality.

Paper 2, Section I
3K Coding and Cryptography

What is a *discrete memoryless channel* (DMC)? State *Shannon's second coding theorem.*

Consider two DMCs of capacities C_1 and C_2, each having input alphabet $\mathcal{A}$ and output alphabet $\mathcal{B}$. The *product* of these channels is a channel whose input and output alphabets are $\mathcal{A} \times \mathcal{A}$ and $\mathcal{B} \times \mathcal{B}$, respectively, with channel probabilities given by

$$\mathbb{P}(y_1 y_2 | x_1 x_2) = \mathbb{P}_1(y_1|x_1)\, \mathbb{P}_2(y_2|x_2),$$

where $\mathbb{P}_i(y|x)$ is the probability that y is received when x is transmitted through the ith channel ($i = 1, 2$). Find the capacity of the product channel in terms of C_1 and C_2.

Paper 3, Section I
3K Coding and Cryptography

(a) Let C_1 and C_2 be (binary) linear codes with $C_2 \subseteq C_1$. Define their *bar product* $C_1|C_2$.

(b) (i) Let $d \geqslant 1$. Identify the Reed–Muller codes $\mathrm{RM}(d, 0)$ and $\mathrm{RM}(d, d)$ as well-known codes of a certain length. [Proofs are not required.]

For $0 < r < d$, identify the Reed–Muller code $\mathrm{RM}(d, r)$ as a bar product of certain Reed–Muller codes. [Proofs are not required.] Use this to compute the rank of $\mathrm{RM}(d, r)$.

(ii) By considering the original definition of Reed-Muller codes, show that every codeword in $\mathrm{RM}(d, d-1)$ has even weight. Deduce that $\mathrm{RM}(d, r)$ has dual code $\mathrm{RM}(d, d-r-1)$.

Paper 4, Section I
3K Coding and Cryptography

In this question we work over $\mathbb{F}_2$.

What is a *general feedback shift register of length d* with *initial fill* $(x_0, \ldots, x_{d-1})$? What does it mean for such a register to be *linear*?

Describe the Berlekamp–Massey method for breaking a cipher stream arising from a linear feedback shift register.

Use the Berlekamp–Massey method to find a linear recurrence with first eight terms $1, 1, 0, 0, 1, 0, 1, 1$.

Paper 1, Section II
11K Coding and Cryptography

(a) Let n be an odd integer. What does it mean to say that a code is a *cyclic code of length n with a defining set?* Define a *BCH code with design distance* δ. Show that a BCH code with design distance δ has minimum distance at least δ. [Properties of the Vandermonde determinant may be assumed.]

(b) Let $\alpha \in \mathbb{F}_{16}$ be a root of $X^4 + X + 1$. Let C be the BCH code of length 15 and design distance 5, with defining set the first few powers of α.

(i) Find the minimal polynomial for each element of the defining set, and hence find the generator polynomial of C.

(ii) Define the *error locator polynomial* $\sigma(X) \in \mathbb{F}_{16}[X]$ for any received word $r(X)$. [Properties of $\sigma(X)$ may be stated without proof.]

(iii) Suppose you receive the word $r(X) = 1 + X + X^7$. Find the error locator polynomial. Hence, either determine the error position or positions of $r(X)$, or explain why this is not possible.

Paper 2, Section II
12K Coding and Cryptography

(a) Consider two large distinct primes $p, q \equiv 3 \pmod 4$ and let $N = pq$. Briefly describe the *Rabin cipher* with modulus N.

I announce that I shall be using the Rabin cipher with modulus N. My friendly agent in Doxfor sends me a message m (with $1 \leqslant m \leqslant N-1$) encoded in the required form. Unfortunately, my cat eats the piece of paper on which the prime factors of N are recorded so I am unable to decipher it. I therefore find a new pair of primes and announce that I shall be using the Rabin code with modulus $N' > N$. My agent now re-encodes the message and sends it to me again.

The enemy agent Omicron intercepts both code messages. Show that Omicron can find m. Can Omicron decipher any other messages sent to me using only one of the coding schemes?

(b) Let p be a large prime and g a primitive root modulo p. What is the *discrete logarithm problem*? Explain what is meant by the *Diffie-Hellmann key exchange* and say briefly how an enemy can break the cipher if she can compute discrete logarithms efficiently.

Extend the Diffie–Hellman key exchange to cover three participants in a way that is likely to be as secure as the two-party system.

Extend the system further to n parties in such a way that they can compute their common secret key in at most $n^2 - n$ communications. (The numbers p and g of our original Diffie-Hellman system are known by everybody in advance.)

Paper 1, Section I

9A Cosmology

Consider the process where protons and electrons combine to form neutral hydrogen atoms at temperature T. Let n_H be the number density of hydrogen atoms, n_e the number density of electrons, m_e the mass of the electron and $E_{\rm bind}$ the binding energy of hydrogen. Derive *Saha's equation* which relates the ratio n_H/n_e^2 to m_e, $E_{\rm bind}$ and T. Clearly describe the steps required.

[*You may use without proof that at temperature T and chemical potential μ, the number density n of a non-relativistic particle species with mass $m \gg k_B T/c^2$ is given by*

$$n = g\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} \exp\left[-\frac{(m\,c^2-\mu)}{k_B\,T}\right] ,$$

where g is the number of degrees of freedom of this particle species and k_B, $\hbar$ and c are the Boltzmann, Planck and speed of light constants, respectively.]

Paper 2, Section I

9A Cosmology

Consider a ball centered on the origin which is initially of uniform energy density ρ and radius L. The ball expands outwards away from the origin. Additionally, take a particle of mass m at some position $\mathbf{x}$ with $r \equiv |\mathbf{x}| \ll L$. Assume that the particle only experiences gravity through Newton's inverse-square law.

Using the above model of the expanding universe, derive the *Friedmann equation* describing the evolution of the scale factor $a(t)$ appearing in the relation $\mathbf{x}(t) = a(t)\,\mathbf{x}_0$.

Describe the two main flaws in this derivation of the Friedmann equation.

Paper 3, Section I

9A Cosmology

Combining the Friedmann and continuity equations

$$H^2 = \frac{8\pi G}{3c^2}\left(\rho - \frac{k\,c^2}{R^2\,a^2}\right) , \qquad \dot\rho + 3\,H\,(\rho + P) = 0\,,$$

derive the *Raychaudhuri equation* (also known as the *acceleration equation*), which expresses $\ddot a/a$ in terms of the energy density ρ and pressure P.

Assume that the strong energy condition $\rho + 3P \geqslant 0$ holds. Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(H^{-1}\right) \geqslant 1.$$

Deduce that $H \to +\infty$ and $a \to 0$ at a finite time in the past or in the future. What property of H distinguishes the two cases? In one sentence, describe the implications for the evolution of this model universe.

Paper 4, Section I

9A Cosmology

Consider a closed Friedmann-Robertson-Walker universe filled with a fluid endowed with an energy density $\rho \geqslant 0$ and pressure $P \geqslant 0$. For such a universe the Friedmann equation reads

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho - \frac{c^2}{R^2\, a^2}\,,$$

where $a(t)$ is the scale factor.

What is the meaning of R? Show that a closed universe cannot expand forever.

[*Hint: Use the continuity equation to show that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho\, a^3) \leqslant 0\,. \qquad]$$

Paper 1, Section II
15A Cosmology

The continuity, Euler and Poisson equations governing how a non-relativistic fluid composed of particles with mass m, number density n, pressure P and velocity $\mathbf{v}$ propagate in an expanding universe take the form

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\boldsymbol{\nabla} \cdot (\rho\,\mathbf{v}) = 0\,,$$
$$\rho a \left(\frac{\partial}{\partial t} + \frac{\mathbf{v}}{a} \cdot \boldsymbol{\nabla}\right)\mathbf{u} = -c^2 \boldsymbol{\nabla} P - \rho \boldsymbol{\nabla}\Phi\,,$$
$$\nabla^2\Phi = \frac{4\pi G}{c^2}\rho\, a^2\,,$$

where $\rho = mc^2 n$, $\mathbf{u} = \mathbf{v} + a\,H\,\mathbf{x}$, $H = \dot{a}/a$, Φ is the gravitational potential and $a(t)$ is the scale factor.

Consider small perturbations about a homogeneous and isotropic flow,

$$n = \bar{n}(t) + \epsilon\,\delta n\,, \quad \mathbf{v} = \epsilon\,\delta\mathbf{v}\,, \quad P = \bar{P}(t) + \epsilon\,\delta P \quad \text{and} \quad \Phi = \bar{\Phi}(t,\mathbf{x}) + \epsilon\,\delta\Phi\,,$$

with $\epsilon \ll 1$.

(a) Show that, to first order in ϵ, the continuity equation can be written as

$$\dot{\delta} + \frac{1}{a}\boldsymbol{\nabla} \cdot \delta\mathbf{v} = 0\,, \qquad (\dagger)$$

where $\delta = \delta n/\bar{n}$ is the *density contrast*.

(b) Show that, to first order in ϵ, the Euler equation can be written as

$$m\,\bar{n}\,a\,(\dot{\delta\mathbf{v}} + H\,\delta\mathbf{v}) = -\boldsymbol{\nabla}\delta P - m\,\bar{n}\boldsymbol{\nabla}\delta\Phi\,. \qquad (\dagger\dagger)$$

(c) Now assume that $\delta P = c_s^2\, m\, \delta n$. Using ($\dagger$), ($\dagger\dagger$) and the perturbed Poisson equation, show that the density contrast δ obeys

$$\ddot{\delta} + 2\,H\,\dot{\delta} - c_s^2\left(\frac{1}{a^2}\nabla^2 + k_{\rm J}^2\right)\delta = 0 \qquad (\star)$$

and express $k_{\rm J}$ as a function of $\bar{n}$, m and c_s^2.

(d) Neglecting the bracketed terms in equation ($\star$), solve it to find the form of the growth of matter perturbations in a radiation-dominated universe.

Paper 3, Section II
14A Cosmology

(a) What are the cosmological *flatness* and *horizon* problems? Explain what forms of time evolution of the cosmological scale factor $a(t)$ must occur during a period of inflationary expansion in a Friedmann-Robertson-Walker universe. How can inflation solve the flatness and horizon problems? [You may assume an equation of state where the pressure P is proportional to the energy density ρ.]

(b) Consider a universe with a Hubble expansion rate $H = \dot{a}/a$ containing a single inflaton field ϕ with a potential $V(\phi) \geqslant 0$. The density and pressure are given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,,$$
$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,.$$

Show that the continuity equation

$$\dot{\rho} + 3H(\rho + P) = 0$$

demands

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0\,. \qquad (\dagger)$$

(c) Consider the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\rho\,, \qquad (\dagger\dagger)$$

and show that

$$\frac{\ddot{a}}{a} = \frac{8\pi G}{3c^2}\left[V(\phi) - \dot{\phi}^2\right]\,.$$

Under what conditions does an inflationary phase occur?

(d) What is *slow roll inflation*? Show that in slow roll inflation, the scalar equation ($\dagger$) and Friedmann equation ($\dagger\dagger$) reduce to

$$3H\dot{\phi} \approx -\frac{\mathrm{d}V}{\mathrm{d}\phi} \quad \text{and} \quad H^2 \approx \frac{8\pi G}{3c^2}V(\phi)\,. \qquad (\star)$$

(e) Using the slow roll equations ($\star$), determine $a(\phi)$ and $\phi(t)$ when $V(\phi) = \frac{1}{4}\lambda\phi^4$, with $\lambda > 0$.

Paper 1, Section II
26I Differential Geometry

Let $S \subset \mathbb{R}^3$ be an oriented surface. Define its *Gauss map* N. For each $p \in S$, show that the derivative of N defines a self-adjoint operator on T_pS, and define the *principal curvatures* of S at a point p. What does it mean for p to be an *umbilical point*? What does it mean for S to be a *minimal surface*?

(a) We say that a smooth map $f : S \to R$ between two surfaces in $\mathbb{R}^3$ is *conformal* if

$$\langle Df_p(u), Df_p(v) \rangle = \lambda(p) \langle u, v \rangle$$

for all $p \in S$ and $u, v \in T_pS$, where $\lambda(p) > 0$.

Show that, if S does not have any umbilical points, then S is a minimal surface if and only if its Gauss map is conformal.

(b) Now drop the assumption about umbilical points. If S is a minimal surface, must its Gauss map be conformal? If the Gauss map is conformal, must S be a minimal surface? Justify your answers.

(c) Suppose S is a connected minimal surface. Can the image of its Gauss map be a great circle in S^2?

Paper 2, Section II
26I Differential Geometry

Define a *k-dimensional smooth manifold*, and a *regular value* of a smooth map between smooth manifolds. State the *inverse function theorem*, and use it to prove the *preimage theorem.*

Suppose X and Y are smooth manifolds and $f : X \to Y$ is a smooth map. If X is compact, show that the set of regular values of f in Y is open.

Consider the space

$$X_a = \{x + y - z^2 - w^2 = a\} \cap \{x^2 + y^2 - z^4/2 = 0\},$$

where x, y, z, w are the standard coordinates on $\mathbb{R}^4$, and $a \in \mathbb{R}$ is a constant. Show that X_a is a 2-dimensional manifold whenever $a \neq 0$. Is X_0 a manifold? Justify your answer.

Paper 3, Section II
25I Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface. Define the *first fundamental form* of S. If $R \subset \mathbb{R}^3$ is also a surface, we say that a smooth map $\phi : S \to R$ is a *local isometry* if $D\phi$ preserves the first fundamental form at each point.

(a) Let $\alpha : I \to S$ be a curve, and let V be a vector field along α. Define the *covariant derivative* of V. What does it mean for α to be *geodesic*? If $\phi : S \to R$ is a local isometry, show that for an arbitrary geodesic $\alpha : I \to S$, $\phi \circ \alpha$ is also a geodesic. [You may use without proof the fact that Christoffel symbols only depend on the first fundamental form.] Must the converse be true? Give a proof or counterexample.

(b) Define the *Gauss curvature* of S. Suppose $\phi : S \to R$ is a local isometry, and let K_S and K_R denote the Gauss curvatures of S and R respectively. Is it true that $K_R \circ \phi = K_S$? State any theorem you use.

(c) Let R be the surface of revolution defined by the curve $\gamma(u) = (e^u, 0, u)$, with $-\infty < u < \infty$. Let S be the surface of revolution defined by the curve $\delta(s) = (\cosh s, 0, s)$, with $0 < s < \infty$.

(i) Show that there is a diffeomorphism $\phi : S \to R$ such that $K_R \circ \phi = K_S$.

(ii) Does there exist a local isometry $\psi : S \to R$? Justify your answer.

[*Hint: You may use without proof that the surface of revolution defined by the curve* $(f, 0, g)$ *has Gauss curvature given by*

$$\frac{(f'g'' - f''g')g'}{((f')^2 + (g')^2)^2 f} .$$

Standard facts about surfaces of revolution may be used without proof if clearly stated.]

Paper 4, Section II
25I Differential Geometry

(a) State *Wirtinger's inequality*. State and prove the *isoperimetric inequality* for domains $\Omega \subset \mathbb{R}^2$ with compact closure and C^1 boundary $\partial\Omega$.

(b) Let $Q \subset \mathbb{R}^2$ be a *cyclic* quadrilateral, meaning that there is a circle through its four vertices. Say its edges have lengths a, b, c and d (in cyclic order). Assume $Q' \subset \mathbb{R}^2$ is another quadrilateral with edges of lengths a, b, c and d (in the same order). Show that $\text{Area}(Q) \geqslant \text{Area}(Q')$. Explain briefly for which Q' equality holds.

Paper 1, Section II
32B Dynamical Systems

(a) Consider a dynamical system of the form

$$\begin{aligned}\dot{x} &= f(x,y)\,,\\ \dot{y} &= g(x,y)+\epsilon p(x,y)\,,\end{aligned}$$

which is Hamiltonian for $\epsilon=0$. Explain the *energy balance method*. What does it tell us about periodic orbits of this system for small ϵ?

(b) (i) For $0<\epsilon\ll 1$, use the energy balance method to seek leading-order approximations to periodic orbits of this system

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= -4x+\epsilon\left[(1-2x^2)ky-(1-3x^2)y^3\right]\,,\end{aligned}$$

where $k>0$.

[*Hint:* $\int_0^{2\pi}\sin^4\theta d\theta=\frac{3}{4}\pi$ *and* $\int_0^{2\pi}\sin^6\theta d\theta=\frac{5}{8}\pi$.]

(ii) For the cases $0<k<6$ and for $k>6$, deduce the stability of any periodic orbits.

(iii) What can we deduce from this approach about the existence of periodic orbits near $k=6$?

Paper 2, Section II
33B Dynamical Systems

(a) Let $F : I \to I$ be a continuous one-dimensional map of an interval $I \in \mathbb{R}$. Define what it means for F to have a *horseshoe*.

Define what it means for F to be *chaotic*. [Glendinning's definition should be used throughout this question.]

Prove that if F has a 3-cycle then F^2 has a horseshoe. [You may assume corollaries of the Intermediate Value Theorem.]

(b) Suppose now that F has a 4-cycle, and consider each of these orderings of the points of the 4-cycle:

(i) $x_0 < x_1 < x_2 < x_3$

(ii) $x_0 < x_1 < x_3 < x_2$

(iii) $x_0 < x_2 < x_1 < x_3$

For each of these orderings, construct a suitable directed graph. Based on each of these directed graphs, determine if the corresponding F must be chaotic and also give the minimum number of distinct 3-cycles that F must have.

Give an explicit example of a continuous map $F : [0,1] \to [0,1]$ which has a 4-cycle and is not chaotic. [*Hint: choose a suitable ordering for the points on the 4-cycle, construct a function which is piece-wise linear between these points, and examine the dynamics of this map.*]

Paper 3, Section II

31B Dynamical Systems

Consider the system

$$\begin{aligned}\dot{x} &= -ax + 3y + x(x^2 + y^2) \\ \dot{y} &= -x - ay + y(x^2 + y^2)\,,\end{aligned}$$

where $a > 0$ is a real constant. Throughout this question, you should state carefully any theorems or standard results used.

(a) Show that the origin is asymptotically stable.

(b) Define the term *Lyapunov function.* For the system above, for what values of k is $V(x, y) = x^2 + ky^2$ a valid Lyapunov function in some neighbourhood of the origin? Give your answer in the form $k_1(a) < k < k_2(a)$ where $k_1(a)$ and $k_2(a)$ should be given explicitly.

(c) By considering $V(x, y)$ for $k = 1$, what can be deduced about the domain of stability (for values of a for which $V(x, y)$ is a valid Lyapunov function)?

(d) State the *Poincaré-Bendixson theorem.* Show that the system above has a periodic orbit.

Paper 4, Section II

32B Dynamical Systems

Consider the dynamical system

$$\begin{aligned}\dot{x} &= x(y - k - 3x + x^2) \\ \dot{y} &= y(y - 1 - x)\,,\end{aligned}$$

where k is a constant.

(a) Find all the fixed points of this system. By considering the existence and location of the fixed points, determine the values of k for which bifurcations occur. For each of these, what types of bifurcation are suggested from this approach?

(b) For the fixed points whose positions are independent of k, determine their linear stability. Verify that these results are consistent with the bifurcations suggested above.

(c) Focusing only on the bifurcations which occur for $0 \leqslant k \leqslant \frac{1}{2}$, use centre manifold theory to analyse these bifurcations. In particular, for each bifurcation derive an equation for the dynamics on the extended centre manifold and hence classify the bifurcation. [*Hint: There are two bifurcations in this range.*]

Paper 1, Section II
37B Electrodynamics

Consider a localised electromagnetic field in vacuum with electric and magnetic fields **E** and **B** respectively in the absence of charges and currents.

(a) Show that the energy density $\epsilon = \frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$ obeys a local conservation law

$$\partial_t \epsilon + \boldsymbol{\nabla} \cdot \mathbf{N} = 0\,.$$

Hence obtain an expression for the vector **N** and remark on its physical significance. Here ε_0 and μ_0 are the electric and magnetic permeabilities of the vacuum.

(b) Show that the momentum density $\mathbf{g} = \varepsilon_0 \mathbf{E} \times \mathbf{B}$ obeys a local conservation law

$$\partial_t g_j + \nabla_i \sigma_{ij} = 0\,.$$

Hence obtain an expression for the second-rank tensor σ_{ij} and remark on its physical significance.

(c) Defining the tensor

$$T^{\mu\nu} = \begin{bmatrix} \epsilon & cg_j \\ N_i/c & \sigma_{ij} \end{bmatrix}$$

show that the results of (a) and (b) can be expressed as $\partial_\mu T^{\mu\nu} = 0$.

(d) Using the fact that the tensor σ_{ij} is symmetric, show that the integral over all space of the angular momentum density $\mathbf{L} = \mathbf{x} \times \mathbf{g}$ is independent of time. Here $\mathbf{x}$ is the position with respect to the origin of an inertial frame.

(e) Show that the symmetry of σ_{ij} in all inertial frames requires $\mu_0 \epsilon_0 = 1/c^2$.

Paper 3, Section II
36B Electrodynamics

Consider a time-dependent localised electromagnetic field in vacuum with a four-current density J^μ and vector potential A^μ.

(a) Determine the differential equation that relates the four-current density to the vector potential in the gauge choice $\partial_\mu A^\mu = 0$.

(b) Show that the solution to the above differential equation can be expressed as

$$A^\mu(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int \frac{J^\mu(\mathbf{x}', t')}{|\mathbf{x} - \mathbf{x}'|} \, d^3x'$$

where you should specify the form of t'.

(c) Show that the time derivative of the dipole moment $\mathbf{p}$ satisfies

$$\dot{\mathbf{p}} = \int \mathbf{J}(\mathbf{x}, t) \, d^3x$$

where $\mathbf{J}$ is the current density.

(d) A small circular loop of radius r is centred at the origin. The unit vector normal to the plane of the loop is $\mathbf{n}$. A current $I(t) = \sum_{n=0}^{\infty} I_n \sin(n\omega t)$ flows in the loop. Find the three vector potential $\mathbf{A}(\mathbf{x}, t)$ to first order in $r/|\mathbf{x}|$.

Paper 4, Section II
36B Electrodynamics

(a) Explain what is meant by a *dielectric material*.

(b) Define the *polarisation* of, and the *bound charge* in, a dielectric material. Explain the reason for the distinction between the electric field $\mathbf{E}$ and the electric displacement $\mathbf{D}$ in a dielectric material.

Consider a sphere of a dielectric material of radius R and permittivity ε_1 embedded in another dielectric material of infinite extent and permittivity ε_2. A point charge q is placed at the centre of the sphere. Determine the bound charge on the surface of the sphere.

(c) Define the *magnetisation* of, and the *bound current* in, a dielectric material. Explain the reason for making a distinction between the magnetic flux density $\mathbf{B}$ and the magnetic intensity $\mathbf{H}$ in a dielectric material.

Consider a cylinder of dielectric material of infinite length, radius R and permeability μ_1 embedded in another dielectric material of infinite extent and permeability μ_2. A line current I is placed on the axis of the cylinder. Determine the magnitude and direction of the bound current density on the surface of the cylinder.

Paper 1, Section II
39C Fluid Dynamics II

A viscous fluid of viscosity μ and density ρ is located in the annulus confined between two long co-axial cylinders of radii R and αR with $\alpha < 1$. The ends of the annular space are open to the atmosphere. The axes of the cylinders are aligned in the vertical direction. We use cylindrical coordinates (r, θ, z) with unit vector $\mathbf{e}_z$ in the downward vertical direction. There is a gravitational force g per unit mass acting on the fluid in the downward direction. In the following you may consider the flow in the long central region of the annulus, far from the ends, and neglect any details of the flow near the ends.

The outer cylinder is fixed and stationary. The inner cylinder steadily translates along its axis with velocity $V\mathbf{e}_z$. The fluid flow between the two cylinders may be assumed to be steady and unidirectional.

(a) Explain why we expect the velocity $\mathbf{u}$ to be of the form $\mathbf{u} = u(r)\mathbf{e}_z$.

(b) Derive the equation satisfied by $u(r)$ and state the corresponding boundary conditions.

(c) Show that the pressure gradient in the z-direction is constant and compute its value.

(d) Solve for the flow $u(r)$ in the annular gap and sketch it for $V = 0$, and for two further values of V, one positive and one negative.

(e) Calculate the force per unit length acting on the inner cylinder and the corresponding force per unit length acting on the outer cylinder. Comment on the sum of these forces.

[*Hint: in cylindrical coordinates* (r, θ, z) *with velocity components* (u_r, u_θ, u_z) *we have*

$$\nabla^2 u_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}.$$

The rz*-component of the rate-of-strain tensor is* $e_{rz} = \dfrac{1}{2}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right)$.]

Paper 2, Section II
39C Fluid Dynamics II

(a) A fluid has kinematic viscosity $\nu > 0$. In flow over a stationary rigid boundary with length scale $\mathcal{L}$, the fluid velocity far from the boundary has typical magnitude $\mathcal{U}$. Define the *Reynolds number*. Explain why even if the Reynolds number is large the effects of viscosity cannot be neglected and explain briefly how boundary layer theory provides a useful approximate approach to including these effects.

(b) A steady high-Reynolds number flow is induced in a semi-infinite fluid otherwise at rest, in the region $y > 0$, by the in-plane motion of an extensible sheet lying along $x \geqslant 0, y = 0$. Points on the sheet move with velocity $\mathbf{V} = \alpha x\, \mathbf{e}_x$, where α is the prescribed constant rate of extension and $\mathbf{e}_x$ is the unit vector in the x-direction.

(i) What should be chosen for the typical flow speed $U(x)$ in the boundary layer? Give an estimate of the corresponding x-dependent Reynolds number and deduce that, for x sufficiently large, the flow is described by the boundary layer equations. Derive the fundamental boundary-layer scaling relating $U(x)$ and the thickness $\delta(x)$ of the boundary layer and deduce the scaling for $\delta(x)$ as a function of x.

(ii) State the two-dimensional boundary layer equations and their boundary conditions for this problem in terms of a streamfunction $\psi(x, y)$.

(iii) Seek a similarity solution to the boundary layer equations using

$$\psi(x, y) = U(x)\delta(x)f(\eta)\,,$$

where $\eta \equiv \frac{y}{\delta(x)}$. Derive the ODE and boundary conditions satisfied by $f(\eta)$.

(iv) Show that the ODE satisfied by f has a solution of the form $A + B\exp(-C\eta)$ and determine the values of the constants A, B and C.

(i) Comment on the behaviour of f as $\eta \to \infty$. What are the implications for the flow external to the boundary layer?

Paper 3, Section II
38C Fluid Dynamics II

A uniform rod in the shape of an elongated cylinder falls through a viscous fluid under the action of gravity. The motion is sufficiently slow that the fluid flow is described by the Stokes equations.

(a) Show that when the long axis of the rod is initially aligned with the horizontal direction the rod falls vertically.

(b) Show that for *any* initial orientation of the rod the motion of the rod occurs with no rotation.

(c) Denoting by $\mathbf{F}$ the hydrodynamic force exerted on the rod and $\mathbf{U}$ its translation speed, explain why we expect a linear relationship of the form $\mathbf{F} = -\mathbf{R} \cdot \mathbf{U}$, where $\mathbf{R}$ is a matrix.

(d) State the reciprocal theorem of Stokes flows. Show that it implies that $\mathbf{R}$ is symmetric.

(e) Use the energy equation, as applied to this steady flow problem, to deduce that the matrix $\mathbf{R}$ is also positive definite.

(f) We denote by $\mathbf{t}$ the unit tangent vector along the rod and by θ the angle between $\mathbf{t}$ and the vertical. Writing $\mathbf{R} = c_1\mathbf{tt} + c_2(\mathbf{1} - \mathbf{tt})$ with $c_2 \geqslant c_1 > 0$, compute the value of $\cos\alpha$ where α is the angle between the vertical and the direction of motion of the rod. Check the case where $c_1 = c_2$ and comment.

Paper 4, Section II
38C Fluid Dynamics II

A thin layer of fluid is flowing down an inclined plane due to the action of gravity. The gravitational acceleration is g, the viscosity of the fluid is μ and the density of the fluid is ρ. The angle between the plane and the horizontal is denoted by α. Cartesian coordinates are defined with x along the plane in the downward direction and y perpendicular to the plane. All quantities may be assumed to be constant in the in-plane direction perpendicular to the slope. The thickness of the fluid layer is denoted by $h(x,t)$.

(a) Assume that the dynamics of the layer is described by the lubrication equations and hence estimate the order of magnitude for the flow speed u in the film. Deduce the two conditions involving h, $\partial h/\partial x$ and the other parameters of the problem that are required for the assumption of the lubrication limit to be self-consistent.

(b) State the momentum equations in the (x,y) coordinates under the lubrication-limit assumption. What are the boundary conditions for the velocity and the pressure?

(c) Solve for the pressure in the fluid and deduce the flow velocity along the plane.

(d) Applying conservation of mass, deduce the partial differential equation satisfied by $h(x,t)$.

(e) Seek a travelling-wave solution $h(x,t) = f(x - ct)$ and hence derive a first-order ODE (containing an unknown constant of integration) satisfied by the function f.

Paper 1, Section I
7E Further Complex Methods

Show that

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{s^{z-1}}{s-t}\, ds = \pi i t^{z-1},$$

where t is real and positive, $0 < \text{Re}(z) < 1$ and the branch of s^z is chosen so that, for z real, s^z is real and positive for s real and positive and $s^z = (-s)^z e^{i\pi z}$ for s real and negative.

Deduce that for z real with $0 < z < 1$

$$\int_0^{\infty} \frac{s^{z-1}}{s+t}\, ds = \pi t^{z-1} \text{cosec} \pi z$$

and

$$\mathcal{P}\int_0^{\infty} \frac{s^{z-1}}{s-t}\, ds = -\pi t^{z-1} \cot \pi z.$$

Why do these results actually hold for a large set of non-real z?

Paper 2, Section I
7E Further Complex Methods

A complex function Arcsinh(z) may be defined by

$$\text{Arcsinh}(z) = \int_0^z \frac{1}{(1+t^2)^{1/2}}\, dt\,,$$

where the integrand $(1+t^2)^{-1/2}$ is equal to $1/\sqrt{2}$ at $t=1$ and has a branch cut along the imaginary axis between the points $\pm i$ (deformed very slightly to the left of the origin).

Explain how to choose the path of integration to ensure that Arcsinh(z) is analytic and single valued in $0 \leqslant \arg z < 2\pi$, except for z on the branch cut specified for $(1+t^2)^{-1/2}$.

Evaluate Arcsinh$(-\sinh(u))$, where u is real and $u > 0$.

Deduce that if arcsinh(z) is an analytic continuation of Arcsinh(z) to the whole complex plane, omitting the branch cut, but without restriction on $\arg(z)$, then it is multivalued. What are the possible values of arcsinh$(\sinh(u))$, with u real and $u > 0$?

Paper 3, Section I
7E Further Complex Methods

Consider the partial differential equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

in $x > 0$ subject to the initial condition $T(x,0) = 0$ for all $x > 0$ and the boundary condition $T(0,t) = \sin \omega t$ for $t > 0$.

Show that the Laplace transform of $T(x,t)$ takes the form

$$\tilde{T}(x,p) = \tilde{T}_0(p) \exp(-(p/\kappa)^{1/2} x)$$

and determine the function $\tilde{T}_0(p)$.

Consider $I(t) = \int_0^\infty T(x,t)\, dx$. Write down an expression for $\tilde{I}(p)$.

Applying the Bromwich contour inversion expression for Laplace transforms gives the result that for $t > 0$

$$I(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{1}{\pi}\int_0^\infty \frac{\omega \kappa^{1/2}}{(s^2+\omega^2)} \frac{e^{-st}}{s^{1/2}}\, ds\,,$$

where A and B are independent of t. Draw a diagram showing the Bromwich contour and explain clearly how the terms appearing in the above expression arise.

Paper 4, Section I
7E Further Complex Methods

What type of equation has solutions described by the following Papperitz symbol?

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ \alpha_1 & \alpha_2 & \alpha_3 & z \\ \beta_1 & \beta_2 & \beta_3 & \end{Bmatrix}$$

Explain the meaning of each of the quantities appearing in the symbol.

The hypergeometric function $F(a,b;c;z)$ is defined by

$$F(a,b;c;z) = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix}$$

with $F(a,b;c;z)$ analytic at $z=0$ and satisfying $F(a,b;c;0) = 1$.

Explain carefully why there are constants A and B such that

$$F(a,b;c;z) = Az^{-a}F(a,1+a-c;1+a-b;z^{-1}) + Bz^{-b}F(b,1+b-c;1+b-a;z^{-1}).$$

[*You may neglect complications associated with special cases such as* $a = b$.]

Paper 1, Section II

14E Further Complex Methods

The polylogarithm function $\mathrm{Li_s}(z)$ is defined for complex values of z ($|z| < 1$) and s (all complex s) by

$$\mathrm{Li_s}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

(a) Briefly justify why the conditions given on z and s given above are appropriate.

Consider the integral

$$I(z,s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{(0+)} \frac{z t^{s-1}}{e^{-t} - z}\, dt\,, \tag{1}$$

where the integral is taken along a Hankel contour, as indicated by the limits.

(b) Show that $I(z,s)$ provides an analytic continuation of $\mathrm{Li_s}(z)$ for all $z \notin (1,\infty)$. [*Hint: You may assume where needed the Hankel representation of the Gamma function,* $\Gamma(z) = (2i\sin \pi z)^{-1} \int_{-\infty}^{(0+)} e^t t^{z-1}\, dt$, *and the result* $\Gamma(z)\Gamma(1-z) = \pi \mathrm{cosec}(\pi z)$.]

Include in your answer a sketch of the Hankel contour, with particular attention to the path of the contour relative to any singularities in the integrand when z is close to, but not on the part $(1,\infty)$ of the real axis.

(c) Describe how to evaluate $I(z,s)$ when s is a non-positive integer. Hence give explicit expressions for $\mathrm{Li_s}(z)$ for $s=0$, $s=-1$ and $s=-2$.

(d) For $s>0$ show that $I(z,s)$ can be expressed in the form

$$I(z,s) = \int_0^{\infty} K(z,s,t)\, dt,$$

where t is a real variable and $K(z,s,t)$ is to be determined. Comment on the required interpretation of the expression (1) when s is a positive integer.

Without detailed calculation, explain (for $s>0$) why $I(z,s)$ jumps by the value $2\pi i (\log x)^{s-1}/\Gamma(s)$ when z moves from just below $(1,\infty)$ to just above $(1,\infty)$ at the point x ($x>1$).

Paper 2, Section II

13E Further Complex Methods

Consider the differential equation

$$\frac{d^3w}{dz^3} - zw = 0\,.$$

Use Laplace's method to find solutions of the form

$$w(z) = \int_\gamma e^{zt}\, f(t)\, dt\,,$$

where γ is a contour in the complex t-plane. Determine the function $f(t)$ and state clearly the condition required for the contour γ.

Draw a sketch of the complex t-plane showing the possible choices of γ, relating these to the behaviour of $f(t)$.

Show that three different suitable contours $\gamma_i, i = 1, 2, 3$, may be formed from the positive real axis plus parts of the real axis or the imaginary axis, with each γ_i defining a function $w_i(z)$. Write down expressions for the values of $w_i(0)$, $w_i'(0)$ and $w_i''(0)$ $(i = 1, 2, 3)$ and evaluate them in terms of Gamma functions.

Give an expression for

$$\det \begin{pmatrix} w_1(0) & w_1'(0) & w_1''(0) \\ w_2(0) & w_2'(0) & w_2''(0) \\ w_3(0) & w_3'(0) & w_3''(0) \end{pmatrix}.$$

Deduce that the functions $w_i(z)$ $(i = 1, 2, 3)$ are linearly independent.

Paper 1, Section II
18H Galois Theory

(a) Let K be a field with $\operatorname{char} K \neq 2, 3$. If $f = x^3 + px + q \in K[x]$, define the *discriminant* of f, and compute it in terms of p and q.

Let L be the splitting field of f and let $G = \mathrm{Aut}(L/K)$ be the Galois group. Describe all possibilities for G. Justify your answer. [Do not assume that f is irreducible.]

Compute all subfields of L when $f = x^3 + 3x + 1 \in \mathbb{Q}[x]$. You may specify the subfields in terms of the roots; you do not need to determine the roots explicitly in terms of radicals.

(b) Let L/K be a Galois extension, and suppose $f \in L[x]$. Show that there exists a non-zero polynomial $g \in L[x]$ such that $fg \in K[x]$.

Now suppose only that L/K is a finite separable extension, and that $f \in L[x]$. Show that there exists a non-zero polynomial $g \in L[x]$ such that $fg \in K[x]$.

Paper 2, Section II
18H Galois Theory

(a) Let L be a finite field of order p^n. Suppose that $\gamma \in L$, and let $f \in \mathbb{F}_p[x]$ be the minimal polynomial of γ over $\mathbb{F}_p$. Show that $\deg f$ divides n. Prove that there is a $\gamma \in L$ for which $\deg f = n$.

Show that for every $r \geqslant 1$, there is an irreducible polynomial $g \in \mathbb{F}_p[x]$ of degree r.

[*You may assume the tower law and the existence of splitting fields, but should prove any results about finite fields that you use.*]

(b) Suppose that K is a field and that L is a finite extension of K. Define what it means for $\alpha \in L$ to be *separable* over K. If $f \in K[x]$ is the minimal polynomial of α and $\gcd(f, f') = 1$ show that α is separable over K.

Now suppose that $L = K(\beta)$ is a finite extension of K and that $\operatorname{char} K = p$. Show there exists a unique intermediate field M with $K \subseteq M \subseteq L$, such that the following conditions hold: M is a separable extension of K, $[L : M] = p^h$ for some h, and $\gamma^{p^h} \in M$ for all $\gamma \in L$. [*Hint: If β is not separable, what is its minimal polynomial?*]

Paper 3, Section II
18H Galois Theory

(a) Let L/K be an extension of fields, and suppose that K contains a primitive nth root of unity ζ. Let $\sigma \in \mathrm{Aut}(L/K)$ be a K-automorphism of L of order n. Prove that there exists a nonzero element $\alpha \in L$ with $\sigma(\alpha) = \zeta\alpha$. What is the minimal polynomial of α over L^{σ}, the fixed field of σ?

(b) Define what it means for L to be an *algebraic closure* of K. Given that

$$\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic over } \mathbb{Q}\}$$

is a field, show that $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. State carefully any results that you use.

(c) Let L be an algebraically closed field of characteristic zero, and $\sigma : L \to L$ a homomorphism of fields. Suppose $\sigma^d = 1$ for some $d > 0$, and let $K = L^{\sigma}$ be the fixed field of σ. If M/K is a finite extension, show that M/K is a Galois extension with cyclic Galois group. [*Hint: Show that there is a K-homomorphism from M to L.*] Give an example showing that the assumption that L is algebraically closed is necessary.

Paper 4, Section II
18H Galois Theory

(a) Stating carefully all the theorems that you use, prove that for every integer $r > 1$ there is a Galois extension $L/\mathbb{Q}$ with Galois group $\mathbb{Z}/r\mathbb{Z}$.

(b) Suppose L_1 and L_2 are two extensions of a field K, and both L_1 and L_2 are subfields of some field M. Let L_1L_2 be the smallest subfield of M containing both L_1 and L_2. If $[L_i : K] = d_i$ and $\gcd(d_1, d_2) = 1$, show that $[L_1L_2 : K] = d_1d_2$.

(c) Let $p \geqslant 3$ be a prime number. Give examples of two non-isomorphic groups G, G' of order $p(p-1)$ containing normal subgroups N, N' of order p such that $G/N \cong G'/N'$.

Fix $p = 3$. For the groups G, G' above, give explicit examples of Galois extensions $L/\mathbb{Q}$ and $L'/\mathbb{Q}$ with $\mathrm{Aut}(L/\mathbb{Q}) \cong G$ and $\mathrm{Aut}(L'/\mathbb{Q}) \cong G'$. Identify the fixed fields L^N and $(L')^{N'}$. Justify your answer.

Now suppose $p > 3$ is an arbitrary prime. Prove that there are extensions L and L' of $\mathbb{Q}$ with $\mathrm{Aut}(L/\mathbb{Q}) \cong G$ and $\mathrm{Aut}(L'/\mathbb{Q}) \cong G'$.

Paper 1, Section II
38D General Relativity

A *Milne universe* is an isotropic, homogeneous model of cosmology which has negative spatial curvature, $k = -1$, and an expanding scale factor, $\dot{a}(t) > 0$, even though there is no matter or radiation ($T_{\alpha\beta} = 0$) and no cosmological constant ($\Lambda = 0$).

(a) Write down the FLRW metric for this cosmological model. Calculate the scale factor $a(t)$ as an explicit function of the proper time t of a stationary observer.

(b) Verify that the singularity as $a \to 0$ is a coordinate singularity by calculating the Kretschmann scalar. [*Hint: You may find it useful to relate the Riemann tensor to the Ricci tensor.*]

(c) By constructing an appropriate coordinate transformation, show that the Milne universe is equivalent to the interior of the future light-cone of a point p in Minkowski space-time. What do the spatial isometries of the hyperbolic $t =$ const. slices correspond to in this Minkowski space-time?

[*Hint: You may wish to use the following formulae:*

$$3\frac{\dot{a} + k}{a^2} - \Lambda = 8\pi\rho\,, \qquad \text{(Friedmann I)}$$

$$2a\ddot{a} + \dot{a}^2 + ka^2 - \Lambda = -8\pi P\,. \qquad \text{(Friedmann II)}$$

Riemann tensor in normal coordinates:

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_\beta\partial_\mu g_{\alpha\nu} + \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\alpha\partial_\mu g_{\beta\nu} - \partial_\beta\partial_\nu g_{\alpha\mu})\,.\,]$$

Paper 2, Section II
38D General Relativity

(a) Consider a 2-sphere with coordinates (θ, ϕ) and metric

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \ .$$

(i) Show that lines of constant longitude ($\phi = \text{constant}$) are geodesics, and that the only line of constant latitude ($\theta = \text{constant}$) that is a geodesic is the equator ($\theta = \pi/2$).

(ii) Take a vector with components $V^\mu = (1, 0)$ in these coordinates, and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector, as functions of θ?

(b) In units where $8\pi G = 1$, the Einstein equation states that $T_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$. Solve for $R_{\alpha\beta}$ in terms of $T_{\alpha\beta}$ and $T = g^{\alpha\beta} T_{\alpha\beta}$, in general space-time dimension $n > 2$.

(c) Using the symmetries of the Riemann curvature tensor, show that in $n = 2$ dimensions, $R_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} R$. [*Hint: Since this is a tensor equation, it only needs to be proved in one particular coordinate system.*] Explain the implications of this if we try to define General Relativity in $n = 2$ space-time dimensions.

Paper 3, Section II

37D General Relativity

Recall that the Schwarzschild metric is

$$ds^2 = -(1 - 2M/r)\, dt^2 + (1 - 2M/r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \ ,$$

in units where $c = G = 1$. An advanced alien civilization builds a static, spherically-symmetrical space station surrounding a non-rotating black hole of mass M. The station itself has mass $M_{\rm st} \ll M$ and is located at a radius $r_{\rm st} > 2M$ (in Schwarzschild coordinates). It occupies a very thin shell of width $\delta r \ll r_{\rm st}$.

(a) Some sodium lamps, which emit photons at a characteristic wavelength λ, are attached to the space station. In terms of $r_{\rm st}$, what is the wavelength of these photons as seen by an observer at radius $r \gg r_{\rm st}$? What happens in the limit that $r_{\rm st}$ approaches the event horizon?

(b) What is the magnitude and direction of the proper acceleration of the space station (*i.e.* the acceleration in its own instantaneous rest frame)? Verify that in the limit $r_{\rm st} \to \infty$, the magnitude is equal to the acceleration due to Newtonian gravity.

Now suppose we wish to take into account the gravitational effects of the space station itself, even though $M_{\rm st} \ll M$. The space station has a mass per unit area of ρ as measured in its own local frame of reference. However, its effective gravitational energy is reduced by the fact that it is in a gravitational potential.

(c) What is an appropriate metric to use outside of the space station? Your answer should indicate how the metric depends on ρ. Why is this justified? [*Hint: You do not need to explicitly solve the Einstein equation in order to answer this problem.*]

Paper 4, Section II
37D General Relativity

(a) Determine whether each of the following spaces is, or is not, a manifold. Justify your answers.

(i) $\mathbb{R}^3$ with points identified if they are related by the transformation $(x, y, z) \to (-x, -y, -z)$.

(ii) $\mathbb{R}^3$, except that the closed ball of all points with $x^2+y^2+z^2 \leqslant 1$ is removed.

(b) Let a tensor $\mathbf{S}$ at point $p \in \mathcal{M}$ be defined as a linear map

$$\mathbf{S} : T_p^*(\mathcal{M}) \to T_p(\mathcal{M}) \times T_p(\mathcal{M}) \,,$$

where T_p is tangent space and T_p^* is cotangent space.

(i) What is the rank of $\mathbf{S}$? Use $\binom{r}{s}$ notation.

(ii) What is the rank of $\mathbf{S} \otimes \nabla \mathbf{S}$, where $\otimes$ is an outer product and ∇ is the covariant derivative?

Consider a spacelike geodesic which goes from point p to point q. As a geodesic, this curve minimizes the action

$$\mathcal{S} = \int_0^1 \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, d\lambda \ ,$$

where $x = x(\lambda)$ with $x(0) = p, x(1) = q$ and $\dot{x}^\mu = dx^\mu / d\lambda$. Show using the Euler-Lagrange equations that

$$\frac{d^2 x^\beta}{ds^2} + \Gamma^\beta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \,,$$

where s is the proper distance along the geodesic and $\Gamma^\beta_{\mu\nu}$ is the Levi-Civita connection.

Paper 1, Section II
17F Graph Theory

(a) Define a *proper k-colouring* of a graph G. Define the *chromatic number* $\chi(G)$ of a graph G. Prove that $\chi(G) \leqslant \Delta(G)+1$ for all graphs G. Do there exist graphs G for which $\chi(G) = \Delta(G)+1$ for each $\Delta(G) = 0, 1, 2, \ldots$?

(b) What does it mean for a graph to be *k-connected*? If G is a non-complete 3-connected graph, show that $\chi(G) \leqslant \Delta(G)$.

(c) State *Euler's formula.* If G is a triangle-free planar graph, prove that $\chi(G) \leqslant 4$.

(d) Define the *edge-chromatic number* $\chi'(G)$ of a graph G. State *Hall's theorem.* If G is a 4-regular bipartite graph, determine $\chi'(G)$.

Paper 2, Section II
17F Graph Theory

(a) For a graph H and a positive integer n, define $ex(n, H)$. Prove that $ex(n, K_3) \leqslant n^2/4$. [You may not assume Turan's theorem without proof.]

(b) For a fixed $\delta > 0$, suppose that G is a graph on n vertices with $e(G) > (1+\delta)n^2/4$. Prove that G must contain at least ϵn^3 triangles, where $\epsilon > 0$ is a constant that does not depend on n or G.

(c) Prove that $ex(n, K_{3,2}) < cn^{3/2}$, for some constant $c > 0$.

(d) Let $x_1, \ldots, x_n$ be distinct points in $\mathbb{R}^2$. Show that there exists a constant $c > 0$ such that at most $cn^{3/2}$ of the ordered pairs (x_i, x_j) can satisfy $|x_i - x_j| = 1$.

Paper 3, Section II
17F Graph Theory

(a) Let G be a graph. Show that G contains a subgraph H with $\chi(H) \leqslant 3$ and

$$e(H) = \lfloor (2/3)e(G) \rfloor .$$

Show that the constant $2/3$ is sharp, in the following sense: for any $\epsilon > 0$ there exists a graph G (with $e(G) > 0$) such that every subgraph H of G with $\chi(H) \leqslant 3$ has $e(H) \leqslant (2/3 + \epsilon)e(G)$.

(b) An *unfriendly partition* of a graph $G = (V, E)$ is a partition $V = A \cup B$, where every $v \in A$ has $|N(v) \cap B| \geqslant |N(v) \cap A|$ and every vertex $v \in B$ has $|N(v) \cap A| \geqslant |N(v) \cap B|$. Show that every finite graph G has an unfriendly partition. [*Hint: Consider a partition $A \cup B = V$ maximizing the number of edges with one end in A and one end in B.*]

(c) Let $G = (\mathbb{N}, E)$ be a countably infinite graph in which all the vertices have finite degree. Show that G has an unfriendly partition.

(d) Let $G = (\mathbb{N}, E)$ be a countably infinite graph in which all the vertices have *infinite* degree. Show that G has an unfriendly partition. (In other words, in this infinite degree case, we want each vertex $v \in A$ to have $N(v) \cap B$ infinite and each $v \in B$ to have $N(v) \cap A$ infinite.)

Paper 4, Section II
17F Graph Theory

Define the *binomial random graph* $G(n,p)$, where $n \in \mathbb{N}$ and $p \in [0,1]$.

Let $G_n \sim G(n,p)$ and let E_n be the event that $\delta(G_n) > 0$. Show that for every $\varepsilon > 0$, if $p = p(n)$ satisfies $p \geqslant (1+\varepsilon)n^{-1}\log n$ then $\mathbb{P}(E_n) \to 1$.

State *Chebyshev's inequality* and show that for every $\varepsilon > 0$, if p is such that $p \leqslant (1-\varepsilon)n^{-1}\log n$ then $\mathbb{P}(E_n) \to 0$.

For $G_n \sim G(n,p)$, let F_n be the event that G_n is connected. Prove that for every $\varepsilon > 0$, if $p \geqslant (1+\varepsilon)n^{-1}\log n$ then $\mathbb{P}(F_n) \to 1$ as $n \to \infty$ and if $p \leqslant (1-\varepsilon)n^{-1}\log n$ then $\mathbb{P}(F_n) \to 0$ as $n \to \infty$. [You may wish to consider separately the case when there is a component of size at most say $n\varepsilon/10$ and the case when there is not.]

[*You may use, without proof, the fact that* $1 - x \leqslant e^{-x}$ *for all* $x \in [0,1]$, *and also that for any fixed* $\delta \in (0,1)$ *we have* $1 - x \geqslant e^{-(1+2\delta)x}$ *for all* $x \in [0,\delta)$. *All logarithms in this question are natural logarithms.*]

Paper 1, Section II
33E Integrable Systems

(a) Show that if L is a symmetric $n \times n$ matrix ($L = L^T$) and B is a skew-symmetric $n \times n$ matrix ($B = -B^T$) then $[B, L] = BL - LB$ is symmetric. If L evolves in time according to

$$\frac{dL}{dt} = [B, L]\,,$$

show that the eigenvalues of L are constant in time.

Write the harmonic oscillator equation $\ddot{q} + \omega^2 q = 0$ in Hamiltonian form. (The frequency ω is a fixed real number). Starting with the symmetric matrix

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}$$

find a Lax pair formulation for the harmonic oscillator and use this formulation to obtain the conservation of energy for the oscillator.

(b) Consider the Airy partial differential equation, given for $-\infty < x < \infty$ and $t \geqslant 0$ by

$$q_t + q_{xxx} = 0. \tag{1}$$

Show that this is a compatibility condition for the pair of linear equations

$$\psi_x - ik\psi = q \tag{2}$$

$$\psi_t - ik^3\psi = -q_{xx} - ikq_x + k^2 q \tag{3}$$

for a function $\psi = \psi(x, t, k) \in \mathbb{C}$. Show that for each t, equation (2) has a solution ψ_+ which is defined for $\operatorname{Im} k \geqslant 0$, analytic in k for $\operatorname{Im} k > 0$, and satisfies

$$\lim_{x \to +\infty} e^{-ikx}\psi_+(x, t, k) = \hat{q}(k, t) = \int_{-\infty}^{+\infty} e^{-ikx} q(x, t)dx\,.$$

Deduce from this and equation (3) that $\hat{q}(k, t)$ evolves in time according to

$$\hat{q}_t - ik^3\hat{q} = 0$$

and hence obtain a representation for the solution of the Airy equation (1).

[*You may assume that q is a smooth function whose derivatives are rapidly decreasing in x.*]

Paper 2, Section II
34E Integrable Systems

It is possible to obtain solutions of the partial differential equation

$$u_{XT} = \sin u\,, \tag{1}$$

at time T from certain discrete scattering data $\{\lambda_m(T), c_m(T)\}_{m=1}^N$ and corresponding eigenfunctions $\psi_m(X,T)$ for an associated linear problem by means of the formula

$$u_X(T,X) = -4\sum_m c_m \psi_m^{(1)}(X,T)e^{i\lambda_m X},$$

where $\psi_m = \begin{pmatrix} \psi_m^{(1)} \\ \psi_m^{(2)} \end{pmatrix}$ and $\tilde{\psi}_m = \begin{pmatrix} -\overline{\psi_m^{(2)}} \\ \overline{\psi_m^{(1)}} \end{pmatrix}$ solve

$$\tilde{\psi}_n(X,T)e^{i\overline{\lambda_n(T)}X} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sum_m \frac{c_m(T)\psi_m(X,T)}{(\overline{\lambda_n(T)} - \lambda_m(T))} e^{i\lambda_m(T)X}\,.$$

Given the fact that the discrete scattering data $\{\lambda_m(T), c_m(T)\}_{m=1}^N$ evolve according to $\lambda_m(T) = \lambda_m(0) = \lambda_m$ and $c_m(T) = c_m(0)e^{-\frac{iT}{2\lambda_n}}$, obtain the solution in the case $N = 1$ with $\lambda_1(T) = il$ purely imaginary and $c_1(0) = c = 2l > 0$. Show that there is a unique *positive* value of l for which the solution is of the form $F(X+T)$ for some function F, which you should give.

Show that

$$g^s : \begin{pmatrix} X \\ T \\ u \end{pmatrix} \mapsto \begin{pmatrix} e^s X \\ e^{-s} T \\ u \end{pmatrix} \tag{2}$$

defines a group of Lie point symmetries of (1). Show that all the solutions to (1) you obtained for $N = 1$ transform under (2) into $F(X+T)$, with F as above.

In the case $N = 2$ and $\lambda_1 = il + m$, $\lambda_2 = il - m$ with real $l > 0, m > 0$ there is a solution of (1) given by

$$u(T,X) = 4\arctan \frac{l\sin\left(2mX - \frac{2mT}{4(l^2+m^2)}\right)}{m\cosh\left(\frac{2lT}{4(l^2+m^2)} + 2lX\right)}\,. \tag{3}$$

Show that if $l^2 + m^2 = \frac{1}{4}$ then this solution is periodic in $t = T - X$ for fixed $x = X + T$; find the period.

Show that for arbitrary $l^2 + m^2$ the solutions (3) may be transformed by (2) into the case $l^2 + m^2 = \frac{1}{4}$.

Paper 3, Section II
32E Integrable Systems

Explain what it means for a vector field $V = V_1(x,u)\partial_x + \phi(x,u)\partial_u$ to generate a *Lie symmetry* for a differential equation $\Delta(x, u, \partial_x u, \ldots, \partial_x^n u) = 0$. State a condition for this to hold in terms of the n^{th} *prolongation* of V, $\mathrm{pr}^{(n)}V$, giving also a definition of this latter concept.

Calculate the second prolongation of the vector field V, and hence show that if V generates an infinitesimal Lie symmetry for the equation

$$u'' = \frac{(u')^2}{u} - u^2 \tag{1}$$

then V_1 must be of the form

$$V_1(x,u) = F(x)\ln|u| + G(x)$$

for some functions F, G.

Show that if c and d are arbitrary real numbers then

$$V = (cx+d)\partial_x - 2cu\partial_u$$

is an infinitesimal Lie symmetry for equation (1), and give the form of the group of symmetries that it generates.

[*Assume* $u > 0$ *throughout.*]

Paper 1, Section II
22G Linear Analysis

Let ℓ^∞ denote the space of bounded real sequences and let ℓ^1 denote the space of summable real sequences. Suppose that $\varphi : \ell^\infty \to \mathbb{R}$ is linear and continuous, that φ is non-negative on non-negative sequences, that $\varphi((x_n)_{n\geqslant 1}) = \varphi((x_{n+1})_{n\geqslant 1})$, and that φ maps the constant sequence equal to one to one.

(a) Prove that $\liminf_{n\to\infty} x_n \leqslant \varphi((x_n)_{n\geqslant 1}) \leqslant \limsup_{n\to\infty} x_n$ for all $(x_n)_{n\geqslant 1} \in \ell^\infty$.

(b) Is there $(y_n)_{n\geqslant 1} \in \ell^1$ so that $\varphi((x_n)_{n\geqslant 1}) = \sum_{n\geqslant 1} x_n y_n$ for all $(x_n)_{n\geqslant 1} \in \ell^\infty$?

(c) Give an example of $(x_n)_{n\geqslant 1} \in \ell^\infty$ that does not converge but for which all φ defined as above give the same value.

(d) Let $y \in \mathbb{R}$. Assume $(x_n)_{n\geqslant 1} \in \ell^\infty$ satisfies $\dfrac{x_{n+1} + x_{n+2} + \cdots + x_{n+p}}{p} \to y$ as $p \to \infty$ uniformly in $n \geqslant 1$. Prove that $\varphi((x_n)_{n\geqslant 1}) = y$.

Paper 2, Section II
22G Linear Analysis

(a) Given a complex Banach space $(V, \|\cdot\|)$, prove that the space of bounded linear maps $(\mathcal{B}(V,V), |||\cdot|||)$ endowed with the norm

$$|||T||| = \sup_{v\in V,\ \|v\|=1} \|Tv\|$$

is a Banach space.

(b) Assume $(V, \|\cdot\|)$ is a complex Hilbert space. State the definitions of a *compact operator* $T : V \to V$ and of a *Hilbertian basis*. Suppose $T \in \mathcal{B}(V,V)$ and V has a Hilbertian basis $(e_n)_{n\geqslant 1}$ such that $T(e_n) = \lambda_n e_n$ for complex numbers λ_n, $n \geqslant 1$. Prove that T is compact if and only if $\lambda_n \to 0$.

(c) Given a complex Hilbert space $(V, \|\cdot\|)$ and $(e_n)_{n\geqslant 1}$ a Hilbertian basis of V, consider $\mathcal{H}(V,V)$, the set of linear operators T such that $\sum_{n\geqslant 1} \|Te_n\|^2 < +\infty$. Prove that operators in $\mathcal{H}(V,V)$ are bounded and compact, and that $(\mathcal{H}(V,V), |||\cdot|||_*)$ with

$$|||T|||_* = \left(\sum_{n\geqslant 1} \|Te_n\|^2\right)^{1/2}$$

is a Hilbert space. Are $|||\cdot|||$ and $|||\cdot|||_*$ equivalent norms on $\mathcal{H}(V,V)$?

Paper 3, Section II
21G Linear Analysis

(a) Prove that any metric space (X,d) is normal for the induced topology.

(b) State the *Urysohn lemma* and the *Tietze extension theorem.*

(c) Prove that a metric space (X,d) is compact if and only if all continuous functions from X to $\mathbb{R}$ are bounded.

Paper 4, Section II
22G Linear Analysis

(a) Define what it means for a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ to be *equi-continuous* on $[0,1]$. State the *Arzelà–Ascoli theorem.*

(b) Given a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, we can inductively define functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n \geqslant 0$ by $f_{n+1}(t) = \int_0^t \varphi(f_n(s))\,\mathrm{d}s$, and $f_0(t) = 0$ for all $t \in \mathbb{R}$. Show that there exists $T_1 > 0$ so that the sequence $(f_n)_{n\geqslant 1}$ is equi-bounded and equi-continuous on $[0, T_1]$.

(c) Deduce the existence of $T_2 \in (0, T_1]$ and a continuously differentiable function $f : [0, T_2] \to \mathbb{R}$ such that $f(0) = 0$ and $f'(t) = \varphi(f(t))$ on $[0, T_2]$. [*Hint: Prove that if* $T_2 \in (0, T_1]$ *is small enough,* $R_n(t) = f_{n+1}(t) - f_n(t) \to 0$ *uniformly on* $[0, T_2]$.]

Paper 1, Section II
16F Logic and Set Theory

State and prove the *Knaster–Tarski fixed-point theorem.*

A subset S of a poset X is called a *down-set* if whenever $x, y \in X$ satisfy $x \in S$ and $y \leqslant x$ then also $y \in S$. Show that the set P of down-sets of X, ordered by inclusion, is a complete poset.

Now let X and Y be totally ordered sets.

(i) Give an example to show that we may have X isomorphic to a down-set in Y, and Y isomorphic to a down-set in X, and yet X is not isomorphic to Y. [*Hint: Consider suitable subsets of the reals.*]

(ii) Show that if X is isomorphic to a down-set in Y, and Y is isomorphic to the complement of a down-set in X, then X is isomorphic to Y.

Paper 2, Section II
16F Logic and Set Theory

(a) Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent.

(b) Which of the following assertions about ordinals α, β and γ are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$.

(ii) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

(iii) If α is a limit ordinal then $\alpha\omega = \omega\alpha$.

(iv) If $\alpha \geqslant \omega_1$ and $\beta < \omega_1$ then $\beta + \alpha = \alpha$.

(v) If $\alpha + \alpha + \beta$ and $\beta + \alpha + \alpha$ are equal then they are both equal to $\alpha + \beta + \alpha$.

Paper 3, Section II
16F Logic and Set Theory

State and prove the *Compactness Theorem* for first-order predicate logic. State and prove the *Upward Löwenheim–Skolem Theorem.*

[*You may assume the Completeness Theorem for first-order predicate logic.*]

For each of the following theories, is the theory axiomatisable (in the language of posets, extended by some set of constants if necessary) or not? Justify your answers.

(i) The theory of posets having only finitely many maximal elements.

(ii) The theory of posets having uncountably many maximal elements.

(iii) The theory of posets having infinitely many maximal elements or infinitely many minimal elements (or both).

(iv) The theory of posets having infinitely many maximal elements or infinitely many minimal elements, but not both.

(v) The theory of the total orders that are isomorphic to a subset of the reals.

Paper 4, Section II
16F Logic and Set Theory

(a) Define the *von Neumann hierarchy* of sets V_α. Show that each V_α is transitive, and explain why $V_\alpha \subset V_\beta$ whenever $\alpha \leqslant \beta$. Prove that every set x is a member of some V_α.

(b) What does it mean to say that a relation r on a set x is *well-founded* and *extensional*? State *Mostowski's Collapsing Theorem.* Give an example of a set x whose rank is greater than ω but for which the Mostowski collapse of x (equipped with the relation $\in$) is equal to ω.

Which of the following statements are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) If a relation r on a set x is isomorphic to the relation $\in$ on some transitive set y then r is well-founded and extensional.

(ii) If a relation r on a set x is isomorphic to the relation $\in$ on some (not necessarily transitive) set y then r is well-founded.

(iii) If a relation r on a set x is isomorphic to the relation $\in$ on some (not necessarily transitive) set y then r is extensional.

Paper 1, Section I
6C Mathematical Biology

Consider the discrete delay equation

$$x_{n+1} = x_n \exp\left[r\left(1 - x_{n-1}\right)\right],$$

with $r > 0$ a constant.

(a) Find the positive fixed point x^* of the model. Setting $x_n = x^* + u_n$, with $|u_n| \ll 1$, determine the linearised stability equation for u_n.

(b) Find the range of r for which the fixed point x^* is stable and for which perturbations decay monotonically in time.

(c) Find the range of r for which the decay of perturbations to x^* is oscillatory.

(d) Find the critical value r^* for x^* to become unstable, and show that at that value of r the system exhibits oscillations of period $p > 1$. Find p.

Paper 2, Section I
6C Mathematical Biology

Two species with populations N_1 and N_2 compete according to the equations

$$\begin{aligned}\frac{dN_1}{dt} &= r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_1}\right)\\ \frac{dN_2}{dt} &= r_2 N_2 \left(1 - b_{21}\frac{N_1}{K_2}\right),\end{aligned}$$

so that only species 1 has limited carrying capacity. Assume that the parameters $r_1, r_2, K_1, K_2, b_{12}$, and b_{21} are all strictly positive.

(a) Rescale the variables N_1, N_2 and t to leave three parameters, $\rho = r_1/r_2$, $\alpha = b_{12}K_2/K_1$ and $\beta = b_{21}K_1/K_2$ and determine the steady states.

(b) Assuming $\beta > 1$, investigate the stability of the biologically relevant steady states and sketch the phase plane trajectories.

(c) Assuming $\beta > 1$, show that irrespective of the size of the parameters the principle of competitive exclusion holds. Briefly describe under what ecological circumstances species 2 becomes extinct.

Paper 3, Section I
6C Mathematical Biology

A biological population contains n individuals. The population increases or decreases according to the transition rates

$$n \xrightarrow{\lambda} n+1 \qquad\qquad n \xrightarrow{\beta n^2} n-2\,.$$

(a) Derive the master equation for $P(n,t)$, the probability that the population contains n individuals at time t, and a corresponding equation for $\langle n \rangle$. What condition does the latter imply on the steady state?

(b) The Fokker-Planck equation has the form:

$$\frac{\partial}{\partial t}P(n,t) = -\frac{\partial}{\partial n}\big[A(n)P(n,t)\big] + \tfrac{1}{2}\frac{\partial^2}{\partial n^2}\big[B(n)P(n,t)\big]\,. \tag{1}$$

Derive the Fokker-Planck equation from your master equation. Deduce the forms of $A(n)$ and $B(n)$ for this system.

(c) Give brief arguments why in the steady state (1) has the approximate solution $(2\pi\sigma^2)^{-1/2}\exp(-(n-\mu)^2/2\sigma^2)$ and derive the corresponding values of σ and μ.

(d) Comment on the relation to the steady-state condition you have derived in (a). Under what conditions on β and λ is the Fokker-Planck equation likely to give an accurate description of the steady state?

Paper 4, Section I

6C Mathematical Biology

An allosteric enzyme E reacts with substrate S to produce a product P according to the mechanism

$$E + S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C_1$$
$$C_1 + S \overset{k_2}{\longrightarrow} C_2 \overset{k_3}{\longrightarrow} C_1 + P \ ,$$

where the k_is are rate constants, and C_1 and C_2 are enzyme-substrate complexes.

(a) With lowercase letters denoting concentrations, write down the differential equation model based on the Law of Mass Action for the dynamics of e, s, c_1, c_2 and p.

(b) Show that the quantity $c_1 + c_2 + e$ is conserved and comment on its physical meaning.

(c) Using the result in (b), assuming initial conditions $s(0) = s_0$, $e(0) = e_0$, $c_1(0) = c_2(0) = p(0) = 0$, and rescaling with $\epsilon = e_0/s_0$, $\tau = k_1 e_0 t$, $u = s/s_0$, and $v_i = c_i/e_0$, show that the reaction mechanism can be reduced to

$$\frac{du}{d\tau} = f(u, v_1, v_2)\,,$$
$$\epsilon\frac{dv_1}{d\tau} = g_1(u, v_1, v_2)\,,$$
$$\epsilon\frac{dv_2}{d\tau} = g_2(u, v_1, v_2)\,.$$

Determine f, g_1 and g_2 and express them in terms of the three dimensionless quantities $\alpha = k_{-1}/k_1 s_0$, $\beta = k_2/k_1$ and $\gamma = k_3/k_1 s_0$.

(d) On time scales $\tau \gg \epsilon$, show that the rate of production of P can be expressed in terms of the rescaled substrate concentration u in the form

$$\frac{dp}{dt} = A\frac{u^2}{\alpha + u + (\beta/\gamma)u^2}\ ,$$

where A is a constant. Compare this relation to the Michaelis-Menten form by means of a sketch.

Paper 3, Section II
13C Mathematical Biology

A chemical species of concentration $C(\mathbf{x},t)$ diffuses in a two-dimensional stationary medium with diffusivity $D(C)$. Write down an expression for the diffusive flux $\mathbf{J}$ that enters Fick's law and then show that C obeys the partial differential equation

$$\frac{\partial C}{\partial t} = \boldsymbol{\nabla} \cdot (D(C)\boldsymbol{\nabla} C) \,. \tag{1}$$

Suppose that at time $t = 0$ an amount $2\pi M$ of the chemical is deposited at the origin and diffuses outward in a circularly symmetric manner, so that $C = C(r,t)$ for $r > 0, t > 0$, where r is the radial coordinate. Assume the diffusivity is $D = kC$ for some constant k. Show, by dimensional analysis or otherwise, that an appropriate similarity solution has the form

$$C = \frac{M^\alpha}{(kt)^\beta} F(\xi)\,, \qquad \xi = \frac{r}{(Mkt)^\gamma} \qquad \text{and} \qquad \int_0^\infty \xi F(\xi)\, d\xi = 1\,,$$

where the exponents α, β, γ are to be determined, and derive the ordinary differential equation satisfied by F.

Solve this ordinary differential equation, subject to appropriate boundary conditions, and deduce that the chemical occupies a finite circular region of radius

$$r_0(t) = (NMkt)^{1/4}\,,$$

with N a constant which you should find.

Still assuming that $D = kC$, show that if a term αC is added to the right-hand side of (1), a solution of the form $C(r,t) = G(r,\tau)e^{\alpha t}$ can be found, where $\tau(t)$ is a time-like variable satisfying $\tau(0) = 0$. Show that a suitable choice of τ reduces the dynamics to

$$\frac{\partial G}{\partial \tau} = k\boldsymbol{\nabla} \cdot (G\boldsymbol{\nabla} G)\,,$$

and that the previous analysis can be applied to find the concentration. Describe the evolution in the cases $\alpha = 0, \alpha > 0$, and $\alpha < 0$.

[*Hint: In plane polar coordinates*

$$\boldsymbol{\nabla} C(r,t) \equiv \left(\frac{\partial C}{\partial r}, 0, 0\right) \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot (V(r,t), 0, 0) \equiv \frac{1}{r}\frac{\partial}{\partial r}(rV)\,.$$]

Paper 4, Section II
14C Mathematical Biology

Consider the standard system of reaction-diffusion equations

$$\begin{aligned} u_t &= D_u \nabla^2 u + f(u,v) \\ v_t &= D_v \nabla^2 v + g(u,v)\,, \end{aligned}$$

where D_u and D_v are diffusion constants and $f(u,v)$ and $g(u,v)$ are such that the system has a stable homogeneous fixed point at $(u,v) = (u_*, v_*)$.

(a) Show that the condition for a Turing instability can be expressed as

$$f_u + d g_v > 2\sqrt{dJ}\,,$$

where $d = D_u/D_v$ is the diffusivity ratio and $J = f_u g_v - f_v g_u > 0$ is the determinant of the stability matrix of the homogeneous system evaluated at (u_*, v_*).

(b) Show that this result implies that a Turing instability at equal diffusivities ($d = 1$) is not possible.

(c) Show that the result in (b) also follows directly from the structure of the reaction-diffusion equations linearised about the homogeneous fixed point in the case $D_u = D_v$.

(d) Using the example

$$\begin{pmatrix} -1 & -1 \\ 1+\delta & 1-\delta \end{pmatrix},$$

for the stability matrix of the homogeneous system, show that the diffusivity ratio at which Turing instability occurs can be made as close to unity as desired by taking δ sufficiently small.

Paper 1, Section II

31J Mathematics of Machine Learning

(a) Let $\mathcal{F}$ be a family of functions $f : \mathcal{X} \to \{0, 1\}$ with $|\mathcal{F}| \geqslant 2$.

Define the *shattering coefficient* $s(\mathcal{F}, n)$ and the *VC dimension* $\mathrm{VC}(\mathcal{F})$ of $\mathcal{F}$.

State the *Sauer–Shelah lemma.*

(b) (i) Let

$$\mathcal{A}_1 = \left\{ \bigcup_{k=1}^{m} [a_k, b_k] : a_k, b_k \in \mathbb{R} \text{ for } k = 1, \ldots, m \right\}.$$

Show that $\mathcal{F}_1 := \{\mathbf{1}_A : A \in \mathcal{A}_1\}$ satisfies $\mathrm{VC}(\mathcal{F}_1) = 2m$.

(ii) Let $\mathcal{F}_2$ be a class of functions from $\mathbb{R}^p$ to $\{0, 1\}$ given by

$$\mathcal{F}_2 := \{x \mapsto \mathbf{1}_{(0,\infty)}(\mu + x^T\beta) : \beta \in \mathbb{R}^p,\ \mu \in \mathbb{R}\}.$$

Stating any result from the course you need, give an upper bound on $\mathrm{VC}(\mathcal{F}_2)$.

(c) (i) Let $\mathcal{G}$ be a family of functions $g : \mathcal{Z} \to \{0, 1\}$ with $|\mathcal{G}| \geqslant 2$ and define $\mathcal{H}$ to be the set of functions $h : \mathcal{X} \times \mathcal{Z} \to \{0, 1\}$ for which $h(x, z) = f(x)g(z)$ for some $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Show that $s(\mathcal{H}, n) \leqslant s(\mathcal{F}, n)s(\mathcal{G}, n)$.

(ii) Now let $\mathcal{G}$ be a family of functions $g : \mathcal{X} \to \{0, 1\}$ with $|\mathcal{G}| \geqslant 2$ and define $\mathcal{H}$ to be the set of functions $h : \mathcal{X} \to \{0, 1\}$ for which $h(x) = f(x)g(x)$ for some $f \in \mathcal{F}$ and $g \in \mathcal{G}$. Show that $s(\mathcal{H}, n) \leqslant s(\mathcal{F}, n)s(\mathcal{G}, n)$.

(d) (i) Let

$$\mathcal{A}_3 = \left\{ \prod_{j=1}^{p} \Big(\bigcup_{k=1}^{m} [a_{jk}, b_{jk}] \Big) : a_{jk}, b_{jk} \in \mathbb{R} \text{ for } j = 1, \ldots, p,\ k = 1, \ldots, m \right\}.$$

Show that $\mathcal{F}_3 := \{\mathbf{1}_A : A \in \mathcal{A}_3\}$ satisfies $s(\mathcal{F}_3, n) \leqslant (n + 1)^{2mp}$.

(ii) For $m \geqslant 3$, let $\mathcal{A}_4$ be the set of all convex polygons in $\mathbb{R}^2$ with m sides, and set $\mathcal{F}_4 := \{\mathbf{1}_A : A \in \mathcal{A}_4\}$. Show that $s(\mathcal{F}_4, n) \leqslant (n + 1)^{3m}$.

Paper 2, Section II

31J Mathematics of Machine Learning

(a)What does it mean for a function $f : \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_n \to \mathbb{R}$ to have the *bounded differences property* with constants $L_1, \ldots, L_n$?

State the *bounded differences inequality.*

(b) Let $\mathcal{X}$ and $\mathcal{Y}$ be input and output spaces respectively. Let H be a machine learning algorithm taking as its argument a dataset $D \in (\mathcal{X} \times \mathcal{Y})^n$ to output a hypothesis $H_D : \mathcal{X} \to \mathbb{R}$. For $D = (x_i, y_i)_{i=1}^n \in (\mathcal{X} \times \mathcal{Y})^n$ and $(x, y) \in \mathcal{X} \times \mathcal{Y}$, for all $i = 1, \ldots, n$ we write

$$D_i(x, y) := ((x_1, y_1), \ldots, (x_{i-1}, y_{i-1}), (x, y), (x_{i+1}, y_{i+1}), \ldots, (x_n, y_n)) .$$

Let $\ell : \mathbb{R} \times \mathcal{Y} \to [0, M]$ be a bounded loss function. Suppose H has the following property: there exists $\beta \geqslant 0$ such that for all $i = 1, \ldots, n$ and for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we have

$$\sup_{(\tilde{x},\tilde{y}) \in \mathcal{X} \times \mathcal{Y}} |\ell(H_{D_i(x,y)}(\tilde{x}), \tilde{y}) - \ell(H_D(\tilde{x}), \tilde{y})| \leqslant \beta.$$

Let $(X, Y) \in \mathcal{X} \times \mathcal{Y}$ be a random input–output pair. Show that $F : (\mathcal{X} \times \mathcal{Y})^n \to \mathbb{R}$ given by

$$F((x_1, y_1), \ldots, (x_n, y_n)) = \mathbb{E}\ell(H_D(X), Y) - \frac{1}{n} \sum_{i=1}^n \ell(H_D(x_i), y_i)$$

satisfies a bounded differences property with constants all equal to $2\beta + M/n$. [In the expectation above, the (x_i, y_i) are considered deterministic.]

(c) Now suppose $D = (X_i, Y_i)_{i=1}^n \in (\mathcal{X} \times \mathcal{Y})^n$ is a collection of i.i.d. input–output pairs independent of, and each having the same distribution as, (X, Y). Show that $\mathbb{E}F(D) \leqslant \beta$. [*Hint: Find an alternative expression for* $\mathbb{E}\ell(H_D(X), Y)$ *as a sum of expectations with the ith term involving* $H_{D_i(X,Y)}$.]

(d) Hence conclude that, given $0 < \delta \leqslant 1$,

$$\frac{1}{n} \sum_{i=1}^n \ell(H_D(X_i), Y_i) + \beta + (2n\beta + M)\sqrt{\frac{\log(1/\delta)}{2n}} \geqslant \mathbb{E}\ell(H_D(X), Y)$$

with probability at least $1 - \delta$.

Paper 4, Section II

30J Mathematics of Machine Learning

Throughout this question, you may assume that the optimum is achieved in any relevant optimisation problems, so for instance in part (a) you may assume $\hat{f}$ is well-defined.

Suppose $(X_1, Y_1), \ldots, (X_n, Y_n) \in \mathcal{X} \times \{-1, 1\}$ are i.i.d. input–output pairs. Let $\mathcal{B}$ be a set of classifiers $h : \mathcal{X} \to \{-1, 1\}$ such that $h \in \mathcal{B} \Rightarrow -h \in \mathcal{B}$.

(a) Write down the *Adaboost* algorithm using $\mathcal{B}$ as the base set of classifiers with tuning parameter M, which produces $\hat{f} : \mathcal{X} \to \mathbb{R}$ of the form $\hat{f} = \sum_{m=1}^{M} \hat{\beta}_m \hat{h}_m$ where $\hat{\beta}_m \geqslant 0$ and $\hat{h}_m \in \mathcal{B}$ for $m = 1, \ldots, M$. [You need not derive explicit expressions for $\hat{\beta}_m$ or $\hat{h}_m$.]

(b) For a set $S \subseteq \mathbb{R}^d$, what is meant by the *convex hull*, $\operatorname{conv} S$? What does it mean for a vector $v \in \mathbb{R}^d$ to be a *convex combination* of vectors $v_1, \ldots, v_m \in \mathbb{R}^d$? State a result relating convex hulls and convex combinations.

(c) Let ϕ denote the exponential loss. What is meant by the *ϕ-risk* $R_\phi(f)$ of $f : \mathcal{X} \to \mathbb{R}$? What is the corresponding *empirical ϕ-risk* $\hat{R}_\phi(f)$? Let $x_{1:n} \in \mathcal{X}^n$. What is meant by the *empirical Rademacher complexity* $\hat{\mathcal{R}}(\mathcal{B}(x_{1:n}))$?

(d) Consider a modification of the Adaboost algorithm where, if at any iteration $m \leqslant M$ we have $\sum_{k=1}^{m} \hat{\beta}_k > 1$, we terminate the algorithm and output $\hat{f} := \sum_{k=1}^{m-1} \hat{\beta}_k \hat{h}_k$, or the zero function if $m = 1$; otherwise we output $\hat{f} = \sum_{k=1}^{M} \hat{\beta}_k \hat{h}_k$ as usual. Let $r_{\mathcal{B}} = \sup_{x_{1:n} \in \mathcal{X}^n} \hat{\mathcal{R}}(\mathcal{B}(x_{1:n}))$. Show that

$$\mathbb{E} R_\phi(\hat{f}) \leqslant \mathbb{E} \hat{R}_\phi(\hat{f}) + 2\exp(1) r_{\mathcal{B}}.$$

[*Hint: Introduce*

$$\mathcal{H} := \left\{ \sum_{m=1}^{M} \beta_m h_m : \sum_{m=1}^{M} \beta_m \leqslant 1,\ \beta_m \geqslant 0,\ h_m \in \mathcal{B} \text{ for } m = 1, \ldots, M \right\}.$$

You may use any results from the course without proof, but should state or name any result you use.]

Paper 1, Section II
20H Number Fields

(a) Let K be a number field of degree n. Show that there are exactly n field embeddings $\sigma_1, \ldots, \sigma_n \colon K \hookrightarrow \mathbb{C}$. [You may assume that $K = \mathbb{Q}(\alpha)$ for some $\alpha \in K$.]

Define the *discriminant* d_K of K. Show that the sign of d_K is $(-1)^s$, where s is the number of pairs of complex conjugate embeddings $(\sigma_i, \bar{\sigma}_i \neq \sigma_i)$. [You may assume that d_K is nonzero.]

(b) If $L = \mathbb{Q}(\theta)$, where $\theta^3 + 2\theta^2 + 1 = 0$, show that $\mathcal{O}_L = \mathbb{Z}[\theta]$.

(c) Let K be as in part (a). Suppose that $\alpha \in K$ and that $|\sigma_j(\alpha)| = 1$ for some j.

(i) Prove that $\left|N_{K/\mathbb{Q}}(\alpha)\right| = 1$.

(ii) Deduce that if $\alpha \in \mathcal{O}_K$, then α is a unit.

(iii) Give an example of a number field K and an element $\alpha \in K \setminus \mathcal{O}_K$ for which $|\sigma_1(\alpha)| = \cdots = |\sigma_n(\alpha)| = 1$.

Paper 2, Section II
20H Number Fields

Let K be a number field.

(a) Let $P_1, \ldots, P_k$ (where $k \geqslant 1$) be distinct nonzero prime ideals of $\mathcal{O}_K$ and let $m_1, \ldots, m_k$ be positive integers. Let I be the product $P_1^{m_1} \cdots P_k^{m_k}$. Explain why $I = P_1^{m_1} \cap \cdots \cap P_k^{m_k}$, and hence show that the map

$$\mathcal{O}_K/I \to \mathcal{O}_K/P_1^{m_1} \times \cdots \times \mathcal{O}_K/P_k^{m_k}$$

taking $\alpha + I$ to $(\alpha + P_1^{m_1}, \ldots, \alpha + P_k^{m_k})$ is an isomorphism of rings.

Deduce that there exists $\alpha \in I$ such that $\alpha \notin P_i I$ for all i. Show that there exists an ideal I' with $I + I' = \mathcal{O}_K$ such that II' is principal. Show also that any ideal of $\mathcal{O}_K$ can be generated by two elements.

(b) State *Dedekind's criterion* for the factorisation of rational primes in $\mathcal{O}_K$. Use it to compute the factorisation of any odd rational prime in $\mathcal{O}_K$ when $K = \mathbb{Q}(\sqrt{d})$ is a quadratic field.

Show that if $d > 0$ and K contains an element α with $N_{K/\mathbb{Q}}(\alpha) = -1$, then no prime $p \equiv 3 \pmod 4$ can ramify in K.

Paper 4, Section II
20H Number Fields

Let K be a number field. What is an *ideal class* of K? Show that the set of ideal classes of K forms an abelian group. [You may use any results about ideals in number fields provided you state them clearly.]

Assuming that there exists a constant c_K such that every nonzero ideal I of $\mathcal{O}_K$ contains a nonzero element α with $\left|N_{K/\mathbb{Q}}(\alpha)\right| \leqslant c_K \, N(I)$, show that the ideal class group of K is finite.

Compute the ideal class group of $\mathbb{Q}(\sqrt{-33})$. [You may assume that the Minkowski constant c_K of an imaginary quadratic field is $\dfrac{2}{\pi}\,|d_K|^{1/2}$.]

Paper 1, Section I
1I Number Theory

A function $f : \mathbb{N} \to \mathbb{C}$ is *multiplicative* if $f(mn) = f(m)f(n)$ for all m, n coprime. Show that if f is multiplicative then so is $g(n) = \sum_{d|n} f(d)$. Define the *Möbius function* μ and *Euler function* ϕ. Establish the identities

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d} \qquad \text{and} \qquad \frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\phi(d)} \,.$$

Paper 2, Section I
1I Number Theory

Explain what it means for a positive definite binary quadratic form to be *reduced*, and what it means for two such forms to be *equivalent*. Prove that every positive definite binary quadratic form is equivalent to a reduced form. Show that any two equivalent forms represent the same set of integers.

Carefully quoting any further results you need, show that $f(x,y) = 6x^2 + 5xy + 2y^2$ and $g(x,y) = 9x^2 + 25xy + 18y^2$ represent the same integers, but are not equivalent.

Paper 3, Section I
1I Number Theory

State *Lagrange's theorem* on the possible number of solutions of a polynomial congruence. State and prove the *Chinese remainder theorem.*

Find the smallest positive integer x satisfying $x^3 + 1 \equiv 0 \pmod{1729}$. Hence, or otherwise, determine the number of solutions of this congruence with $1 \leqslant x \leqslant 1729$.

Paper 4, Section I
1I Number Theory

Compute the continued fraction expansion of $\sqrt{29}$.

Find integers x and y satisfying $x^2 - 29y^2 = -1$.

Paper 3, Section II

11I Number Theory

(a) Define what it means for an integer to be a *primitive root* mod n.

(b) Let p be an odd prime, and b a primitive root mod p. Prove the following are equivalent.

(i) b is a primitive root mod p^2.

(ii) b is a primitive root mod p^m for all $m \geqslant 2$.

(iii) No pseudoprime to the base b is divisible by p^2.

(c) Find the three smallest positive integers b with the property that b is a primitive root mod 5^m for all $m \geqslant 1$.

(d) Let $P(n)$ be the number of primitive roots mod n. Show that for each $k \geqslant 1$ there are only finitely many integers n with $P(n) = k$.

Paper 4, Section II

11I Number Theory

(a) Define the *Legendre symbol* and state *Euler's criterion.* State and prove *Gauss' lemma.* Determine the primes p for which the congruence $x^2 \equiv 2 \pmod{p}$ is soluble.

(b) Let $\pi_k(x)$ be the number of primes p less than or equal to x with $p \equiv k \pmod{8}$.

(i) By considering the prime factorisation of $n^2 - 2$ for suitable n, show that $\pi_7(x) \to \infty$ as $x \to \infty$.

(ii) By considering the prime factorisation of $n^2 - 2$ for all n in a suitable range, show that for all x sufficiently large we have

$$\pi_1(x) + \pi_7(x) + 1 \geqslant \frac{\log x}{6 \log 3}\,.$$

Paper 1, Section II
41C Numerical Analysis

(a) Let $H \in \mathbb{R}^{n\times n}$ be diagonalisable. Show that the sequence defined by $\mathbf{z}^{(k+1)} = H\mathbf{z}^{(k)}$ converges to 0 for all initial vectors $\mathbf{z}^{(0)} \in \mathbb{C}^n$ if, and only if, $\rho(H) < 1$ where $\rho(H)$ is the spectral radius of H.

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, and let $\mathbf{b} \in \mathbb{R}^n$.

(b) Prove that the solution to $A\mathbf{x} = \mathbf{b}$ is the unique minimiser of the function $f(\mathbf{x}) = (1/2)\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}$.

(c) The *steepest descent method* with constant step size α is defined by

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \alpha\nabla f(\mathbf{x}^{(k)}).$$

Applying the method to the function f given in (b), write down the iterations explicitly in terms of A and $\mathbf{b}$. Under what conditions on α does the sequence $\mathbf{x}^{(k)}$ converge to $A^{-1}\mathbf{b}$?

(d) Consider the steepest descent method with *exact line search*, where at each iteration k, the constant $\alpha = \alpha^{(k)}$ is chosen so that $f(\mathbf{x}^{(k+1)})$ is as small as possible. Give an explicit expression for the step size $\alpha^{(k)}$. Show that, in this case, the residuals $\mathbf{r}^{(k)} = \mathbf{b} - A\mathbf{x}^{(k)}$ satisfy $(\mathbf{r}^{(k)})^T\mathbf{r}^{(k+1)} = 0$ for all k.

Paper 2, Section II
41C Numerical Analysis

(a) Consider a linear recurrence relation

$$\sum_{k=r}^{s} a_k u_{m+k}^{n+1} = \sum_{k=r}^{s} b_k u_{m+k}^{n} \qquad n \geqslant 0, \;\; m \in \mathbb{Z}\,,$$

where (a_k) and (b_k) are fixed coefficients.

(i) Show that if we define the Fourier transform of $\boldsymbol{u}^n = (u_m^n)_{m\in\mathbb{Z}}$ by $\widehat{u^n}(\theta) = \sum_{m\in\mathbb{Z}} e^{-im\theta} u_m^n$, then the linear recurrence relation takes the form

$$\widehat{u^{n+1}}(\theta) = H(\theta)\widehat{u^n}(\theta)\,,$$

where $H(\theta)$ is a function that you should specify.

(ii) Show that the sequence $(\boldsymbol{u}^n)_{n\geqslant 0}$ is bounded in the ℓ_2 norm, for all $\boldsymbol{u}^0$, if and only if $|H(\theta)| \leqslant 1$ for all $\theta \in [-\pi, \pi]$.

[*You may assume Parseval's identity:*

$$\|u\|_{\ell_2}^2 = \sum_{m\in\mathbb{Z}} |u_m|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{u}(\theta)|^2\, d\theta. \qquad]$$

(b) Consider the following three recurrence relations:

(i) $u_m^{n+1} = u_m^n + \mu(u_m^n - u_{m-1}^n)$

(ii) $u_m^{n+1} = \frac{1}{2}\mu(1+\mu)u_{m-1}^n + (1-\mu^2)u_m^n - \frac{1}{2}\mu(1-\mu)u_{m+1}^n$

(iii) $u_m^{n+1} - \frac{1}{2}(\mu-\alpha)(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}) = u_m^n + \frac{1}{2}(\mu+\alpha)(u_{m-1}^n - 2u_m^n + u_{m+1}^n)$

where $n \in \mathbb{N}$ is the time discretization index, $m \in \mathbb{Z}$ is the spatial discretization index, $\mu \geqslant 0$ is the Courant number, and, for (iii), $\alpha \geqslant 0$ is a parameter. In each case give an expression for the amplification factor $H(\theta)$, and deduce the set of values μ (and α for (iii)) for which we have stability.

Paper 3, Section II
40C Numerical Analysis

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix with real eigenvalues $\lambda_1, \ldots, \lambda_n$ ordered by their magnitudes in nonincreasing order, $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n|$.

(a) Define the *power method* to compute the leading eigenvalue of A. Show that, under suitable assumptions, the iterates $(\mathbf{x}_k)$ of the power method satisfy

$$r(\mathbf{x}_k) - \lambda_1 = O(|\lambda_2/\lambda_1|^{2k})$$

as $k \to \infty$, where $r(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}/\mathbf{x}^T\mathbf{x}$ is the Rayleigh quotient.

(b) Let

$$A = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix}$$

to which we apply the power method with starting vector $\mathbf{x}_0 = (1/\sqrt{2}, -1/\sqrt{2}, 0)$. Compute $\mathbf{x}_k$ and $r(\mathbf{x}_k)$ explicitly, and find the limit value $\lim_{k\to\infty} r(\mathbf{x}_k)$. Compare with the result in (a) and comment. [*Hint: The eigenvalues of A are 9, 6 and 2.*]

(c) Define the *inverse iteration with shift*, and describe (without proof) the convergence of the method, clearly stating the assumptions.

Paper 4, Section II
40C Numerical Analysis

(a) State and prove the *Gershgorin circle theorem.*

(b) Consider the diffusion equation on the square $[0,1]^2$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x,y)\frac{\partial}{\partial x}u(x,y)\right) + \frac{\partial}{\partial y}\left(a(x,y)\frac{\partial}{\partial y}u(x,y)\right),$$

where $0 < a(x,y) < a_{\max}$ for all $(x,y) \in [0,1]^2$ is the diffusion coefficient, and with Dirichlet boundary conditions $u(x,y,t) = 0$ for (x,y) on the boundary of $[0,1]^2$.

Consider a uniform grid of size $M \times M$ with step $h = 1/(M+1)$ and let $u_{i,j} = u(ih, jh)$ for $1 \leqslant i \leqslant M$ and $1 \leqslant j \leqslant M$.

(i) Using finite differences, show that the right-hand side of the diffusion equation can be discretised by an expression of the form

$$\frac{1}{h^2}(\alpha u_{i-1,j} + \beta u_{i+1,j} + \gamma u_{i,j-1} + \delta u_{i,j+1} - (\alpha+\beta+\gamma+\delta)u_{i,j})$$

for some $\alpha, \beta, \gamma, \delta$ which you should specify, and which depend on i, j and the diffusion coefficient. Show that the error of this discretisation is $O(h^2)$.

(ii) The time derivative is discretised using a forward Euler scheme with a time step $\Delta t = k$. Use Gershgorin's theorem, clearly justifying all your steps, to show that the resulting scheme is stable when $0 < \mu \leqslant 1/(4a_{\max})$, where $\mu = k/h^2$ is the Courant number.

Paper 1, Section II
34A Principles of Quantum Mechanics

Let A and $A^\dagger$ respectively be the lowering and raising operator for a one-dimensional quantum harmonic oscillator, with $[A, A^\dagger] = 1$. Also let $|n\rangle$ be the n^{th} excited state of the oscillator, obeying $N|n\rangle = n|n\rangle$ where $N = A^\dagger A$ is the number operator.

(a) Show that $A|n\rangle \propto |n-1\rangle$ and find the constant of proportionality.

(b) For any $z \in \mathbb{C}$, define the coherent state $|z\rangle$ by

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \ .$$

Show that $\langle z|z\rangle = 1$ and that $A|z\rangle = z|z\rangle$.

(c) Calculate the expectation value $\langle N\rangle$ and uncertainty ΔN of the number operator in the state $|z\rangle$. Show that the relative uncertainty $\Delta N/\langle N\rangle \to 0$ as $\langle N\rangle \to \infty$.

(d) A harmonic oscillator is prepared to be in state $|z\rangle$ at time $t = 0$. Using the properties of the Hamiltonian of the one-dimensional harmonic oscillator, show that the state evolved to time $t > 0$ is still an eigenstate of A and find its eigenvalue. Calculate the probability that the oscillator is found to be in the original state $|z\rangle$ at time t, and show that this probability is 1 whenever $t = kT$, where $k \in \mathbb{N}$ and T is the classical period of the oscillator.

Paper 2, Section II
35A Principles of Quantum Mechanics

(a) Let $\{|\uparrow\rangle, |\downarrow\rangle\}$ be a basis of S_z eigenstates for a spin-$\frac{1}{2}$ particle. Find the eigenstates $|{\uparrow_\theta}\rangle$ and $|{\downarrow_\theta}\rangle$ of $\mathbf{n}\cdot\mathbf{S}$, where $\mathbf{n} = (\sin\theta, 0, \cos\theta)$, and give their corresponding eigenvalues.

(b) Two spin-$\frac{1}{2}$ particles are in the combined spin state

$$|\psi\rangle = \frac{|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}}\,.$$

Show that this state is unchanged under the substitution

$$(|\uparrow\rangle, |\downarrow\rangle) \mapsto (|{\uparrow_\theta}\rangle, |{\downarrow_\theta}\rangle).$$

Hence show that $|\psi\rangle$ is an eigenstate, with eigenvalue zero, of each Cartesian component of the combined spin operator $\mathbf{S} = \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$, where $\mathbf{S}^{(i)}$ is the spin operator of the i^{th} particle.

(c) Two spin-$\frac{1}{2}$ particles are in the spin state

$$|\chi\rangle = \frac{|\uparrow\rangle|{\downarrow_\theta}\rangle - |\downarrow\rangle|{\uparrow_\theta}\rangle}{\sqrt{2}}\,.$$

A measurement of S_z for the first particle is carried out, followed by a measurement of S_z for the second particle. List the possible outcomes for this pair of measurements and find the total probability, in terms of θ, for each pair of outcomes to occur. For which of these outcomes is the system left in an eigenstate of the combined total spin operator $\mathbf{S}\cdot\mathbf{S}$, and what are the corresponding eigenvalues?

[*Hint: The Pauli sigma matrices are given by*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad]$$

Paper 3, Section II

33A Principles of Quantum Mechanics

(a) Show that $[L_z, z] = 0$ and hence that $\langle n', \ell', m'|z|n, \ell, m\rangle$ vanishes unless $m' = m$, where $|n, \ell, m\rangle$ is a simultaneous eigenstate of H, L^2 and L_z.

(b) Given that $[L^2, [L^2, z]] = 2\hbar^2(L^2 z + zL^2)$, show that $\langle n', \ell', m'|z|n, \ell, m\rangle$ vanishes unless $|\ell' - \ell| = 1$ or $\ell' = \ell = 0$. By considering parity, show that this matrix element also vanishes if $\ell' = \ell$.

(c) A hydrogen atom in its ground state $|n, \ell, m\rangle = |1, 0, 0\rangle$ is placed in a constant, uniform electric field $\mathbf{E}$. With reference to the atom's charge distribution, but without detailed calculation, give a physical explanation of why there is no correction of first-order (in $\mathbf{E}$) to the ground state energy, but higher-order corrections are possible.

(d) Show that the second-order correction to the energy of the ground state caused by the electric field is

$$\frac{e^2|\mathbf{E}|^2}{\mathcal{R}} \sum_{n=2}^{\infty} \frac{n^2}{1-n^2} \left|\langle n, 1, 0|z|1, 0, 0\rangle\right|^2 ,$$

where $-\mathcal{R}$ is the unperturbed energy of $|1, 0, 0\rangle$.

[*You may assume that, when a Hamiltonian is perturbed by* ΔH, *the second-order correction to the ground state energy is*

$$\sum_{\alpha} \frac{|\langle\alpha|\Delta H|\phi\rangle|^2}{E_\phi - E_\alpha} ,$$

where $\{|\alpha\rangle\}$ *is a complete set of unperturbed eigenstates states orthogonal to the unperturbed ground state* $|\phi\rangle$, *and* E_α, E_ϕ *are their unperturbed energies.*]

Paper 4, Section II
33A Principles of Quantum Mechanics

A particle travels in one dimension subject to the Hamiltonian

$$H_0 = \frac{P^2}{2m} - U\,\delta(x)\,,$$

where U is a positive constant. Let $|0\rangle$ be the unique bound state of this potential and E_0 its energy. Further let $|k,\pm\rangle$ be unbound H_0 eigenstates of even/odd parity, each with energy E_k, chosen so that $\langle k',+|k,+\rangle = \langle k',-|k,-\rangle = \delta(k'-k)$.

(a) At times $t \leqslant 0$ the particle is trapped in the well. From $t=0$ it is disturbed by a time-dependent potential $v(x,t) = -Fx\,e^{-i\omega t}$ and subsequently its state may be expressed as

$$|\psi(t)\rangle = a(t)\,e^{-iE_0t/\hbar}|0\rangle + \int_0^\infty \Big(b_k(t)|k,+\rangle + c_k(t)|k,-\rangle\Big)\,e^{-iE_kt/\hbar}\,dk\,.$$

Show that

$$\dot{a}(t)\,e^{-iE_0t/\hbar}|0\rangle + \int_0^\infty e^{-iE_kt/\hbar}\Big(\dot{b}_k(t)|k,+\rangle + \dot{c}_k(t)|k,-\rangle\Big)\,dk = \frac{iF}{\hbar}e^{-i\omega t}\,x|\psi(t)\rangle$$

for all $t>0$.

(b) Working to first order in F, hence show that $b_k(t)=0$ and that

$$c_k(t) = \frac{iF}{\hbar}\langle k,-|x|0\rangle\,e^{i\Omega_k t/2}\,\frac{\sin(\Omega_k t/2)}{\Omega_k/2}\,,$$

where $\Omega_k = (E_k - E_0 - \hbar\omega)/\hbar$.

(c) The original bound state has position space wavefunction $\langle x|0\rangle = \sqrt{K}\,e^{-K|x|}$ where $K = mU/\hbar^2$, while the position space wavefunction of the odd parity unbound state is $\langle x|k,-\rangle = \sin(kx)\,/\sqrt{\pi}$ and its energy $E_k = \hbar^2k^2/2m$. Show that at late times the probability that the particle escapes from the original potential well is

$$P_{\text{free}}(t) = \frac{8\hbar F^2 t}{mE_0^2}\frac{\sqrt{E_f/|E_0|}}{(1+E_f/|E_0|)^4}$$

to lowest order in F, where $E_f>0$ is the final energy. [*You may assume that as* $t\to\infty$, *the function* $\sin^2(\lambda t)/(\lambda^2 t) \to \pi\,\delta(\lambda)$.]

Paper 1, Section II
29K Principles of Statistics

(a) Suppose that Θ is an open subset of $\mathbb{R}^p$, that $\Phi : \Theta \to \mathbb{R}$ is continuously differentiable at some $\theta_0 \in \Theta$, and that $\{\widehat{\theta}_n\}_{n\geqslant 1}$ is a sequence of random vectors in $\mathbb{R}^p$ satisfying $\sqrt{n}(\widehat{\theta}_n - \theta_0) \xrightarrow{d} Z$, where $Z \in \mathbb{R}^p$. Prove that

$$\sqrt{n}(\Phi(\widehat{\theta}_n) - \Phi(\theta_0)) \xrightarrow{d} \nabla_\theta \Phi(\theta_0)^T Z\,.$$

For the remainder of this problem, consider the $N(0,\sigma^2)$ model, where $\sigma \in (0,\infty)$.

(b) Derive the maximum likelihood estimator $\widehat{\sigma}_{\text{MLE}}$ of σ based on an i.i.d. sample of size n from the model. What is the asymptotic distribution of $\sqrt{n}(\widehat{\sigma}_{\text{MLE}} - \sigma)$? [*Hint: You may use, without proof, the fact that* $\mathbb{E}[Z^4] = 3$ *when* $Z \sim N(0,1)$.]

(c) What is the Fisher information $I(\sigma)$ (for the sample size $n = 1$)?

(d) Now consider the alternative parametrization of the model in terms of $\rho = \sigma^2$, where $\rho \in (0,\infty)$. What is the maximum likelihood estimator $\widehat{\rho}_{\text{MLE}}$ of ρ?

Paper 2, Section II
29K Principles of Statistics

Suppose $X_1,\ldots,X_n$ are i.i.d. samples from a $N(\theta,1)$ distribution. Consider an estimator $\widehat{\theta}_{a,b}$ of the form $a\bar{X}_n + b$, where $a, b \in \mathbb{R}$ and $\bar{X}_n$ denotes the sample mean. Throughout this question, we will consider risks computed with respect to the quadratic loss.

(a) Compute the risk of $\widehat{\theta}_{a,b}$ for estimating θ.

(b) Use the formula in part (a) to show that when $a > 1$, the estimator $\widehat{\theta}_{a,b}$ is inadmissible for estimating θ.

(c) Now use the formula in part (a) to show that when $a < 0$, the estimator $\widehat{\theta}_{a,b}$ is also inadmissible for estimating θ. [*Hint: Compare the estimator with the constant estimator* $\delta := \frac{-b}{a-1}$.]

(d) Prove that $\bar{X}_n$ is admissible for estimating θ. [*Hint: You may use, without proof, the general Cramér–Rao lower bound, and the facts that* $I(\theta) = 1$ *and* $\mathbb{E}_\theta[\delta(X)]$ *is differentiable for any estimator* δ *under the Gaussian model.*]

(e) Can any of the estimators considered in parts (b) and (c) be minimax for estimating θ?

Paper 3, Section II
28K Principles of Statistics

Suppose T_n is an estimator computed from n i.i.d. observations $X_1, \ldots, X_n$. Recall that the jackknife bias-corrected estimate of T_n is given by $\widetilde{T}_{\mathrm{JACK}} = T_n - \widehat{B}_n$, where

$$\widehat{B}_n = (n-1)\left(\frac{1}{n}\sum_{i=1}^{n} T_{(-i)} - T_n\right).$$

(a) Suppose that as $n \to \infty$ the bias function $B_n(\theta) = \mathbb{E}_\theta[T_n] - \theta$ can be approximated as

$$B_n(\theta) = \frac{a}{n} + \frac{b}{n^2} + O\left(\frac{1}{n^3}\right),$$

for some $a, b \in \mathbb{R}$. Prove that

$$|\mathbb{E}[\widetilde{T}_{\mathrm{JACK}}] - \theta| = O\left(\frac{1}{n^2}\right).$$

For the remainder of this problem, suppose $X_i \overset{i.i.d.}{\sim} N(\mu, 1)$.

(b) Consider the estimator $T_n = (\bar{X}_n)^2$ for $\theta = \mu^2$, where $\bar{X}_n$ denotes the sample mean. Compute the biases of T_n and $\widetilde{T}_{\mathrm{JACK}}$.

(c) What is the asymptotic distribution of $\sqrt{n}(T_n - \mu^2)$?

(d) Show that $\sqrt{n}(\widetilde{T}_{\mathrm{JACK}} - \mu^2)$ has the same asymptotic distribution as $\sqrt{n}(T_n - \mu^2)$. [*Hint: Define* $g(t) = t^2$ *and define* $\bar{X}_{n-1,i}$ *to be the sample mean of the observations with* X_i *excluded. Note that*

$$\widetilde{T}_{\mathrm{JACK}} = T_n - \frac{n-1}{n}\sum_{i=1}^{n}\left(g(\bar{X}_{n-1,i}) - g(\bar{X}_n)\right)$$

and use the identities

$$\sum_{i=1}^{n}(\bar{X}_{n-1,i} - \bar{X}_n) = 0 \quad \textit{and} \quad \bar{X}_{n-1,i} - \bar{X}_n = \frac{1}{n-1}(\bar{X}_n - X_i). \qquad]$$

Paper 4, Section II
28K Principles of Statistics

(a) Suppose it is possible to generate samples from a Uniform$[0,1]$ distribution. Describe a method for generating samples from an exponential distribution with rate parameter 1, and prove that the method is valid.

(b) Recall that the accept/reject algorithm, which operates on two pdfs f and h satisfying $f \leqslant Mh$, proceeds as follows:

1. Generate $X \sim h$ and $U \sim$ Uniform$[0,1]$.
2. If $U \leqslant \frac{f(X)}{Mh(X)}$, take $Y = X$. Otherwise, return to Step 1.

Prove that the output Y has pdf f.

(c) Suppose the pdf f is given by

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for all } x \geqslant 0.$$

Let h be the pdf of an exponential distribution with rate parameter 1. Explain how to apply the accept/reject algorithm in this special case. Identify an appropriate value for M.

(d) Compute the expected number of steps required to generate one sample from the pdf f in part (c) using the accept/reject algorithm.

(e) Let Y be a random variable generated according to the algorithm in (c). Now suppose we generate a random variable X using the following additional steps:

1. Generate $V \sim$ Uniform$[0,1]$.
2. If $V \leqslant \frac{1}{2}$, take $Z = Y$. Otherwise, take $Z = -Y$.

What is the distribution of Z?

(f) Suppose the final goal is to generate samples from the distribution of Z in part (e). Following the steps outlined in parts (a)–(e), could the efficiency of the algorithm be improved by choosing X to be an exponential random variable with rate parameter $\lambda \neq 1$?

Paper 1, Section II
27G Probability and Measure

(a) State and prove *Kolmogorov's zero-one law.*

(b) Consider the product space $E = \mathbb{R}^{\mathbb{N}}$ equipped with the σ-algebra $\sigma(\mathcal{C})$ generated by the cylinder sets

$$\mathcal{C} = \left\{ A = \times_{n=1}^{\infty} A_n \,|\, A_n \subseteq \mathbb{R}, A_n \text{ Borel for } n \leqslant N, A_n = \mathbb{R} \text{ for } n > N, \text{ some } N \in \mathbb{N} \right\}.$$

For m a probability measure on $\mathbb{R}$, show that there exists a unique product measure μ on $(E, \sigma(\mathcal{C}))$ for which $\mu(A) = \prod_{n=1}^{\infty} m(A_n)$ for all $A \in \mathcal{C}$. Show further that the shift map θ defined on E by $\theta((x_1, x_2, \dots)) = (x_2, x_3, \dots)$ is measure-preserving and ergodic for μ.

[*You may use without proof the existence of an infinite sequence of i.i.d. real random variables defined on any probability space.*]

Paper 2, Section II
27G Probability and Measure

(a) State and prove the *monotone convergence theorem.*

(b) Let f_1 be a μ-integrable function and let f be a measurable function defined on some measure space $(E, \mathcal{E}, \mu)$. Suppose the sequence $(f_n : n \in \mathbb{N})$ of measurable functions on E is such that $f_n \uparrow f$ pointwise on E as $n \to \infty$. Show that $\mu(f_n) \uparrow \mu(f)$ as $n \to \infty$. Show that the conclusion may fail if f_1 is not integrable.

Paper 3, Section II
26G Probability and Measure

Suppose that as $n \to \infty$, a sequence of real random variables $X_n \to^d X$, i.e. X_n converges in distribution to some limiting random variable X. Suppose further that as $n \to \infty$ a sequence of real random variables $Y_n \to^P c$, i.e. Y_n converges in probability to some constant (non-random) limit $c > 0$. Show that $X_n Y_n \to^d cX$ as $n \to \infty$.

Now let $(Z_n : n \in \mathbb{N})$ be i.i.d. real random variables with $\mathbb{E}Z_i = 0$ and finite variance $\mathrm{Var}(Z_i) = 1$ for all i. Show that

$$\frac{\sqrt{n}\sum_{i=1}^{n} Z_i}{\sum_{i=1}^{n} Z_i^2} \to^d N(0,1)$$

as $n \to \infty$, where $N(0,1)$ denotes the standard normal distribution.

[*You may use the strong law of large numbers and the central limit theorem without proof, provided they are clearly stated. You may further use without proof the equivalence of weak convergence of laws of probability measures and convergence in distribution for real random variables.*]

Paper 4, Section II
26G Probability and Measure

Denote by L^1 the space of real-valued functions on $\mathbb{R}$ that are integrable with respect to Lebesgue measure. For $f \in L^1$ and g_t the probability density function of a normal $N(0,t)$ random variable with variance $t > 0$, show that their convolution

$$f * g_t(x) = \int_{\mathbb{R}} f(x-y) g_t(y) dy \,, \;\; x \in \mathbb{R},$$

defines another element of L^1. Show carefully that the Fourier inversion theorem holds for $f * g_t$.

Now suppose that the Fourier transform of f is also in L^1. Show that $f * g_t(x) \to f(x)$ for almost every $x \in \mathbb{R}$ as $t \to 0$.

[*You may use Fubini's theorem and the translation invariance of Lebesgue measure without proof.*]

Paper 1, Section I

10D Quantum Information and Computation

Alice and Bob are separated in space and possess local quantum systems A and B respectively.

(a) State the *no-signalling theorem* for quantum states of the composite system AB.

(b) State and prove the *no-cloning theorem* (for unitary processes) for a set $\mathcal{S}$ of quantum states.

(c) Now let $\mathcal{S} = \{|\,0\rangle\,, |\,1\rangle\,, |+\rangle\,, |-\rangle\}$ where $|\pm\rangle = \frac{1}{\sqrt{2}}(|\,0\rangle \pm |\,1\rangle)$. Starting with a suitable state for a 2-qubit composite system AB, show how the no-cloning theorem for the set $\mathcal{S}$ can be seen as a consequence of the no-signalling theorem for AB.

Paper 2, Section I

10D Quantum Information and Computation

(a) Suppose that Alice and Bob are distantly separated in space and they can communicate classically publicly. They also have available a noiseless quantum channel on which there is no eavesdropping. Describe the steps of the BB84 protocol that results in Alice and Bob sharing a secret key of expected length $n/2$. [Note that the steps of information reconciliation and privacy amplification will not be needed in this idealised situation.]

(b) Suppose now that an eavesdropper Eve taps into the quantum channel. Eve also possesses a supply of ancilla qubits each in state $|\,0\rangle_E$. For each passing qubit $|\,\psi\rangle_A$ sent by Alice, Eve intercepts it and applies a CX operation to it and one of her ancilla qubits $|\,0\rangle_E$ with Alice's qubit being the control i.e. Eve applies CX_{AE}. After this action Eve sends Alice's qubit on to Bob while retaining her ancilla qubit.

(i) Show that for two choices of Alice's sent qubits, the qubit received by Bob will be entangled with Eve's corresponding ancilla qubit.

(ii) Calculate the bit error rate for Alice and Bob's final key in part (a) that results from Eve's action.

Paper 3, Section I
10D Quantum Information and Computation

Let $\mathbf{x} = x_0 x_1 \ldots x_{N-1}$ be an N-bit string with $N = 2K$ being even. Let $\mathcal{H}_M$ denote a state space of dimension M with orthonormal basis $\{|\,k\rangle \,:\, k \in \mathbb{Z}_M\}$. A quantum oracle $O_{\mathbf{x}}$ for $\mathbf{x}$ is a unitary operation on $\mathcal{H}_N \otimes \mathcal{H}_2$ whose action is defined by $O_{\mathbf{x}}\,|\,i\rangle\,|\,y\rangle = |\,i\rangle\,|\,y \oplus x_i\rangle$, where $y \in \{0,1\}$ and $\oplus$ denotes addition modulo 2.

Consider the following oracle problem, called Problem A:
Input: an oracle $O_{\mathbf{x}}$ for some N-bit string $\mathbf{x}$.
Promise: $\mathbf{x}$ is either a constant string, or a balanced string (the latter meaning that $\mathbf{x}$ contains exactly K 0's and K 1's).
Problem: decide if $\mathbf{x}$ is balanced.

(a) Suppose we have a universal set of quantum gates available and any desired unitary operation that is independent of $\mathbf{x}$ may be exactly implemented. Also, we may perform measurements in the basis $\{|\,i\rangle : i \in \mathbb{Z}_N\}$ of an N-dimensional register.

Show that Problem A can be solved with certainty by a quantum algorithm that makes only one query to the oracle $O_{\mathbf{x}}$. The algorithm should begin with each register initially in the state $|\,0\rangle$ (in the appropriate state space).

(b) Suppose now that in addition to $O_{\mathbf{x}}$ and measurements in the basis $\{|\,i\rangle : i \in \mathbb{Z}_N\}$, we can implement only the Pauli Z gate on a qubit register and gates F and F^{-1} on an N-dimensional register, where F has the property that $F\,|\,0\rangle = \frac{1}{\sqrt{N}} \sum_{i \in \mathbb{Z}_N} |\,i\rangle$.

By considering the action of Z on a qubit register $|\,y\rangle$, or otherwise, show that with the restricted set of operations, Problem A can be solved with certainty by a quantum algorithm that makes two queries to the oracle $O_{\mathbf{x}}$, and as before, with each register starting in the state $|\,0\rangle$ (in the appropriate state space).

Paper 4, Section I

10D Quantum Information and Computation

(a) Let B_n denote the set of all n-bit strings and write $N = 2^n$. The *Grover iteration operator* on n qubits is given by

$$Q = -H_n I_0 H_n I_{x_0}.$$

Give a definition of the constituent operators H_n, I_0 and I_{x_0} and state a geometrical interpretation of the action of Q on the space of n qubits.

(b) The quantum oracle for the identity function $\mathcal{I} : B_n \to B_n$, $\mathcal{I}(x) = x$ is the unitary operation $U_{\mathcal{I}}$ on $2n$ qubits defined by $U_{\mathcal{I}}(|\,x\rangle\,|\,y\rangle) = |\,x\rangle\,|\,y \oplus \mathcal{I}(x)\rangle$ for all $x, y \in B_n$. Here $\oplus$ denotes the sum of n-bit strings bitwise mod 2 separately at each of the n positions in the string, *i.e.* the group operation in $(\mathbb{Z}_2)^n$.

Show how the action of $U_{\mathcal{I}}$ can be represented by a circuit of CX gates.

(c) Suppose we are given a quantum oracle for $\mathcal{I}$ but it is known to be faulty on one of its inputs. Instead of the full identity function it implements instead the function $f : B_n \to B_n$ given by

$$f(x) = \begin{cases} x & \text{for all } x \neq x_0 \\ x \oplus a & \text{for } x = x_0 \end{cases}$$

where $a \in B_n$ is the n-bit string $00\ldots01$ and where $x_0 \in B_n$ is unknown, *i.e.* the given quantum oracle actually implements U_f. By providing a suitable input state for a circuit involving U_f and further gates independent of f, show how I_{x_0} on n qubits may be implemented in terms of U_f.

(d) Hence or otherwise show that for sufficiently large N, x_0 may be determined with some constant probability greater than $\frac{1}{2}$ using $O(\sqrt{N})$ queries to the oracle U_f.

Paper 2, Section II
15D Quantum Information and Computation

(a) (i) Define the *Bell measurement* on two qubits.

(ii) In terms of the Bell measurement and the Bell state $|\phi^+\rangle$ give the steps of the quantum teleportation protocol. You need not give a derivation of the steps but you should clearly state all inputs and outputs of the protocol.

(iii) Suppose now that the $|\phi^+\rangle$ state used in the protocol is replaced by $|\xi\rangle = I \otimes U |\phi^+\rangle$, where U is any 1-qubit unitary and all steps of the protocol remain otherwise the same as in part (ii) above. State the outputs of this modified protocol and give a justification of your answer. [You may quote any statements from part (ii) above.]

(b) A *programmable 1-qubit gate* $\mathcal{G}$ is defined to be a device acting on two registers A and B, where A is a 1-qubit register called the input register and B is a K-qubit register (for some fixed $K \in \mathbb{N}$) called the program register. For any given state of AB the action of $\mathcal{G}$ is a fixed unitary operation G on the $K+1$ qubits, which is required to satisfy the following condition called (PROG):

> For any 1-qubit unitary U there is a K-qubit state $|P_U\rangle$ such that for any 1-qubit state $|\alpha\rangle$ we have
>
> $$|\alpha\rangle \otimes |P_U\rangle \longmapsto G(|\alpha\rangle \otimes |P_U\rangle) = (U|\alpha\rangle) \otimes \left|\tilde{P}_U\right\rangle.$$

Here $\left|\tilde{P}_U\right\rangle$ is some K-qubit state (which could generally depend on $|\alpha\rangle$ too). Thus $|P_U\rangle$ serves as a "program" for the application of U to any 1-qubit state $|\alpha\rangle$ via the fixed unitary action G.

(i) By considering suitable inner products or otherwise, show that if (PROG) holds then $\left|\tilde{P}_U\right\rangle$ must be independent of the state $|\alpha\rangle$.

(ii) Suppose that $|P_U\rangle$ and $|P_V\rangle$ implement 1-qubit unitaries U and V that have physically different actions i.e. $U \neq Ve^{i\theta}$ for any phase θ. Show that $|P_U\rangle$ and $|P_V\rangle$ must then be orthogonal if (PROG) holds. [*Hint: It may be helpful to show that for any unitary W, if $\langle\alpha| W |\alpha\rangle$ is independent of $|\alpha\rangle$ then W must be the identity gate (up to an overall phase).*]

(iii) Show that a programmable 1-qubit gate $\mathcal{G}$ satisfying (PROG) cannot exist.

(iv) Suppose now that (PROG) is extended to allow the action of $\mathcal{G}$ to involve quantum measurements as well as unitary operations and we require of the "program" $|P_U\rangle$ only that it succeeds in applying U to $|\alpha\rangle$ with at least some constant probability $0 < p < 1$ independent of U and $|\alpha\rangle$, *i.e.* the action of $\mathcal{G}$ on $|\alpha\rangle \otimes |P_U\rangle$ results in $U|\alpha\rangle$ in the first register with probability at least p for each U and $|\alpha\rangle$. Can such a probabilistic programmable 1-qubit gate exist? Give a reason for your answer.

Paper 3, Section II

15D Quantum Information and Computation

For any positive integer N, let QFT_N denote the quantum Fourier transform mod N.

(a) Consider an N-dimensional state space equipped with an orthonormal basis $\mathcal{B} = \{|\,k\rangle : k \in \mathbb{Z}_N\}$. You may assume that QFT_N, measurements in the basis $\mathcal{B}$, and the basic arithmetic operations of addition and multiplication modulo N may all be performed in time $O(\mathrm{poly}(\log N))$.

Consider the function $f : \mathbb{Z}_N \to \mathbb{Z}_N$ defined by $f(x) = a^x \mod N$, where we have fixed a choice of $a \in \mathbb{Z}_N$ with $a \neq 0$. It is promised that f is periodic with period r which divides N exactly, and f is one-to-one within each period.

Describe a quantum algorithm which runs in time $O(\mathrm{poly}(\log N))$ that will identify r with success probability at least $1/2$. The algorithm should start with each quantum register (of suitable dimension) being in state $|\,0\rangle$ and it should have the property that in any run, we also learn whether it has succeeded or not. For any step of your algorithm that is not one of the operations listed above, give a brief justification that it can be performed in time $O(\mathrm{poly}(\log N))$. [You may use without proof any results from classical number theory or classical probability theory but they must be stated clearly.]

(b) Consider an N-dimensional state space with orthonormal basis $\{|\,i\rangle : i \in \mathbb{Z}_N\}$. Let S be the operation defined by $S\,|\,i\rangle = |\,i+1\rangle$ for all $i \in \mathbb{Z}_N$ (and $+$ being addition modulo N). Show that the states $\mathrm{QFT}_N\,|\,k\rangle$ for $k \in \mathbb{Z}_N$ are eigenvectors of S.
Now let $N = 4$ and represent each basis state $|\,j\rangle$ with two qubits as $|\,x\rangle\,|\,y\rangle$ where the 2-bit string xy is j written in binary. Suppose we can implement only the gates QFT_4, its inverse and any 1-qubit phase gate $P(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$. Show how S may be implemented on any input 2-qubit state and sketch the circuit for S.

Paper 1, Section II
19H Representation Theory

Let G be a finite group.

State *Maschke's theorem* for complex representations of G. Deduce that every representation of G is isomorphic to a direct sum of irreducible representations.

Define the *character* χ_V of a complex representation V of G. Suppose that G acts on a finite set X. What is the *permutation representation* $\mathbb{C}X$? Describe its character $\chi_{\mathbb{C}X}$.

Show that if $V_1, \ldots, V_r$ are all the irreducible representations of G up to isomorphism then the regular representation decomposes as

$$\mathbb{C}G \cong \bigoplus_{i=1}^{r} (\dim V_i) V_i.$$

If V is a complex representation of G, let $\mathrm{Hom}_G(V,V)$ be the space of G-linear maps from V to V. If

$$V \cong \bigoplus_{i=1}^{r} n_i V_i,$$

what is the dimension of $\mathrm{Hom}_G(V,V)$? What is the dimension when $V = \mathbb{C}G$?

Now suppose V is a complex representation of G with character χ such that $\chi(g) = 0$ for all non-identity elements $g \in G$. Show that V is a direct sum of copies of the regular representation $\mathbb{C}G$.

Deduce that if W is any complex representation of G then

$$W \otimes \mathbb{C}G \cong \bigoplus_{i=1}^{\dim W} \mathbb{C}G.$$

[*You may assume that the irreducible complex characters of a finite group form an orthonormal basis of the space of class functions.*]

Paper 2, Section II
19H Representation Theory

Suppose that G is a group of order 16. Let $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_r$ be the degrees of the irreducible characters of G. What are the possible values of r and $d_1, \ldots, d_r$? For each such collection $d_1, \ldots, d_r$ find a group of order 16 with these character degrees and construct the character table of the group. [You may assume any general results from the course provided that you state them clearly. You may restrict yourself to brief justifications of the values in each character table.]

Paper 3, Section II
19H Representation Theory

Let $G = SU(2)$ and let V_n be the complex vector space of homogeneous polynomials of degree n in two variables x, y. Construct a continuous homomorphism $\rho_n \colon G \to GL(V_n)$ so that (ρ_n, V_n) is an irreducible representation of G. Prove that (ρ_n, V_n) is indeed irreducible.

What is the character of V_n? Show that every irreducible representation of $SU(2)$ is isomorphic to (ρ_n, V_n) for some $n \geqslant 0$.

Suppose that χ is the character of a representation V of G. State a formula for the character of $\Lambda^2 V$ in terms of χ. Use it to decompose $\Lambda^2 V_4$ as a direct sum of irreducible representations up to isomorphism.

Express the character of $\Lambda^3 V$ in terms of χ. Justify your answer. Decompose $\Lambda^3 V_4$ as a direct sum of irreducible representations up to isomorphism.

Paper 4, Section II
19H Representation Theory

Suppose that H is a subgroup of a group G and χ is a complex character of H.

State *Mackey's restriction formula* and *Frobenius reciprocity* for characters. Use them to deduce Mackey's irreducibility criterion for an induced representation.

Suppose that k is a finite field of order $q \geqslant 4$, $G = SL_2(k)$ and

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a, b \in k, a \neq 0 \right\}.$$

Describe the degree 1 complex characters χ of B and explain, with justification, for which of them $\mathrm{Ind}_B^G \chi$ is irreducible.

Paper 1, Section II
24F Riemann Surfaces

(a) State the *Uniformisation theorem*, and deduce the Riemann mapping theorem.

(b) Let
$$E = \{x + iy \mid x, y \in \mathbb{R}\,,\ -\pi < x < \pi\}$$
be an infinite vertical strip in $\mathbb{C}$, and let $U \subseteq \mathbb{C}$ consist of $\mathbb{C}$ with the negative real axis (and zero) removed. A *Mercator projection* is a conformal equivalence $f : U \to E$ such that $\operatorname{Im} f(z) \to -\infty$ as $z \to 0$ and $\operatorname{Im} f(z) \to +\infty$ as $z \to \infty$. Exhibit an explicit Mercator projection.

(c) Consider a conformal equivalence $\phi : E \to E$ such that $\operatorname{Im}\phi(z) \to +\infty$ as $\operatorname{Im} z \to +\infty$ and $\operatorname{Im}\phi(z) \to -\infty$ as $\operatorname{Im} z \to -\infty$. Prove that ϕ is translation by an imaginary number, stating carefully any results that you use.

(d) Characterise all Mercator projections.

Paper 2, Section II
24F Riemann Surfaces

(a) Let $D = \{p_1, \ldots, p_n\}$ be a finite (possibly empty) subset of a Riemann surface R, and let $m_1, \ldots, m_n$ be strictly positive integers. Let V be the set of meromorphic functions f on R such that each pole of f is at some p_i, and the order of a pole at p_i is at most m_i. Prove that V is a vector space over $\mathbb{C}$.

(b) For any compact Riemann surface R, prove that
$$\dim_{\mathbb{C}} V \leqslant 1 + \sum_{i=1}^{n} m_i$$
by considering Laurent expansions about the p_i, or otherwise.

(c) Let $R = \mathbb{C}/\Lambda$ be a complex torus. For any meromorphic function f on R with poles $p_1, \ldots, p_n$, prove that
$$\sum_{i=1}^{n} \operatorname{res}_f(p_i) = 0\,.$$
Assuming that $n \geqslant 1$, deduce that $\dim_{\mathbb{C}} V = \sum_i m_i$.

Paper 3, Section II
23F Riemann Surfaces

(a) Consider a finite group H of conformal equivalences of the Riemann sphere $\mathbb{C}_\infty$ such that H fixes a point $p \in \mathbb{C}_\infty$. Prove that H is cyclic and that there is a neighbourhood U of p, invariant under H, so that the quotient $V = H \backslash U$ has the structure of a Riemann surface. Show furthermore that there are charts on U and V so that the quotient map takes the form $z \mapsto z^n$ for some $n \in \mathbb{N}$.

[*You may use without proof the fact that every Möbius transformation is conjugate to either a dilation* $z \mapsto \lambda z$ *or a translation* $z \mapsto z + c$.]

(b) Let G be a finite group of conformal automorphisms of $\mathbb{C}_\infty$. Prove that the quotient $R = G \backslash \mathbb{C}_\infty$ has a conformal structure such that the quotient map $\mathbb{C}_\infty \to R$ is holomorphic.

(c) For each positive integer $n \geqslant 2$, construct a faithful action of the dihedral group D_{2n} on $\mathbb{C}_\infty$. Furthermore, exhibit a rational function f such that z_1 and z_2 are in the same D_{2n}-orbit if and only if $f(z_1) = f(z_2)$.

Paper 1, Section I
5J Statistical Modelling

Let Y_μ be the Poisson distribution with mean μ. Show that the transformation $g(y) = 2\sqrt{y}$ is "variance stabilising" for Y_μ in the sense that the variance of $g(Y_\mu)$ is approximately 1 when μ is large.

Suppose we fit a linear model to the transformed response $\sqrt{Y}$. How does this differ from using the square root link in the Poisson regression?

Paper 2, Section I
5J Statistical Modelling

(a) Give the definition of an *exponential family* of probability distributions. [You may assume the natural parameter is one-dimensional.]

(b) Suppose $Y_1, \ldots, Y_n \overset{i.i.d.}{\sim} f(y;\theta)$ where $f(y;\theta)$ is the density function of an exponential family with natural parameter θ and sufficient statistic Y. Show that $\bar{Y} = \sum_{i=1}^n Y_i/n$ is a sufficient statistic for θ.

(c) In the setting above, show that the maximum likelihood estimator of θ is given by setting the theoretical mean $\mu(\theta) = \mathbb{E}_\theta(Y_1)$ to the empirical mean $\bar{Y}$.

Paper 3, Section I
5J Statistical Modelling

The density function of the Laplace distribution Laplace(μ, σ) with mean μ and scale parameter σ is given by

$$f(y;\mu,\sigma) = (2\sigma)^{-1} \exp\left\{-\frac{|y-\mu|}{\sigma}\right\}.$$

Briefly comment on why the Laplace distribution cannot be written in exponential dispersion family form.

Consider the linear model where $(X_i, Y_i), i = 1, \ldots, n$ are assumed independent and

$$Y_i \mid X_i \sim \text{Laplace}(X_i^T\beta, \sigma).$$

Show that the maximum likelihood estimator $\hat{\beta}$ of β is obtained by minimising

$$S(\beta) = \sum_{i=1}^n |Y_i - X_i^T\beta|.$$

Obtain the maximum likelihood estimator of σ in terms of $S(\hat{\beta})$.

Paper 4, Section I

5J Statistical Modelling

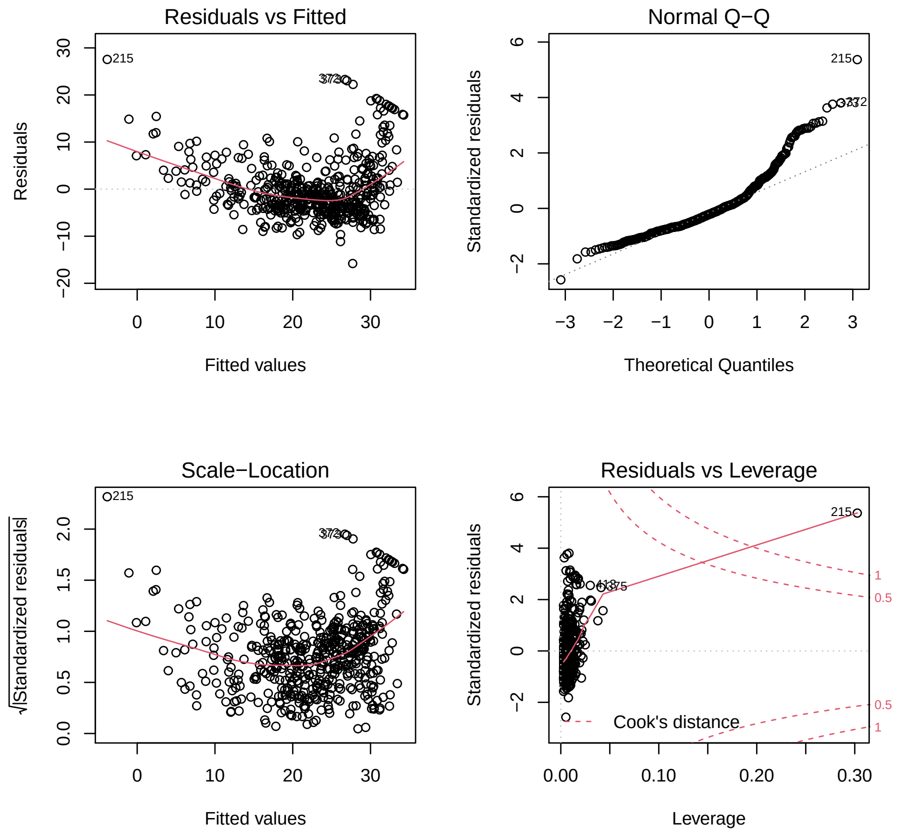
The `Boston` dataset records `medv` (median house value), `age` (average age of houses), `lstat` (percent of households with low socioeconomic status), and other covariates for 506 census tracts in Boston.

```
> head(Boston[, c("medv", "age", "lstat")])
  medv  age lstat
1 24.0 65.2  4.98
2 21.6 78.9  9.14
3 34.7 61.1  4.03
4 33.4 45.8  2.94
5 36.2 54.2  5.33
6 28.7 58.7  5.21
```

Describe the mathematical model fitted in the `R` code below and give three observations from the output of the code that you think are the most noteworthy.

```
> summary(fit <- lm(medv ~ lstat * age , data = Boston))

Residuals:
    Min      1Q  Median      3Q     Max
-15.806  -4.045  -1.333   2.085  27.552

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) 36.0885359  1.4698355  24.553  < 2e-16 ***
lstat       -1.3921168  0.1674555  -8.313 8.78e-16 ***
age         -0.0007209  0.0198792  -0.036   0.9711
lstat:age    0.0041560  0.0018518   2.244   0.0252 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6.149 on 502 degrees of freedom
Multiple R-squared:  0.5557,    Adjusted R-squared:  0.5531
F-statistic: 209.3 on 3 and 502 DF,  p-value: < 2.2e-16

>
> par(mfrow = c(2, 2))
> plot(fit)
```

[QUESTION CONTINUES ON THE NEXT PAGE]


Residuals vs Fitted
Residuals
Fitted values
215
372
Normal Q−Q
Standardized residuals
Theoretical Quantiles
215
372
Scale−Location
√|Standardized residuals|
Fitted values
215
372
Residuals vs Leverage
Standardized residuals
Leverage
Cook's distance
215
375
1
0.5
0.5
1

Paper 1, Section II
13J Statistical Modelling

The following dataset contains information about some of the passengers on RMS *Titanic* when it sank on 15th April, 1912.

```
> head(titanic)
  Survived Pclass    Sex Age SibSp Parch    Fare Cabin Embarked
1        0      3   male  22     1     0  7.2500  <NA>        S
2        1      1 female  38     1     0 71.2833   C85        C
3        1      3 female  26     0     0  7.9250  <NA>        S
4        1      1 female  35     1     0 53.1000  C123        S
5        0      3   male  35     0     0  8.0500  <NA>        S
6        0      3   male  NA     0     0  8.4583  <NA>        Q
> nrow(titanic)
[1] 889
```

We would like to predict which passengers were more likely to survive (`Survived`, 0 = No, 1 = Yes) using the other covariates, including ticket class (`Pclass`, 1 = 1st, 2 = 2nd, 3 = 3rd), sex (`Sex`), age (`Age`), number of siblings/spouses aboard (`SibSp`), number of parents/children aboard (`Parch`), passenger fare (`Fare`), cabin number (`Cabin`), port of embarkation (`Embarked`, C = Cherbourg, Q = Queenstown, S = Southampton).

(a) Describe what the following chunk of `R` code does.

```
> apply(titanic, 2, function(x) sum(is.na(x)))
Survived   Pclass      Sex      Age    SibSp    Parch     Fare    Cabin
       0        0        0      177        0        0        0      687
Embarked
       0
> titanic$Cabin <- NULL
> titanic$Age[is.na(titanic$Age)] <- mean(titanic$Age, na.rm = TRUE)
```

(b) Write down the generalised linear model fitted (including the likelihood function maximised) by the code below. Define *Akaike's information criterion* (AIC) and explain, in words, how you can use the backward stepwise algorithm and AIC to select a model.

```
> summary(fit <- glm(Survived ~ ., family = binomial, data = titanic))

Deviance Residuals:
    Min       1Q   Median       3Q      Max
-2.6445  -0.5907  -0.4227   0.6214   2.4432
```

[QUESTION CONTINUES ON THE NEXT PAGE]

```
Coefficients:
             Estimate Std. Error z value Pr(>|z|)    
(Intercept)  5.284055   0.564696   9.357  < 2e-16 ***
Pclass      -1.100033   0.143530  -7.664 1.80e-14 ***
Sexmale     -2.718736   0.200779 -13.541  < 2e-16 ***
Age         -0.039885   0.007855  -5.078 3.82e-07 ***
SibSp       -0.325732   0.109368  -2.978   0.0029 ** 
Parch       -0.092470   0.118702  -0.779   0.4360    
Fare         0.001919   0.002376   0.808   0.4192    
EmbarkedQ   -0.035043   0.381920  -0.092   0.9269    
EmbarkedS   -0.418564   0.236788  -1.768   0.0771 .  
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 1182.82  on 888  degrees of freedom
Residual deviance:  784.21  on 880  degrees of freedom
AIC: 802.21

Number of Fisher Scoring iterations: 5
```

(c) The model summary above says “Dispersion parameter for binomial family taken to be 1”. Do you think that is reasonable based on the model summary? Justify your answer. You might find the following information useful.

```
> qnorm(0.25) # 25th-percentile of the standard normal distribution
[1] -0.6744898
```

(d) Give an estimator of the dispersion parameter in this model when it is not fixed at 1.

Paper 4, Section II
13J Statistical Modelling

Consider the following R code:

```
> n <- 1000000
> sigma_z <- 1; sigma_x1 <- 0.5; sigma_x2 <- 1; sigma_y <- 2; beta <- 2
> Z <- sigma_z * rnorm(n)
> X1 <- Z + sigma_x1 * rnorm(n)
> X2 <- Z + sigma_x2 * rnorm(n)
> Y <- beta * Z + sigma_y * rnorm(n)
> lm(Y ~ Z)

Call:
lm(formula = Y ~ Z)

Coefficients:
(Intercept)            Z
  -0.003089     1.999780

> lm(Y ~ X1)

Call:
lm(formula = Y ~ X1)

Coefficients:
(Intercept)           X1
  -0.002904     1.600521

> lm(Y ~ X2)

Call:
lm(formula = Y ~ X2)

Coefficients:
(Intercept)           X2
  -0.002672     0.997499
```

Describe the phenomenon you see in the output above, then give a mathematical explanation for this phenomenon. Do you expect the slope coefficient in the second model to be generally smaller than that in the first model? Do you think modifying (for example, doubling) the value of `sigma_y` will substantially alter the slope coefficient in the second model? Justify your answer.

Paper 1, Section II
36A Statistical Physics

(a) What systems are described by a *grand canonical ensemble*? If there are N_n particles in microstate n each with energy E_n, write down an expression for the *grand canonical partition function* $\mathcal{Z}$ in terms of the temperature T, the chemical potential μ and the Boltzmann constant k_B.

(b) Define the *grand canonical potential* Φ in terms of the average energy E, T, the entropy S, μ, and the average number of particles $\langle N\rangle$. Write down the relation between Φ and $\mathcal{Z}$.

(c) Using scaling arguments, express $\Phi(T, V, \mu)$ in terms of the pressure p and the volume V.

(d) Consider the grand canonical ensemble for a classical ideal gas of non-relativistic particles of mass m in a fixed 3-dimensional volume V.

(i) Compute $\mathcal{Z}$ and Φ.

(ii) Calculate $\langle N\rangle$ and $\Delta N/\langle N\rangle$, where $(\Delta N)^2 = \langle N^2\rangle - \langle N\rangle^2$. Comment on the latter result.

(iii) Derive the equation of state for the gas.

[*You may assume that* $\int_{-\infty}^{\infty} e^{-a\,x^2} dx = \sqrt{\pi/a}$ *for* $a > 0$.]

(e) Using the grand canonical ensemble and your results from part (d), derive the equation of state for a classical ideal gas of relativistic particles with energies $\sqrt{|\mathbf{p}|^2\, c^2 + m^2 c^4}$. Compute $\Delta N/\langle N\rangle$.

Paper 2, Section II
37A Statistical Physics

(a) What systems are described by a *microcanonical ensemble* and which by a *canonical ensemble*?

(b) Starting from the Gibbs formula for entropy, $S = -k_B \sum_n p(n) \ln p(n)$, where $p(n)$ is the probability of being in microstate n and k_B is the Boltzmann constant, show how maximising entropy subject to appropriate constraints leads to the correct forms of the probability distributions for (i) the microcanonical ensemble and (ii) the canonical ensemble.

(c) Derive an expression for the entropy in the canonical ensemble in terms of the partition function Z and temperature T.

(d) A system consists of N non-interacting particles fixed at points in a lattice in thermal contact with a reservoir at temperature T. Each particle has three possible states with energies $-\epsilon, 0, \epsilon$, where $\epsilon > 0$ is a constant. Compute the average energy E and the entropy S. Evaluate E and S in the limits $T \to \infty$ and $T \to 0$.

(e) For the system in part (d), describe a configuration that would have negative temperature. Justify your answer.

Paper 3, Section II

35A Statistical Physics

(a) What distinguishes bosons from fermions? What are the implications for the occupation number of states and for the ground state at low temperatures?

(b) Consider a gas of N non-interacting ultra-relativistic electrons in a large fixed 3-dimensional cubic volume V.

(i) Using the grand partition function, show that $pV = AE$, where p is the pressure, E is the average energy and A is a constant that you should determine.

(ii) Show that the Fermi energy, $E_F = D\,(N/V)^{1/3}$, where D is a constant that you should determine.

(iii) Show that at zero temperature $pV^a = K$, where a and K are constants that you should determine. How does this compare to an ultra-relativistic classical ideal gas?

(c) Now consider the same system as in part (b) with a magnetic field B, so the energy of an electron is $\pm\mu_B B$ depending on whether the spin is parallel or anti-parallel to the magnetic field, and μ_B is a constant. Assuming that $\mu_B B \ll E_F$, show that at zero temperature the total magnetic moment

$$M \approx \alpha \mu_B^{\gamma} B^{\delta} g(E_F)\,,$$

where $g(E_F)$ is the density of states at energy E_F and α, γ and δ are numerical constants that you should find. Then find the magnetic susceptibility χ of the gas at zero temperature. Comment on the result.

Paper 4, Section II
35A Statistical Physics

(a) State *Carnot's theorem.* Show how it can be used to define a thermodynamic temperature.

(b) Consider a solid body with heat capacity at constant volume C_V. Assume that the solid's volume remains constant throughout the following three scenarios:

(i) If the temperature changes from T_i to T_f, show that the entropy change is $\Delta S = S_f - S_i = C_V \ln(T_f/T_i)$.

(ii) Two identical such bodies (both with heat capacity C_V) with initial temperatures T_1 and T_2 are brought into equilibrium in a reversible process. What are the final temperatures of the bodies?

(iii) Now suppose that the two bodies are instead brought directly into thermal contact (irreversibly). What are the final temperatures of the bodies? Compute the entropy change and show that it is positive.

(c) The Gibbs free energy is given by $G = E + pV - TS$, where E is energy, p is pressure, V is volume and S is entropy. Explain why $G = \mu(T,p)N$, where μ is the chemical potential and N is the number of particles.

(d) What is a *first-order phase transition*?

(e) Consider a system at constant pressure where phase I is stable for $T > T_0$, phase II is stable for $T < T_0$, and there is a first-order phase transition at $T = T_0$. Show that in a transition from phase II to phase I, $S_I - S_{II} > 0$, where S_I is the entropy in phase I and S_{II} is the entropy in phase II. [*Hint: Consider* $S = -\left(\frac{\partial G}{\partial T}\right)_{p,N}$ *for each phase.*]

Paper 1, Section II
30K Stochastic Financial Models

Fix a positive integer N and consider the problem of minimising

$$\mathbb{E}\left(X_N^2 + \sum_{n=1}^{N} u_n^2\right),$$

where X_0 is given and

$$X_n = X_{n-1} + u_n + \xi_n$$

for $1 \leqslant n \leqslant N$. Here $(\xi_n)_{1\leqslant n\leqslant N}$ is an IID sequence of random variables with $\mathbb{E}(\xi_1) = 0$ and $\mathrm{Var}(\xi_1) = \sigma^2$, and the controls $(u_n)_{1\leqslant n\leqslant N}$ are previsible with respect to the filtration generated by $(\xi_n)_{1\leqslant n\leqslant N}$.

(a) Write down the Bellman equation for this problem.

(b) Show that the value function can be expressed as

$$V(n,x) = A_n + B_n x + C_n x^2$$

for constants $(A_n, B_n, C_n)_{0\leqslant n\leqslant N}$ to be found.

(c) Show that the optimal control is

$$u_n^* = -\frac{X_0}{N+1} - \frac{\xi_1}{N} - \frac{\xi_2}{N-1} - \cdots - \frac{\xi_{n-1}}{N-n+2}$$

for $1 \leqslant n \leqslant N$.

Paper 2, Section II
30K Stochastic Financial Models

Consider a one-period market model with constant interest rate r and d risky assets. For $n \in \{0, 1\}$ let S_n denote the vector of time-n prices of the risky assets and let X_n be the time-n wealth of an investor. Let $\mu = \mathbb{E}(S_1)$ and $V = \mathrm{Cov}(S_1)$. Assume $\mu \neq (1+r)S_0$.

(a) Suppose V is non-singular. Find, with proof, the minimum of $\mathrm{Var}(X_1)$ subject to the constraints that $X_0 = x$ and $\mathbb{E}(X_1) = m$ for given constants x and m. Show that the optimal portfolio of risky assets is of the form $\theta^* = \lambda V^{-1}[\mu - (1+r)S_0]$ for a constant λ to be found. Now find the minimum of $\mathrm{Var}(X_1)$ subject to $X_0 = x$ and $\mathbb{E}(X_1) \geqslant m$.

(b) Again suppose V is non-singular. Find, with proof, the maximum of the quantity

$$\frac{\mathbb{E}(X_1) - (1+r)X_0}{\sqrt{\mathrm{Var}(X_1)}},$$

subject to $X_0 = x$. Show that all optimal portfolios are mean-variance efficient.

(c) Now suppose V is singular and that there exists *no* vector $\theta \in \mathbb{R}^d$ such that $V\theta = \mu - (1+r)S_0$. Show that for any m and x,

$$\min\{\mathrm{Var}(X_1) : \mathbb{E}(X_1) = m \text{ and } X_0 = x\} = 0\,.$$

Show that there exists an arbitrage in this market.

Paper 3, Section II
29K Stochastic Financial Models

(a) Let W be a Brownian motion and c a constant. Let $M_t = e^{cW_t - c^2 t/2}$ for $t \geqslant 0$. Show that M is a martingale in the filtration generated by W.

For the rest of the question, consider the Black–Scholes model with constant interest rate r and time-t stock price $S_t = S_0 e^{\mu t + \sigma W_t}$ for $0 \leqslant t \leqslant T$, where μ, σ, T are constants with $\sigma > 0$.

(b) Show that there exists a risk-neutral measure for the Black–Scholes model. [You may use the Cameron–Martin theorem without proof if it is clearly stated.]

(c) Find the time-0 Black–Scholes price of a European claim with time-T payout $Y_0 = S_T^p$ where the exponent p is a constant.

(d) Consider two European claims with time-T payouts

$$Y_1 = \max_{0 \leqslant t \leqslant T} S_t \text{ and } Y_2 = \frac{S_0^{2-p} S_T^p}{\min_{0 \leqslant t \leqslant T} S_t}\,.$$

Find, with proof, the exponent p such that these two claims have the same time-0 Black–Scholes price.

Paper 4, Section II
29K Stochastic Financial Models

Consider a discrete-time market with constant interest rate r and a stock with time-n price S_n for $0 \leqslant n \leqslant N$.

(a) Suppose a self-financing investor holds θ_n shares of the stock between times $n-1$ and n for $1 \leqslant n \leqslant N$. Explain why the investor's wealth process $(X_n)_{0 \leqslant n \leqslant N}$ evolves as

$$X_n = (1+r)X_{n-1} + \theta_n[S_n - (1+r)S_{n-1}] \quad \text{for } 1 \leqslant n \leqslant N.$$

For the rest of the question, suppose $S_n = S_{n-1}\xi_n$ where

$$\mathbb{P}(\xi_n = 1+b) = p$$
$$\mathbb{P}(\xi_n = 1+a) = 1-p$$

for all $n \geqslant 1$, for given constants $0 < p < 1$ and $a < r < b$.

(b) Show that

$$\mathbb{Q}\Big(S_N = S_0(1+b)^i(1+a)^{N-i}\Big) = \binom{N}{i} q^i(1-q)^{N-i}$$

for all $0 \leqslant i \leqslant N$, where $\mathbb{Q}$ is the unique risk-neutral measure and q is a constant which you should find.

(c) Now introduce a European contingent claim into this market with time-N payout $g(S_N)$ for a given function g. Find, with proof, the constant x and the previsible process $\theta = (\theta_n)_{1 \leqslant n \leqslant N}$ such that if an investor has time-0 wealth $X_0 = x$ and employs the trading strategy θ then the time-N wealth is $X_N = g(S_N)$ almost surely. Express your answer in terms of the function V defined by

$$V(n,s) = (1+r)^{-(N-n)}\mathbb{E}^{\mathbb{Q}}[g(S_N)|S_n = s] \quad \text{for } 0 \leqslant n \leqslant N, s > 0.$$

(d) Suppose the claim in part (c) is a European call option with strike K. Show that the corresponding initial cost x of the claim is of the form

$$S_0\widehat{\mathbb{Q}}(S_N > K) - K(1+r)^{-N}\mathbb{Q}(S_N > K)\,,$$

where $\widehat{\mathbb{Q}}$ is a probability measure such that

$$\widehat{\mathbb{Q}}\Big(S_N = S_0(1+b)^i(1+a)^{N-i}\Big) = \binom{N}{i} \hat{q}^i(1-\hat{q})^{N-i}$$

for $0 \leqslant i \leqslant N$ and a constant $\hat{q}$ which you should find.

Paper 1, Section I

2G Topics in Analysis

Show that if a, A, B, C, D are non-negative integers and $AD - BC = 1$, then

$$a + \frac{At + B}{Ct + D} = \frac{\alpha t + \beta}{\gamma t + \delta}$$

for some α, β, γ, δ non-negative integers with $\alpha\delta - \beta\gamma = 1$.

If $N, a_1, a_2, \ldots$ are strictly positive integers with $a_{N+k} = a_k$ for all k and

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ldots}}}}$$

show that x is a root of a quadratic (or linear) equation with integer coefficients.

Give the quadratic equation explicitly in the case when $N = 2$, $a_1 = a$, $a_2 = b$. Explain how you know which root gives the continued fraction.

Paper 2, Section I
2G Topics in Analysis

In this question you should work in $\mathbb{R}^n$ with the usual Euclidean distance.

Define a set of *first Baire category.*

For each of the following statements, say whether it is true or false and give an appropriate proof or counterexample.

(i) The countable union of sets of first category is of first category.

(ii) If A is of first category in $\mathbb{R}^2$ and $y \in \mathbb{R}$, then

$$C_y = \{x \,:\, (x, y) \in A\}$$

is of first category in $\mathbb{R}$.

(iii) If C is of first category in $\mathbb{R}$, then

$$A = \{(x, y) \,:\, x \in C,\, y \in \mathbb{R}\}$$

is of first category in $\mathbb{R}^2$.

(iv) If A and B are sets of first category in $\mathbb{R}^2$, then

$$A + B = \{\mathbf{a} + \mathbf{b} \,:\, \mathbf{a} \in A,\, \mathbf{b} \in B\}$$

is of first category.

[*You may use results about complete metric spaces provided you state them precisely.*]

Paper 3, Section I
2G Topics in Analysis

Let Ω be a non-empty bounded open subset of $\mathbb{R}^2$ with closure $\mathrm{Cl}\,\Omega$ and boundary $\partial\Omega$. We take $\phi : \mathrm{Cl}\,\Omega \to \mathbb{R}$ to be a continuous function which is twice differentiable on Ω.

If $\nabla^2\phi > 0$ on Ω show that ϕ attains a maximum on $\partial\Omega$.

By giving proofs or counterexamples establish which of the following are true and which are false.

(i) If $\nabla^2\phi = 0$ on Ω, then ϕ attains a maximum on $\partial\Omega$.

(ii) If $\nabla^2\phi = 0$ on Ω, then ϕ attains a minimum on $\partial\Omega$.

(iii) If $\nabla^2\phi = f$ on Ω for some continuous function $f : \mathrm{Cl}\,\Omega \to \mathbb{R}$, then ϕ attains a maximum on $\partial\Omega$.

Paper 4, Section I
2G Topics in Analysis

Consider the continuous map $f : [0,1] \to \mathbb{C}$ given by $f(t) = t - 1/2$. Show that there does not exist a continuous function $\phi : [0,1] \to \mathbb{R}$ with $f(t) = |f(t)| \exp(i\phi(t))$.

Show that, if $g : [0,1] \to \mathbb{C} \setminus \{0\}$ is continuous, there exists a continuous function $\theta : [0,1] \to \mathbb{R}$ with $g(t) = |g(t)| \exp(i\theta(t))$. [You may assume that this result holds in the special case when $\Re g(t) > 0$ for all $t \in [0,1]$.]

Show that $r(g) = \theta(1) - \theta(0)$ is uniquely defined.

If $u(t) = g(t^2)$ and $v(t) = g(t)^2$, find $r(u)$ and $r(v)$ in terms of $r(g)$.

Give an example with $g_1, g_2 : [0,1] \to \mathbb{C} \setminus \{0\}$ continuous such that $g_1(0) = g_2(0)$ and $g_1(1) = g_2(1)$, but $r(g_1) \neq r(g_2)$.

Paper 2, Section II
11G Topics in Analysis

Suppose $f : [0,1]^2 \to \mathbb{R}$ is continuous. Show, quoting carefully any theorems that you use, that

$$\sum_{j=0}^{n} \sum_{k=0}^{n} \binom{n}{j} \binom{n}{k} f(j/n, k/n) t^j (1-t)^{n-j} s^k (1-s)^{n-k} \to f(t,s)$$

uniformly on $[0,1]^2$ as $n \to \infty$.

Deduce that

$$\int_0^1 \left(\int_0^1 f(s,t)\, ds \right) dt = \int_0^1 \left(\int_0^1 f(s,t)\, dt \right) ds$$

whenever $f : [0,1]^2 \to \mathbb{R}$ is continuous.

By giving proofs or counterexamples establish which of the following statements are true and which are false. You may not use the Stone–Weierstrass theorem without proof.

(i) If $f : [0,1]^2 \to \mathbb{R}$ is continuous and $\int_0^1 \left(\int_0^1 s^n t^m f(s,t)\, ds \right) dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

(ii) Suppose $a < b$. If $f : [a,b]^2 \to \mathbb{R}$ is continuous and $\int_a^b \left(\int_a^b s^n t^m f(s,t)\, ds \right) dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

(iii) If $f : [-1,1]^2 \to \mathbb{R}$ is continuous and $\int_{-1}^1 \left(\int_{-1}^1 s^{2n} t^{2m} f(s,t)\, ds \right) dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

(iv) If $f : [0,1]^2 \to \mathbb{R}$ is continuous and $\int_0^1 \left(\int_0^1 s^{2n} t^{2m} f(s,t)\, ds \right) dt = 0$ for all integers $n, m \geqslant 0$, then $f = 0$.

Paper 4, Section II
12G Topics in Analysis

(a) State Brouwer's fixed point theorem for the closed unit disc D. For which of the following $E \subset \mathbb{R}^2$ is it the case that every continuous function $f : E \to E$ has a fixed point? Give a proof or a counterexample.

(i) E is the union of two disjoint closed discs.

(ii) $E = \{(x,0) : 0 < x < 1\}$.

(iii) $E = \{(x,0) : 0 \leqslant x \leqslant 1\}$.

(iv) $E = \{\mathbf{x} \in \mathbb{R}^2 : 1 \leqslant |\mathbf{x}| \leqslant 2\}$.

(b) Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function with the property that $|f(\mathbf{x})| \leqslant 1$ whenever $|\mathbf{x}| = 1$, then f has a fixed point.

[*Hint: Consider $T \circ f$ where for $\mathbf{x} \in \mathbb{R}^2$, $T\mathbf{x}$ is the element of D closest to $\mathbf{x}$.*]

(c) Let

$$E = \{(p_1, p_2, q_1, q_2) : 0 \leqslant p_i, q_i \leqslant 1 \text{ and } p_1 + p_2 = 1, q_1 + q_2 = 1\}$$

and suppose A, $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ are given by

$$A(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij} p_i q_j \text{ and } B(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{2}\sum_{j=1}^{2} b_{ij} p_i q_j$$

with a_{ij} and b_{ij} constant. Let

$$u_1(\mathbf{p},\mathbf{q}) = \max\{0, A((1,0),\mathbf{q}) - A(\mathbf{p},\mathbf{q})\}\,,\ u_2(\mathbf{p},\mathbf{q}) = \max\{0, A((0,1),\mathbf{q}) - A(\mathbf{p},\mathbf{q})\}\,.$$

By considering $(\mathbf{p}', \mathbf{q}')$ with

$$\mathbf{p}' = \frac{\mathbf{p} + \mathbf{u}(\mathbf{p},\mathbf{q})}{1 + u_1(\mathbf{p},\mathbf{q}) + u_2(\mathbf{p},\mathbf{q})}$$

and $\mathbf{q}'$ defined appropriately, show that we can find a $(\mathbf{p}^*, \mathbf{q}^*) \in E$ with

$$\forall (\mathbf{p},\mathbf{q}) \in E, \quad A\big(\mathbf{p}^*, \mathbf{q}^*\big) \geqslant A\big(\mathbf{p}, \mathbf{q}^*\big) \text{ and } B\big(\mathbf{p}^*, \mathbf{q}^*\big) \geqslant B\big(\mathbf{p}^*, \mathbf{q}\big).$$

Carefully explain the result in terms of a two-person game.

Paper 1, Section II
40C Waves

(a) Starting from the equations for mass and momentum conservation and a suitable equation of state, derive the linearised wave equation for perturbation pressure $\tilde{p}(\mathbf{x}, t)$ for 3-dimensional sound waves in a compressible gas with sound speed c_0 and density ρ_0.

(b) For a 1-dimensional wave of given frequency ω propagating in the x-direction, the perturbation pressure $\tilde{p}(x,t)$ may be written in the form $\Re(\hat{p}(x)e^{i\omega t})$. What is the form of $\hat{p}$ for a harmonic plane wave of frequency ω propagating in the positive x-direction? Express the perturbation fluid speed $\tilde{u}(x,t)$ in terms of $\tilde{p}(x,t)$.

(c) The gas occupies the region $x < L$, with a rigid boundary at $x = L$. A thin flexible membrane of mass m per unit area is located within the gas at equilibrium position $x = 0$. A plane wave of unit amplitude of the form specified in part (b) is incident from $x = -\infty$. The combined effects of the membrane and the rigid boundary result in a reflected wave of complex amplitude R, where R is the ratio between the individual complex amplitudes at $x = 0^-$ of the reflected and incident waves.

(i) Show that

$$R = \frac{\cos\beta + (\alpha - i)\sin\beta}{\cos\beta + (\alpha + i)\sin\beta} \quad \text{where } \alpha = \frac{\omega m}{\rho_0 c_0} \quad \text{and} \quad \beta = \frac{\omega L}{c_0}\,.$$

Deduce that $|R| = 1$ in general and briefly discuss this result physically.

(ii) Identify a condition on β so that the membrane is stationary and there is non-trivial pressure perturbation in $0 < x < L$. Briefly discuss this result physically.

(iii) Identify and interpret a limit for α in which the pressure perturbation in $0 < x < L$ becomes very small relative to that in $x < 0$.

Paper 2, Section II
40C Waves

Infinitesimal displacements $\boldsymbol{u}(\boldsymbol{x},t)$ in a uniform, linear isotropic elastic solid with density ρ_0 and Lamé moduli λ and μ satisfy the linearised Cauchy momentum equation:

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda+\mu)\,\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \mu\nabla^2\boldsymbol{u}.$$

(a) Show that the dilatation $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ and the rotation $\boldsymbol{\nabla}\times\boldsymbol{u}$ satisfy wave equations, and find the wave-speeds c_P and c_S.

(b) A plane harmonic P-wave with wavevector $\boldsymbol{k}$ lying in the (x,z) plane is incident from $z<0$ at an oblique angle on the planar horizontal interface $z=0$ between two elastic solids with different densities and elastic moduli. Show in a diagram the directions of all the reflected and transmitted waves, labelled with their polarisations, assuming that none of these waves is evanescent. State the boundary conditions on components of $\boldsymbol{u}$ and the stress tensor $\boldsymbol{\sigma}$ and explain why these are sufficient to determine the amplitudes. (You do not need to calculate the directions or amplitudes explicitly.)

(c) Now consider a plane harmonic P-wave of unit amplitude, with $\boldsymbol{k} = k(\sin\theta, 0, \cos\theta)$, incident from $z<0$ on the interface $z=0$ between two elastic (and inviscid) liquids with modulus λ, density ρ and wave-speed c_P in $z<0$ and modulus λ', density ρ' and wave-speed c'_P in $z>0$, with $\rho' \neq \rho$.

(i) Under what conditions is there a propagating transmitted wave in $z>0$?

(ii) Assume from here on that these conditions are met. Obtain solutions for the reflected and transmitted waves.

(iii) Show that the amplitude of the reflected wave is

$$R = \frac{\lambda'\sin 2\theta - \lambda\sin 2\theta'}{\lambda'\sin 2\theta + \lambda\sin 2\theta'},$$

where θ' is the angle the wave vector of the transmitted wave makes with the vertical.

(iv) Hence obtain an expression for θ in terms of the wave-speeds and densities of the two liquids that implies no reflection (i.e. $R=0$).

Paper 3, Section II
39C Waves

Waves propagating in a slowly-varying medium satisfy the local dispersion relation $\omega = \Omega(\boldsymbol{k}; \boldsymbol{x}, t)$ in the standard notation.

(a) Derive the ray-tracing equations:

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i}, \quad \frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i}, \quad \frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t},$$

governing the evolution of a wave packet specified by

$$\phi(\boldsymbol{x}, t) = A(\boldsymbol{x}, t; \varepsilon) \exp\left(\frac{i\theta(\boldsymbol{x}, t)}{\varepsilon}\right),$$

where $0 < \varepsilon \ll 1$. A rigorous derivation is not required, but assumptions should be clearly stated and the meaning of the d/dt notation should be carefully explained.

(b) The dispersion relation for two-dimensional, small amplitude, internal gravity waves of wavenumber vector $\boldsymbol{k} = (k, 0, m)$, relative to Cartesian coordinates (x, y, z) with z vertical, propagating in an inviscid, incompressible, stratified fluid with a slowly-varying mean flow $\boldsymbol{U}$ is

$$\omega = \frac{Nk}{\sqrt{k^2 + m^2}} + \boldsymbol{k} \cdot \boldsymbol{U},$$

where N is the buoyancy frequency. Consider the specific flow $\boldsymbol{U} = \gamma(x, 0, -z)$. N and γ are positive constants.

(i) Calculate $k(t)$ and $m(t)$, applying the initial conditions $k(0) = k_0 > 0$, $m(0) = m_0$.

(ii) Consider a wave packet with initial wave vector $(k_0, 0, m_0)$, released from $(x_0, 0, z_0)$ where $x_0 > 0$ and $z_0 > 0$. Show that the wave packet can initially propagate upwards provided $z_0 < z_m$, where z_m is a function of k_0 and m_0.

(iii) Demonstrate that such a wave packet eventually approaches $z = 0$, but takes an infinite amount of time to do so. [*Hint: It is not essential to solve for an explicit expression for the position of the wave packet at arbitrary time t.*]

Paper 4, Section II
39C Waves

Consider finite amplitude, one-dimensional sound waves in a perfect gas with ratio of specific heats γ.

(a) Show that the fluid speed u and local sound speed c satisfy

$$\left(\frac{\partial}{\partial t} + (u \pm c)\frac{\partial}{\partial x}\right) R_{\pm} = 0\,,$$

where the *Riemann invariants* $R_{\pm}(x,t)$ should be defined carefully. Write down parametric equations for the paths on which these quantities are actually invariant.

(b) At time $t = 0$ the gas occupies the region $x > 0$. It is at rest and has uniform density ρ_0, pressure p_0 and sound speed c_0. A piston initially at $x = 0$ starts moving backwards at time $t = 0$ with displacement $x = -\varepsilon t(1-t)$, where $\varepsilon > 0$ is constant.

(i) Show that prior to any shock forming $c = c_0 + \frac{1}{2}(\gamma - 1)u$.

(ii) For small ε, derive an expression for the relative pressure fluctuation $\delta p/p_0 = p/p_0 - 1$ to second order in the relative sound speed fluctuation $\delta c/c_0 = c/c_0 - 1$.

(iii) Calculate the time average over the interval $0 \leqslant t \leqslant 1$ of the relative pressure fluctuation, measured on the piston, and briefly discuss your result physically.

END OF PAPER