UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2022

List of Courses

Paper 2, Section I
2G Analysis and Topology

Let $f : (M, d) \to (N, e)$ be a homeomorphism between metric spaces. Show that $d'(x, y) = e(f(x), f(y))$ defines a metric on M that is equivalent to d. Construct a metric on $\mathbb{R}$ which is equivalent to the standard metric but in which $\mathbb{R}$ is not complete.

Paper 4, Section I
2G Analysis and Topology

Define the *closure* of a subspace Z of a topological space X, and what it means for Z to be *dense*. What does it mean for a topological space Y to be *Hausdorff*?

Assume that Y is Hausdorff, and that Z is a dense subspace of X. Show that if two continuous maps $f, g : X \to Y$ agree on Z, they must agree on the whole of X. Does this remain true if you drop the assumption that Y is Hausdorff?

Paper 1, Section II
10G Analysis and Topology

Let X and Y be metric spaces. Determine which of the following statements are always true and which may be false, giving a proof or a counterexample as appropriate.

(a) Let f_n and f be real-valued functions on X and let A, B be two subsets of X such that $X = A \cup B$. If f_n converges uniformly to f on both A and B, then f_n converges uniformly to f on X.

(b) If the sequences of real-valued functions f_n and g_n converge uniformly on X to f and g respectively, then $f_n g_n$ converges uniformly to fg on X.

(c) Let X be the rectangle $[1, 2] \times [1, 2] \subset \mathbb{R}^2$ and let $f_n : X \to \mathbb{R}$ be given by

$$f_n(x, y) = \frac{1 + nx}{1 + ny}.$$

Then f_n converges uniformly on X.

(d) Let A be a subset of X and x_0 a point such that any neighbourhood of x_0 contains a point of A different from x_0. Suppose the functions $f_n : A \to Y$ converge uniformly on A and, for each n, $\lim_{x \to x_0} f_n(x) = y_n$. If Y is complete, then the sequence y_n converges.

(e) Let f_n converge uniformly on X to a bounded function f and let $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then the composition $g \circ f_n$ converges uniformly to $g \circ f$ on X.

Paper 2, Section II
10G Analysis and Topology

State the inverse function theorem for a function $F : \mathbb{R}^n \to \mathbb{R}^n$. Suppose F is a differentiable bijection with F^{-1} also differentiable. Show that the derivative of F at any point in $\mathbb{R}^n$ is a linear isomorphism.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable map such that its derivative is invertible at any point in $\mathbb{R}^n$. Is $F(\mathbb{R}^n)$ open? Is $F(\mathbb{R}^n)$ closed? Justify your answers.

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$F(x, y, z) = (x + y + z, zy + zx + xy, xyz).$$

Determine the set C of points $p \in \mathbb{R}^3$ for which F fails to admit a differentiable local inverse around p. Is the set $\mathbb{R}^3 \setminus C$ connected? Justify your answer.

Paper 3, Section II
11G Analysis and Topology

Define a *contraction mapping* between two metric spaces. State and prove the contraction mapping theorem. Use this to show that the equation $x = \cos x$ has a unique real solution.

State the mean value inequality. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by

$$f(x, y) = \left(\frac{\cos x + \cos y - 1}{2}, \cos x - \cos y \right).$$

Prove that f has a fixed point. [*Hint: Find a suitable subset of* $\mathbb{R}^2$ *on which* f *is a contraction mapping.*]

Paper 4, Section II
10G Analysis and Topology

Define what it means for a topological space to be *connected*. Describe without proof the connected subspaces of $\mathbb{R}$ with the standard topology. Define what it means for a topological space to be *path connected*, and show that path connectedness implies connectedness.

Given metric spaces A and B, let $C(A, B)$ be the space of continuous bounded functions from A to B with the topology induced by the uniform metric.

(a) For $n \in \mathbb{N}$, let $I_n \subset \mathbb{R}$ be

$$I_n = [1, 2] \cup [3, 4] \cup \ldots \cup [2n - 1, 2n]$$

with the subspace topology. For fixed $m, n \in \mathbb{N}$, how many connected components does $C(I_n, I_m)$ have?

(b) (i) Give an example of a closed bounded subspace of $\mathbb{R}^2$ which is connected but not path connected, justifying your answer. Call your example S.

(ii) Show that $C([0, 1], S)$ is not path connected.

(iii) Is $C([0, 1], S)$ connected? Briefly justify your answer.

Paper 4, Section I
3G Complex Analysis

Show that there is no bijective holomorphic map $f : D(0,1) \setminus \{0\} \to A$, where $D(0,1)$ is the disc $\{z \in \mathbb{C} : |z| < 1\}$ and A is the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$.

[*Hint: Consider an extension of f to the whole disc.*]

Paper 3, Section II
13G Complex Analysis

Let $U \subset \mathbb{C}$ be a (non-empty) connected open set and let f_n be a sequence of holomorphic functions defined on U. Suppose that f_n converges uniformly to a function f on every compact subset of U. Show that f is holomorphic in U. Furthermore, show that f_n' converges uniformly to f' on every compact subset of U.

Suppose in addition that f is not identically zero and that for each n, there is a unique $c_n \in U$ such that $f_n(c_n) = 0$. Show that there is at most one $c \in U$ such that $f(c) = 0$. Find an example such that f has no zeros in U. Give a necessary and sufficient condition on the c_n for this to happen in general.

Paper 1, Section I
3G Complex Analysis or Complex Methods

Show that $f(z) = \frac{z}{\sin z}$ has a removable singularity at $z = 0$. Find the radius of convergence of the power series of f at the origin.

Paper 1, Section II
12G Complex Analysis or Complex Methods

(a) Let $\Omega \subset \mathbb{C}$ be an open set such that there is $z_0 \in \Omega$ with the property that for any $z \in \Omega$, the line segment $[z_0, z]$ connecting z_0 to z is completely contained in Ω. Let $f : \Omega \to \mathbb{C}$ be a continuous function such that

$$\int_\Gamma f(z)\, dz = 0$$

for any closed curve Γ which is the boundary of a triangle contained in Ω. Given $w \in \Omega$, let

$$g(w) = \int_{[z_0,w]} f(z)\, dz.$$

Explain briefly why g is a holomorphic function such that $g'(w) = f(w)$ for all $w \in \Omega$.

(b) Fix $z_0 \in \mathbb{C}$ with $z_0 \neq 0$ and let $\mathcal{D} \subset \mathbb{C}$ be the set of points $z \in \mathbb{C}$ such that the line segment connecting z to z_0 does not pass through the origin. Show that there exists a holomorphic function $h : \mathcal{D} \to \mathbb{C}$ such that $h(z)^2 = z$ for all $z \in \mathcal{D}$. [You may assume that the integral of $1/z$ over the boundary of any triangle contained in $\mathcal{D}$ is zero.]

(c) Show that there exists a holomorphic function f defined in a neighbourhood U of the origin such that $f(z)^2 = \cos z$ for all $z \in U$. Is it possible to find a holomorphic function f defined on the disk $|z| < 2$ such that $f(z)^2 = \cos z$ for all z in the disk? Justify your answer.

Paper 2, Section II
12A Complex Analysis or Complex Methods

(a) Let $R = P/Q$ be a rational function, where deg $Q \geqslant$ deg $P + 2$, and Q has no real zeros. Using the calculus of residues, write a general expression for

$$\int_{-\infty}^{\infty} R(x)e^{ix}\, dx$$

in terms of residues. Briefly justify your answer.

[You may assume that the polynomials P and Q do not have any common factors.]

(b) Explicitly evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1 + x^4}\, dx\,.$$

Paper 3, Section I
3A Complex Methods

The function $f(x)$ has Fourier transform

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = \frac{-2ki}{p^2 + k^2},$$

where $p > 0$ is a real constant. Using contour integration, calculate $f(x)$ for $x > 0$.

[Jordan's lemma and the residue theorem may be used without proof.]

Paper 4, Section II
12A Complex Methods

The Laplace transform $F(s)$ of a function $f(t)$ is defined as

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

(a) For $f(t) = t^n$ for n a non-negative integer, show that

$$\begin{aligned} L\{f(t)\} &= F(s) = \frac{n!}{s^{n+1}}, \\ L\{e^{at} f(t)\} &= F(s-a) = \frac{n!}{(s-a)^{n+1}}. \end{aligned}$$

(b) Use contour integration to find the inverse Laplace transform of

$$F(s) = \frac{1}{s^2 (s+2)^2}.$$

(c) Verify the result in part (b) by using the results in part (a) and the convolution theorem.

(d) Use Laplace transforms to solve the differential equation

$$\frac{d^4}{dt^4}[f(t)] + 4\frac{d^3}{dt^3}[f(t)] + 4\frac{d^2}{dt^2}[f(t)] = 0,$$

subject to the initial conditions

$$f(0) = \frac{d}{dt} f(0) = \frac{d^2}{dt^2} f(0) = 0 \text{ and } \frac{d^3}{dt^3} f(0) = 1.$$

Paper 2, Section I
4D Electromagnetism

A uniformly charged sphere of radius R has total charge Q. Find the electric field inside and outside the sphere.

A second uniformly charged sphere of radius R has total charge $-Q$. The centre of the second sphere is displaced from the centre of the first by the vector $\mathbf{d}$, where $|\mathbf{d}| < R$. Show that the electric field in the overlap region is constant and find its value.

Paper 4, Section I
5D Electromagnetism

(a) Use the Maxwell equations to show that, in the absence of electric charges and currents, the magnetic field obeys

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \nabla^2 \mathbf{B}$$

for some appropriate speed c that you should express in terms of ϵ_0 and μ_0.

(b) Show that

$$\mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \cos(kz - \omega t)$$

satisfies the Maxwell equations given appropriate conditions on the constants B_1, B_2, B_3, ω and k that you should find. What is the corresponding electric field $\mathbf{E}$?

(c) Compute and interpret the Poynting vector $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$.

Paper 1, Section II
15D Electromagnetism

(a) Use Gauss' law to compute the electric field $\mathbf{E}$ and electric potential ϕ due to an infinitely long, straight wire with charge per unit length $\lambda > 0$.

(b) Two infinitely long wires, both lying parallel to the z-axis, intersect the $z = 0$ plane at $(x, y) = (\pm a, 0)$. They carry charge per unit length $\pm\lambda$ respectively. Show that the equipotentials on the $z = 0$ plane form circles and determine the centres and radii of these circles as functions of a and

$$k = \frac{2\pi\epsilon_0\phi}{\lambda} \ ,$$

where ϵ_0 is the permittivity of free space.

Sketch the equipotentials and the electric field. What happens in the case $\phi = 0$?

Find the electric field in the limit $a \to 0$ with $\lambda a = p$ fixed.

Paper 2, Section II
16D Electromagnetism

(a) Starting from an appropriate Maxwell equation, derive Faraday's law of induction relating electromotive force to the change of flux for a static circuit.

(b) An infinite wire lies along the z-axis and carries current $I > 0$ in the positive z-direction.

(i) Use Ampère's law to calculate the magnetic field $\mathbf{B}$.

(ii) In addition to the infinite wire described above, a square loop of wire, with sides of length $2a$ and total resistance R, is restricted to lie in the $x = 0$ plane. The centre of the square initially sits at point $y = d > a$. The square loop is pulled away from the wire in the direction of increasing y at speed v. Calculate the current that flows in the loop and draw a diagram indicating the direction of the current.

(iii) The square loop is instead pulled in the z-direction, parallel to the infinite wire, at a speed u. Calculate the current in the loop.

Paper 3, Section II
15D Electromagnetism

(a) A Lorentz transformation is given by

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. How does a 4-vector $X^{\mu} = (ct, x, y, z)$ transform?

(b) The electromagnetic field is an anti-symmetric tensor with components

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1/c & -E_2/c & -E_3/c \\ E_1/c & 0 & B_3 & -B_2 \\ E_2/c & -B_3 & 0 & B_1 \\ E_3/c & B_2 & -B_1 & 0 \end{pmatrix}.$$

Determine how the components of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ transform under the Lorentz transformation given in part (a).

(c) An infinite, straight wire has uniform charge per unit length λ and carries no current. Determine the electric field and magnetic field. By applying a Lorentz boost, find the fields seen by an observer who travels with speed v in the direction parallel to the wire. Interpret your results using the appropriate Maxwell equation.

Paper 2, Section I
5C Fluid Dynamics

An unsteady fluid flow has velocity field given in cartesian coordinates (x, y, z) by $\mathbf{u} = (2t, xt, 0)$, where $t > 0$ denotes time. Dye is continuously released into the fluid from the origin.

(a) Determine if this fluid flow is incompressible.

(b) Find the distance from the origin at time t of the dye particle that was released at time s, where $s < t$.

(c) Determine the equation of the curve formed by the dye streak in the (x, y)-plane.

Paper 3, Section I
7C Fluid Dynamics

A two-dimensional flow has velocity given by

$$\mathbf{u}(\mathbf{x}) = 2\,\frac{\mathbf{x}(\mathbf{d}\cdot\mathbf{x})}{r^4} - \frac{\mathbf{d}}{r^2}$$

as a function of the position vector $\mathbf{x}$, with $r = |\mathbf{x}|$, where $\mathbf{d}$ is a fixed vector.

(a) Show that this flow is incompressible for $r \neq 0$.

(b) Compute the stream function ψ for this flow in polar coordinates (r, θ) with $\theta = 0$ aligned with the vector $\mathbf{d}$.

[*Hint: in polar coordinates*

$$\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$$

for a vector $\mathbf{F} = (F_r, F_\theta)$.]

Paper 1, Section II
16C Fluid Dynamics

Consider a steady viscous flow (with viscosity μ) of constant density ρ through a long pipe of circular cross-section with radius R. The flow is driven by a constant pressure gradient $\partial p/\partial z$ along the pipe (z is the coordinate along the pipe).

The Navier-Stokes equation describing this flow is

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right]=-\nabla p+\mu\nabla^2\mathbf{u}.$$

(a) Using cylindrical coordinates (r,θ,z) aligned with the pipe, determine the velocity $\mathbf{u}=(0,0,w(r))$ of the flow.

[*Hint: in cylindrical coordinates*

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\,.]$$

(b) The viscous stress exerted on the flow by the pipe boundaries is equal to

$$\mu\left(\frac{\partial w}{\partial r}\right)\bigg|_{r=R}.$$

Demonstrate the overall force balance for the (cylindrical) volume of the fluid enclosed within the section of the pipe $z_0 \leqslant z \leqslant z_0+L$.

(c) Compute the mass flux through the pipe.

Paper 3, Section II
16C Fluid Dynamics

Consider an axisymmetric, two-dimensional, incompressible flow $\mathbf{u}(r) = (u_r, u_\theta)$ in polar coordinates (r, θ).

(a) Determine the behaviour of u_r if it is finite everywhere in space.

(b) Representing $u_\theta = \Omega(r)r$, express the vorticity of the flow $\boldsymbol{\omega}$ in terms of Ω.

(c) Starting from the Navier-Stokes equation

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla p + \mu\nabla^2\mathbf{u}$$

derive the vorticity evolution equation

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}$$

for a general incompressible flow with kinematic viscosity $\nu = \mu/\rho$.

(d) Deduce the form of the evolution equation for the scalar vorticity $\omega = |\boldsymbol{\omega}|$ for the axisymmetric two-dimensional flow of part (a).

(e) Show that the equation derived in part (d) adopts a self-similar form $\omega(r,t) = \omega(\xi)$, where $\xi = r/\sqrt{\nu t}$ is the similarity variable.

[*You may use the fact that, in polar coordinates,*

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

and

$$\boldsymbol{\nabla}\times\mathbf{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right]\mathbf{e}_z$$

for a vector $\mathbf{F} = (F_r, F_\theta)$, *where* $\mathbf{e}_z$ *is a unit vector normal to the flow plane.*]

Paper 4, Section II
16C Fluid Dynamics

A fluid of density ρ_1 occupies the region $z > 0$ and a second fluid of density ρ_2 occupies the region $z < 0$. The system is perturbed so that the subsequent motion is irrotational and the interface is at $z = \zeta(x,t)$. State the equations and nonlinear boundary conditions that are satisfied by the corresponding velocity potentials ϕ_1 and ϕ_2 and pressures p_1 and p_2.

Obtain a set of linearised equations and boundary conditions when the perturbations are small and proportional to $e^{i(kx-\omega t)}$. Hence derive the dispersion relation

$$\omega^2 = gk\,F\left(\frac{\rho_1}{\rho_2}\right),$$

where g is the gravitational acceleration and F is a function to be determined.

Paper 1, Section I

2E Geometry

Give a characterisation of the geodesics on a smooth embedded surface in $\mathbb{R}^3$.

Write down all the geodesics on the cylinder $x^2 + y^2 = 1$ passing through the point $(x, y, z) = (1, 0, 0)$. Verify that these satisfy your characterisation of a geodesic. Which of these geodesics are closed?

Can $\mathbb{R}^2 \setminus \{(0,0)\}$ be equipped with an abstract Riemannian metric such that every point lies on a unique closed geodesic? Briefly justify your answer.

Paper 3, Section I

2F Geometry

Consider the space $S_{a,b} \subset \mathbb{R}^3$ defined by

$$x^2 + y^2 + z^3 + az + b = 0$$

for unknown real constants a, b with $(a, b) \neq (0, 0)$.

(a) Stating any result you use, show that $S_{a,b}$ is a smooth surface in $\mathbb{R}^3$ whenever $4a^3 + 27b^2 \neq 0$.

(b) What about the cases where $4a^3 + 27b^2 = 0$? Briefly justify your answer.

Paper 1, Section II

11E Geometry

(a) Let $\mathbb{H}$ be the upper half plane model of the hyperbolic plane. Let G be the group of orientation preserving isometries of $\mathbb{H}$. Write down the general form of an element of G. Show that G acts transitively on (i) the points in $\mathbb{H}$, (ii) the boundary $\mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$, and (iii) the set of hyperbolic lines in $\mathbb{H}$.

(b) Show that if $P \in \mathbb{H}$ then $\{g \in G \,|\, g(P) = P\}$ is isomorphic to $\mathrm{SO}(2)$.

(c) Show that for any two distinct points $P, Q \in \mathbb{H}$ there exists a unique $g \in G$ with $g(P) = Q$ and $g(Q) = P$.

(d) Show that if ℓ, m are hyperbolic lines meeting at $P \in \mathbb{H}$ with angle θ then the points of intersection of ℓ, m with the boundary of $\mathbb{H}$, when taken in a suitable order, have cross ratio $\cos^2(\theta/2)$.

Paper 2, Section II
11F Geometry

Consider the surface $S \subset \mathbb{R}^3$ given by

$$(\sinh u \cos v, \sinh u \sin v, v) \qquad \text{for } u, v > 0.$$

Sketch S. Calculate its first fundamental form.

(a) Find a surface of revolution S' such that there is a local isometry between S and S'. Do they have the same Gauss curvature?

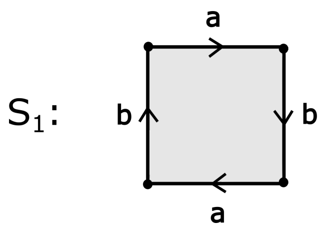
(b) Given an oriented surface $R \subset \mathbb{R}^3$, define the *Gauss map* of R. Describe the image of the Gauss map for S' equipped with the orientation associated to the outward-pointing normal. Use this to calculate the total Gaussian curvature of S'.

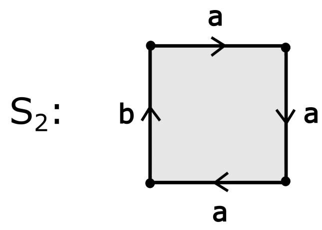
(c) By considering the total Gaussian curvature of S, or otherwise, show that there does not exist a global isometry between S and S'.

You should carefully state any result(s) you use.

Paper 3, Section II
12F Geometry

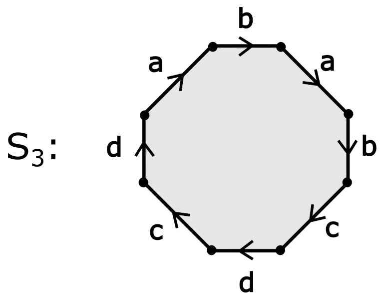
(a) Define a *topological surface*. Consider the topological spaces S_1 and S_2 given by identifying the sides of a square as drawn. Show that S_1 is a topological surface. [*Hint: It may help to find a finite group G acting on the 2-sphere S^2 such that S^2/G is homeomorphic to S_1.*]

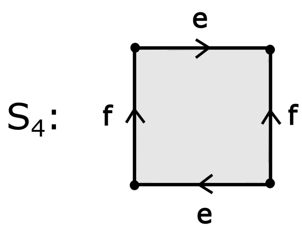




Is S_2 a topological surface? Briefly justify your answer.

(b) By cutting each along a suitable diagonal, show that the two topological surfaces S_3 and S_4 defined by gluing edges of polygons as shown are homeomorphic.





If you delete an open disc from S_4, can the resulting surface be embedded in $\mathbb{R}^3$? Briefly justify your answer. Can S_4 itself be embedded in $\mathbb{R}^3$? State any result you use.

Paper 4, Section II
11E Geometry

(a) Write down the metric on the unit disc model $\mathbb{D}$ of the hyperbolic plane. Let C be the Euclidean circle centred at the origin with Euclidean radius r. Show that C is a hyperbolic circle and compute its hyperbolic radius.

(b) Let Δ be a hyperbolic triangle with angles α, β, γ, and side lengths (opposite the corresponding angles) a, b, c. State the hyperbolic sine formula. The hyperbolic cosine formula is $\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos \alpha$. Show that if $\gamma = \pi/2$ then

$$\tan \alpha = \frac{\sinh a}{\cosh a \sinh b} \qquad \text{and} \qquad \tan \alpha \tan \beta \cosh c = 1.$$

(c) Write down the Gauss–Bonnet formula for a hyperbolic triangle. Show that the hyperbolic polygon in $\mathbb{D}$ with vertices at $re^{2\pi ik/n}$ for $k = 0, 1, 2, \ldots, n-1$ has hyperbolic area

$$A_n(r) = 2n \left[\cot^{-1} \left(\frac{1 - r^2}{1 + r^2} \cot \left(\frac{\pi}{n} \right) \right) - \frac{\pi}{n} \right].$$

(d) Show that there exists a hyperbolic hexagon with all interior angles a right angle. Draw pictures illustrating how such hexagons may be used to construct a closed hyperbolic surface of any genus at least 2.

Paper 2, Section I

1E Groups, Rings and Modules

(a) Let R be an integral domain and M an R-module. Let $T \subset M$ be the subset of torsion elements, i.e., elements $m \in M$ such that $rm = 0$ for some $0 \neq r \in R$. Show that T is an R-submodule of M.

(b) Let $\phi : M_1 \to M_2$ be a homomorphism of R-modules. Let $T_1 \leqslant M_1$ and $T_2 \leqslant M_2$ be the torsion submodules. Show that there is a homomorphism of R-modules $\Phi : M_1/T_1 \to M_2/T_2$ satisfying $\Phi(m + T_1) = \phi(m) + T_2$ for all $m \in M_1$.

Does ϕ injective imply Φ injective?

Does Φ injective imply ϕ injective?

Paper 3, Section I

1E Groups, Rings and Modules

State the first isomorphism theorem for rings.

Let R be a subring of a ring S, and let J be an ideal in S. Show that $R + J$ is a subring of S and that

$$\frac{R}{R \cap J} \cong \frac{R+J}{J}.$$

Compute the characteristics of the following rings, and determine which are fields.

$$\frac{\mathbb{Q}[X]}{(X+2)} \qquad \frac{\mathbb{Z}[X]}{(3, X^2+X+1)}$$

Paper 1, Section II

9E Groups, Rings and Modules

Define a *Euclidean domain*. Briefly explain how $\mathbb{Z}[i]$ satisfies this definition.

Find all the units in $\mathbb{Z}[i]$. Working in this ring, write each of the elements 2, 5 and $1 + 3i$ in the form $u\, p_1^{\alpha_1} \dots p_t^{\alpha_t}$ where u is a unit, and $p_1, \dots, p_t$ are pairwise non-associate irreducibles.

Find all pairs of integers x and y satisfying $x^2 + 4 = y^3$.

Paper 2, Section II

9E Groups, Rings and Modules

Define a Sylow subgroup and state the Sylow theorems. Prove the third theorem, concerning the number of Sylow subgroups.

Quoting any general facts you need about alternating groups, show that A_n has no subgroup of index m if $1 < m < n$ and $n \geqslant 5$. Hence, or otherwise, show that there is no simple group of order 90.

Paper 3, Section II
10E Groups, Rings and Modules

Let R be a Euclidean domain. What does it mean for two matrices with entries in R to be *equivalent*? Prove that any such matrix is equivalent to a diagonal matrix. Under what further conditions is the diagonal matrix said to be in *Smith normal form*?

Let $M \leqslant \mathbb{Z}^n$ be the subgroup generated by the rows of an $n \times n$ matrix A. Show that $G = \mathbb{Z}^n/M$ is finite if and only if $\det A \neq 0$, and in that case the order of G is $|\det A|$.

Determine whether the groups G_1 and G_2 corresponding to the following matrices are isomorphic.

$$A_1 = \begin{pmatrix} 5 & 0 & 4 \\ 0 & 1 & 2 \\ 2 & 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 7 & 2 & -1 \\ 6 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

Paper 4, Section II
9E Groups, Rings and Modules

(a) Let R be a unique factorisation domain with field of fractions F. What does it mean for a polynomial $f \in R[X]$ to be *primitive*? Prove that the product of two primitive polynomials is primitive. Let $f, g \in R[X]$ be polynomials of positive degree. Show that if f and g are coprime in $R[X]$ then they are coprime in $F[X]$.

(b) Let $I \subset \mathbb{C}[X, Y]$ be an ideal generated by non-zero coprime polynomials f and g. By running Euclid's algorithm in a suitable ring, or otherwise, show that $I \cap \mathbb{C}[X] \neq \{0\}$ and $I \cap \mathbb{C}[Y] \neq \{0\}$. Deduce that $\mathbb{C}[X, Y]/I$ is a finite dimensional $\mathbb{C}$-vector space.

Paper 1, Section I
1F Linear Algebra

Define the determinant of a matrix $A \in M_n(\mathbb{C})$.

(a) Assume A is a block matrix of the form $\begin{pmatrix} M & X \\ 0 & N \end{pmatrix}$, where M and N are square matrices. Show that $\det A = \det M \det N$.

(b) Assume A is a block matrix of the form $\begin{pmatrix} 0 & M \\ N & 0 \end{pmatrix}$, where M and N are square matrices of sizes k and $n-k$. Express $\det A$ in terms of $\det M$ and $\det N$.

[You may assume properties of column operations if clearly stated.]

Paper 4, Section I
1F Linear Algebra

What is a *Hermitian form* on a complex vector space V? If φ and ψ are two Hermitian forms and $\varphi(v,v) = \psi(v,v)$ for all $v \in V$, prove that $\varphi(v,w) = \psi(v,w)$ for all $v, w \in V$.

Determine whether the Hermitian form on $\mathbb{C}^2$ defined by the matrix

$$A = \begin{pmatrix} 4 & 2i \\ -2i & 3 \end{pmatrix}$$

is positive definite.

Paper 1, Section II
8F Linear Algebra

(a) Let V be a finite dimensional complex inner product space, and let α be an endomorphism of V. Define its adjoint α^*.

Assume that α is normal, i.e. α commutes with its adjoint: $\alpha\alpha^* = \alpha^*\alpha$.

(i) Show that α and α^* have a common eigenvector $\mathbf{v}$. What is the relation between the corresponding eigenvalues?

(ii) Deduce that V has an orthonormal basis of eigenvectors of α.

(b) Now consider a real matrix $A \in \mathrm{Mat}_n(\mathbb{R})$ which is skew-symmetric, i.e. $A^T = -A$.

(i) Can A have a non-zero real eigenvalue?

(ii) Use the results of part (a) to show that there exists an orthogonal matrix $R \in O(n)$ such that $R^T AR$ is block-diagonal with the non-zero blocks of the form $\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

Paper 2, Section II
8F Linear Algebra

Let V be a real vector space (not necessarily finite-dimensional). Define the *dual space* V^*. Prove that if $f_1,\ f_2 \in V^*$ are such that $f_1(v)f_2(v) = 0$ for all $v \in V$, then f_1 or f_2 is the zero element in V^*.

Now suppose that V is a finite-dimensional real vector space.

Let ϕ be a symmetric bilinear form on V. State Sylvester's law of inertia for ϕ.

Let q be a quadratic form on V, let r denote its rank and σ its signature. Show that q can be factorised as $q(v) = f_1(v)f_2(v)$ with $f_1, f_2 \in V^*$ for all $v \in V$ if and only if $r + |\sigma| \leqslant 2$.

A vector $v_0 \in V$ is called isotropic if $q(v_0) = 0$. Show that if there exist v_1 and v_2 in V such that $q(v_1) > 0$ and $q(v_2) < 0$, then one can construct a basis of V consisting of isotropic vectors.

Paper 3, Section II
9F Linear Algebra

Suppose that α is an endomorphism of an n-dimensional complex vector space. Define the *minimal polynomial* m_α of α. State the Cayley–Hamilton theorem, and explain why m_α exists and is unique.

(a) If α has minimal polynomial $m_\alpha(x) = x^m$, what is the minimal polynomial of α^3?

(b) If $\lambda \neq 0$ is an eigenvalue for α, show that λ^3 is an eigenvalue for α^3. Describe the λ^3–eigenspace of α^3 in terms of eigenspaces of α.

(c) Assume α is invertible with minimal polynomial $m_\alpha(x) = \prod_{i=1}^{k}(x - \lambda_i)^{c_i}$.

(i) Show that the minimal polynomial m_{α^3} of α^3 must divide $\prod_{i=1}^{k}(x - \lambda_i^3)^{c_i}$.

(ii) Prove that equality holds if in addition all λ_i are real (in other words, we have $m_{\alpha^3}(x) = \prod_{i=1}^{k}(x - \lambda_i^3)^{c_i}$).

Paper 4, Section II
8F Linear Algebra

Let V and W be finite dimensional vector spaces, and α a linear map from V to W. Define the *rank* $r(\alpha)$ and *nullity* $n(\alpha)$ of α. State and prove the rank-nullity theorem.

Assume now that α and β are linear maps from V to itself, and let $n = \dim V$. Prove the following inequalities for the linear maps $\alpha + \beta$ and $\alpha\beta$:

$$|r(\alpha) - r(\beta)| \leqslant r(\alpha + \beta) \leqslant \min\{r(\alpha) + r(\beta), n\}$$

and

$$\max\{r(\alpha) + r(\beta) - n, 0\} \leqslant r(\alpha\beta) \leqslant \min\{r(\alpha), r(\beta)\}.$$

For arbitrary values of n and $0 \leqslant r(\alpha), r(\beta) \leqslant n$, show that each of the four bounds can be attained for some (α, β). Can both upper bounds always be attained simultaneously?

Paper 3, Section I
8H Markov Chains

Let X be an irreducible, positive recurrent and reversible Markov chain taking values in S and let π be its invariant distribution. For $A \subseteq S$, we write

$$T_A = \min\{n \geqslant 0 : X_n \in A\} \quad \text{and} \quad T_A^+ = \min\{n \geqslant 1 : X_n \in A\}.$$

(a) Prove that for all $A \subseteq S$ and $z \in A$, we have

$$\mathbb{P}_\pi(X_{T_A} = z) = \pi(z)\mathbb{E}_z\left[T_A^+\right].$$

(b) Let π_A be the probability measure defined by $\pi_A(x) = \pi(x)/\pi(A)$ for $x \in A$. Prove that

$$\mathbb{E}_{\pi_A}\left[T_A^+\right] = \frac{1}{\pi(A)}.$$

Paper 4, Section I
7H Markov Chains

Let X be an irreducible Markov chain with transition matrix P and values in the set S. For $i \in S$, let $T_i = \min\{n \geqslant 1 : X_n = i\}$ and $V_i = \sum_{n=0}^{\infty} \mathbf{1}(X_n = i)$.

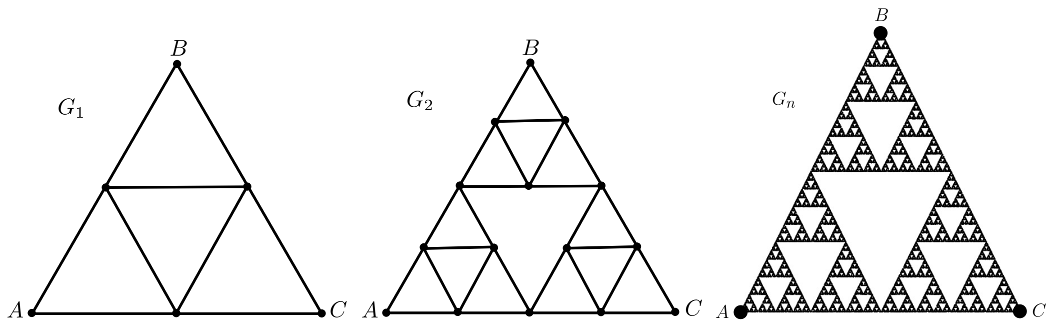
(a) Suppose $X_0 = i$. Show that V_i has a geometric distribution.

(b) Suppose X is transient. Prove that for all $i, j \in S$, we have

$$P^n(i,j) \to 0 \quad \text{as } n \to \infty.$$

Paper 1, Section II
19H Markov Chains

The n-th iteration of the Sierpinski triangle is constructed as follows: start with an equilateral triangle, subdivide it into 4 congruent equilateral triangles, and remove the central one. Repeat the same procedure $n-1$ times on each smaller triangle that is not removed. We call G_n the graph whose vertices are the corners of the triangles and edges the segments joining them, as shown in the figure:



Let A, B, and C be the corners of the original triangle. Let X be a simple random walk on G_n, i.e., from every vertex, it jumps to a neighbour chosen uniformly at random. Let

$$T_{BC} = \min\{i \geqslant 0 : X_i \in \{B, C\}\}.$$

(a) Suppose $n = 1$. Show that $\mathbb{E}_A[T_{BC}] = 5$.

(b) Suppose $n = 2$. Show that $\mathbb{E}_A[T_{BC}] = 5^2$.

(c) Show that $\mathbb{E}_A[T_{BC}] = 5^n$ when X is a simple random walk on G_n, for $n \in \mathbb{N}$.

Paper 2, Section II
18H Markov Chains

Let X be a random walk on $\mathbb{N} = \{0, 1, 2, \ldots\}$ with $X_0 = 0$ and transition matrix given by

$$P(i, i+1) = \frac{1}{3} = 1 - P(i, i-1), \quad \text{for } i \geqslant 1, \quad \text{and} \quad P(0,0) = \frac{2}{3} = 1 - P(0,1).$$

(a) Prove that X is positive recurrent.

(b) Let Y be an independent walk with matrix P and suppose that $Y_0 = 0$. Find the limit

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0, Y_n = 1),$$

stating clearly any theorems you use.

(c) Let $T = \min\{n \geqslant 1 : (X_n, Y_n) = (0,0)\}$. Find the expected number of times that Y visits 1 by time T.

Paper 2, Section I
3B Methods

The function $u(x,y)$ satisfies

$$x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = 0\,,$$

with boundary data $u(x,0) = f(x^2)$. Find and sketch the characteristic curves. Hence determine $u(x,y)$.

Paper 3, Section I
5A Methods

The Legendre polynomial $P_n(x)$ satisfies

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0, \quad n = 0,1,\ldots, \text{ for } -1 \leqslant x \leqslant 1.$$

Show that $Q_n(x) = P_n'(x)$ satisfies an equation which can be recast in self-adjoint form with eigenvalue $(n-1)(n+2)$. Write down the orthogonality relation for $Q_n(x)$, $Q_m(x)$ for $n \neq m$.

Paper 1, Section II
13B Methods

A uniform string of length l and mass per unit length μ is stretched horizontally under tension $T = \mu c^2$ and fixed at both ends. The string is subject to the gravitational force μg per unit length and a resistive force with value

$$-2k\mu \frac{\partial y}{\partial t}$$

per unit length, where $y(x,t)$ is the transverse, vertical displacement of the string and k is a positive constant.

(a) Derive the equation of motion of the string assuming that $y(x,t)$ remains small.

[In the remaining parts of the question you should assume that gravity is negligible.]

(b) Find $y(x,t)$ for $t > 0$, given that

$$y(x,0) = 0, \quad \frac{\partial y}{\partial t}(x,0) = A \sin\left(\frac{\pi x}{l}\right) \qquad (\star)$$

with A constant, and $k = \pi c/l$.

(c) An extra transverse force

$$\alpha\mu \sin\left(\frac{3\pi x}{l}\right) \cos kt$$

per unit length is applied to the string, where α is a constant. With the initial conditions $(\star)$, find $y(x,t)$ for $t > 0$ and comment on the behaviour of the string as $t \to \infty$.

Compute the total energy E of the string as $t \to \infty$.

Paper 2, Section II
14A Methods

(a) Verify that $y = e^{-x}$ is a solution of the differential equation

$$(x + \lambda + 1)y'' + (x + \lambda)y' - y = 0,$$

where λ is a constant. Find a second solution of the form $y = ax + b$.

(b) Let $\mathcal{L}$ be the operator

$$\mathcal{L}[y] = y'' + \frac{(x + \lambda)}{(x + \lambda + 1)}y' - \frac{1}{(x + \lambda + 1)}y$$

acting on functions $y(x)$ satisfying

$$y(0) = \lambda y'(0) \quad \text{and} \quad \lim_{x \to \infty} y(x) = 0. \qquad (\star)$$

The Green's function $G(x; \xi)$ for $\mathcal{L}$ satisfies

$$\mathcal{L}[G] = \delta(x - \xi),$$

with $\xi > 0$. Show that

$$G(x; \xi) = -\frac{(x + \lambda)}{(\xi + \lambda + 1)}$$

for $0 \leqslant x < \xi$, and find $G(x; \xi)$ for $x > \xi$.

(c) Hence or otherwise find the solution when $\lambda = 2$ for the problem

$$\mathcal{L}[y] = -(x + 3)e^{-x},$$

for $x \geqslant 0$ and $y(x)$ satisfying the boundary conditions given in $(\star)$.

Paper 3, Section II
14A Methods

(a) Prove Green's third identity for functions $u(\boldsymbol{r})$ satisfying Laplace's equation in a volume V with surface S, namely

$$u(\boldsymbol{r}_0) = \int_S \left(u\frac{\partial G_{fs}}{\partial n} - \frac{\partial u}{\partial n} G_{fs} \right) dS,$$

where $G_{fs}(\boldsymbol{r};\boldsymbol{r}_0) = -1/(4\pi|\boldsymbol{r}-\boldsymbol{r}_0|)$ is the free space Green's function.

(b) A solution is sought to the Neumann problem for $\nabla^2 u = 0$ in the half-space $z > 0$ with boundary condition

$$\left.\frac{\partial u}{\partial z}\right|_{z=0} = p(x,y),$$

where both u and its spatial derivatives decay sufficiently rapidly as $|\boldsymbol{r}| \to \infty$.

(i) Explain why it is necessary to assume that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)dx\,dy\ = 0.$$

(ii) Using the method of images or otherwise, construct an appropriate Green's function $G(\boldsymbol{r};\boldsymbol{r}_0)$ satisfying $\partial G/\partial z = 0$ at $z = 0$.

(iii) Hence find the solution in the form

$$u(x_0,y_0,z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) f(x-x_0, y-y_0, z_0) dx\,dy\,,$$

where f is to be determined.

(iv) Now let

$$p(x,y) = \begin{cases} \sin(x) & \text{for } |x|,|y| < \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

By expanding f in inverse powers of z_0, determine the leading order term for u (proportional to z_0^{-3}) as $z_0 \to \infty$.

Paper 4, Section II
14B Methods

(a) Let $h(x) = m'(x)$. Express the Fourier transform $\tilde{h}(k)$ of $h(x)$ in terms of the Fourier transform $\tilde{m}(k)$ of $m(x)$, given that $m \to 0$ as $|x| \to \infty$. [You need to show an explicit calculation.]

(b) Calculate the inverse Fourier transform of

$$\tilde{m}(k) = -i\pi \,\mathrm{sgn}(k) e^{-\alpha|k|} \,,$$

with $\mathrm{Re}\,\alpha > 0$.

(c) The function $u(x, y)$ obeys Laplace's equation $\nabla^2 u = 0$ in the region defined by $-\infty < x < \infty$ and $0 < y < a$, with real positive a, where $u(x, 0) = f(x)$, $u(x, a) = g(x)$ and $u \to 0$ as $|x| \to \infty$.

(i) By performing a suitable Fourier transform of Laplace's equation, determine the ordinary differential equation satisfied by $\tilde{u}(k, y)$. Hence express $\tilde{u}(k, y)$ in terms of the Fourier transforms $\tilde{f}(k)$, $\tilde{g}(k)$ of $f(x)$ and $g(x)$.

(ii) Find $\tilde{u}(k, y)$ for

$$f(x) = 0, \qquad g(x) = \frac{x}{x^2 + a^2} - \frac{x}{x^2 + 9a^2} .$$

Hence, determine $u(x, y)$.

[*The following convention is used in this question:*

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \quad \text{and} \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \,.]$$

Paper 1, Section I
5C Numerical Analysis

Use the Gram–Schmidt algorithm to compute a reduced QR factorization of the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & -4 \\ 2 & 2 & 2 \\ -2 & 0 & 2 \end{bmatrix},$$

i.e. find a matrix $Q \in \mathbb{R}^{4\times 3}$ with orthonormal columns and an upper triangular matrix $R \in \mathbb{R}^{3\times 3}$ such that $A = QR$.

Paper 4, Section I
6C Numerical Analysis

(a) Suppose that $w(x) > 0$ for all $x \in [a,b]$. The weights $b_1, \ldots, b_n$ and nodes $c_1, \ldots, c_n$ are chosen so that the Gaussian quadrature formula for a function $f \in C[a,b]$

$$\int_a^b w(x) f(x) dx \approx \sum_{k=1}^{n} b_k f(c_k)$$

is exact for every polynomial of degree $2n-1$. Show that the b_i, $i = 1, \ldots, n$ are all positive.

(b) Evaluate the coefficients b_k and c_k of the Gaussian quadrature of the integral

$$\int_{-1}^{1} x^2 f(x) dx,$$

which uses two evaluations of the function $f(x)$ and is exact for all f that are polynomials of degree 3.

Paper 1, Section II
17C Numerical Analysis

For a function $f \in C^3[-1,1]$ consider the following approximation of $f''(0)$:

$$f''(0) \approx \eta(f) = a_{-1} f(-1) + a_0 f(0) + a_1 f(1)\,,$$

with the error

$$e(f) = f''(0) - \eta(f).$$

We want to find the smallest constant c such that

$$|e(f)| \leqslant c \max_{x\in[-1,1]} \left| f'''(x) \right|. \qquad (\star)$$

(a) State the necessary conditions on the approximation scheme η for the inequality $(\star)$ to be valid with some $c < \infty$. Hence, determine the coefficients a_{-1}, a_0, a_1.

(b) State the Peano kernel theorem and use it to find the smallest constant c in the inequality $(\star)$.

(c) Explain briefly why this constant is sharp.

Paper 2, Section II
17C Numerical Analysis

A scalar, autonomous, ordinary differential equation $y' = f(y)$ is solved using the Runge–Kutta method

$$\begin{aligned} k_1 &= f(y_n)\,, \\ k_2 &= f(y_n + (1-a)hk_1 + ahk_2)\,, \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2)\,, \end{aligned}$$

where h is a step size and a is a real parameter.

(a) Determine the order of the method and its dependence on a.

(b) Find the range of values of a for which the method is A-stable.

Paper 3, Section II
17C Numerical Analysis

(a) The equation $y' = f(t, y)$ is solved using the following multistep method with s steps,

$$\sum_{k=0}^{s} \rho_k y_{n+k} = h \sum_{k=0}^{s} \sigma_k f(t_{n+k}, y_{n+k}) \,,$$

where h is the step size and ρ_k, σ_k are specified constants with $\rho_s = 1$. Prove that this method is of order p if and only if

$$\sum_{k=0}^{s} \rho_k P(t_{n+k}) = h \sum_{k=0}^{s} \sigma_k P'(t_{n+k}) \,,$$

for all polynomials P of degree p.

(b) State the Dahlquist equivalence theorem regarding the convergence of a multistep method. Consider a multistep method

$$y_{n+3} + (2a-3)(y_{n+2} - y_{n+1}) - y_n = ha(f_{n+2} + f_{n+1}) \,,$$

where $a \neq 0$ is a real parameter. Determine the values of a for which this method is convergent, and find its order.

Paper 1, Section I
7H Optimisation

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function. Briefly describe the steps of the *gradient descent* method for minimizing f.

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a twice-differentiable function satisfying $\alpha I \preceq \nabla^2 f(x) \preceq \beta I$ for some $\alpha, \beta > 0$ and all $x \in \mathbb{R}^n$. Suppose the gradient descent method is run with step size $\eta = \frac{1}{\beta}$. How does the rate of convergence of the gradient descent method depend on the condition number $\frac{\beta}{\alpha}$?

Now let $f(x, y, z) = x^2 + 100y^2 + 10000z^2$. Compute a condition number for f. Find a linear transformation $A : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f \circ A$ has a condition number of 1.

[*For two matrices $A, B \in \mathbb{R}^n$, we write $A \preceq B$ to denote the fact that $B - A$ is a positive semidefinite matrix.*]

Paper 2, Section I
7H Optimisation

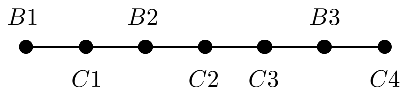
State the Lagrange sufficiency theorem. Using the Lagrange sufficiency theorem, solve the following optimisation problem:

$$\begin{aligned} \text{minimise} \quad & -x_1 - 3x_2 \\ \text{subject to} \quad & x_1^2 + x_2^2 \leqslant 25 \\ & -x_1 + 2x_2 \leqslant 5. \end{aligned}$$

Paper 3, Section II
19H Optimisation

Explain what is meant by a *transportation problem* with n suppliers and m consumers.

A straight road contains three bakeries, B1, B2, and B3, and four cafes, C1, C2, C3, and C4. They are arranged in the following order:



The distance between consecutive establishments is 1 mile: For example, the distance between B1 and C2 is 3 miles. Bakeries B1, B2, and B3 produce 6, 4, and 8 cakes daily, respectively. Cafes C1, C2, C3, and C4 consume 3, 5, 7, and 3 cakes daily, respectively. The cost of transporting one cake from a bakery to a cafe is equal to the distance between the two locations, measured in miles. Cakes may be cut into arbitrary pieces before transporting. The resulting cost matrix is

$$C = \begin{pmatrix} 1 & 3 & 4 & 6 \\ 1 & 1 & 2 & 4 \\ 4 & 2 & 1 & 1 \end{pmatrix}.$$

(a) Use the north-west corner rule to find a basic feasible solution. Is this solution degenerate? If not, find a degenerate basic feasible solution to this problem.

(b) Consider the following transportation plan:

- B1 delivers 3 cakes each to C1 and C3,
- B2 delivers 4 cakes to C2, and
- B3 delivers 1 cake to C2, 4 cakes to C3, and 3 cakes to C4.

Explain why this is a basic feasible solution. Calculate the complete transportation tableau for this solution. Is the solution optimal? If not, perform one step of the transportation algorithm. Is the solution optimal now?

Paper 4, Section II
18H Optimisation

(a) Explain what is meant by a *two player zero-sum game*. What are *pure* and *mixed* strategies?

(b) Let $0 < a < b < c < d$, and let

$$A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A_2 = \begin{pmatrix} a & b \\ d & c \end{pmatrix}, \quad \text{and } A_3 = \begin{pmatrix} a & c \\ d & b \end{pmatrix}.$$

Which of the three games with the payoff matrices given above admit optimal strategies that are pure?

(c) Consider the payoff matrix

$$A = \begin{pmatrix} 1 & 5 \\ 7 & 3 \end{pmatrix}.$$

Let $p = [p_1, p_2]^T$ be the strategy of player 1, and let v be the value of the game. Show that $v > 0$. Setting $x = [p_1/v, p_2/v]^T$, show that the optimal strategy for player 1 can be found by solving the problem

$$\begin{aligned} \text{minimize} \quad & e^T x \\ \text{subject to} \quad & A^T x \geqslant e \\ & x \geqslant 0, \end{aligned}$$

where $e = [1, 1]^T$.

(d) Find the dual of the linear program in part (c). Is the dual a linear program in standard form? Solve the dual using the simplex method and identify the optimal strategies for both players.

Paper 3, Section I
6B Quantum Mechanics

(a) A beam of identical, free particles, each of mass m, moves in one dimension. There is no potential. Show that the wavefunction $\chi(x) = Ae^{ikx}$ is an energy eigenstate for any constants A and k.

What is the energy E and the momentum p in terms of k? What can you say about the sign of E?

(b) Write down expressions for the probability density ρ and the probability current J in terms of the wavefunction $\psi(x,t)$. Use the current conservation equation, i.e.

$$\frac{\partial \rho}{\partial t} + \frac{\partial J}{\partial x} = 0$$

to show that, for a stationary state of fixed energy E, the probability current J is independent of x.

(c) A beam of particles in a stationary state is incident from $x \to -\infty$ upon a potential $U(x)$ with $U(x) \to 0$ as $x \to \pm\infty$. Given the asymptotic behaviour of the form

$$\psi(x) = \begin{cases} e^{ikx} + Re^{-ikx}\,, & x \to -\infty\,, \\ \\ Te^{ikx}\,, & x \to \infty\,, \end{cases}$$

show that $|R|^2 + |T|^2 = 1$. Interpret this result.

Paper 4, Section I
4B Quantum Mechanics

The radial wavefunction $g(r)$ for the hydrogen atom satisfies the equation

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{d}{dr}g(r)\right) - \frac{e^2}{4\pi\epsilon_0 r}g(r) + \frac{l(l+1)\hbar^2}{2mr^2}g(r) = Eg(r)\,. \qquad (\dagger)$$

(a) Explain the origin of each of the terms in (†). What are the allowed values of l?

(b) For a given l, the lowest energy bound state solution of (†) takes the form $r^a e^{-br}$. Find a, b, and the corresponding value of E, in terms of l.

(c) A hydrogen atom makes a transition between two such states, corresponding to $l+1$ and l. What is the frequency of the photon emitted?

Paper 1, Section II

14B Quantum Mechanics

(a) Write down the time-dependent Schrödinger equation for a harmonic oscillator of mass m, frequency ω and coordinate x.

(b) Show that a wavefunction of the form

$$\psi(x,t) = N(t)\, \exp\Big(- F(t)x^2 + G(t)x \Big)\,,$$

where F, G and N are complex functions of time, is a solution to the Schrödinger equation, provided that F, G, N satisfy certain conditions which you should establish.

(c) Verify that

$$F(t) = A \tanh(a + i\omega t), \quad G(t) = \sqrt{\frac{m\omega}{\hbar}}\, \mathrm{sech}(a + i\omega t)\,,$$

where a is a real positive constant, satisfy the conditions you established in part (b). Hence determine the constant A. [You do not need to find the time-dependent normalization function $N(t)$.]

(d) By completing the square, or otherwise, show that $|\psi(x,t)|^2$ is peaked around a certain position $x = h(t)$ and express $h(t)$ in terms of F and G.

(e) Find $h(t)$ as a function of time and describe its behaviour.

(f) Sketch $|\psi(x,t)|^2$ for a fixed value of t. What is the value of $\langle \hat{x} \rangle_{\psi}$?

[*You may find the following identities useful:*

$$\cosh(\alpha + i\beta) = \cosh\alpha \cos\beta + i \sinh\alpha \sin\beta\,,$$

$$\sinh(\alpha + i\beta) = \sinh\alpha \cos\beta + i \cosh\alpha \sin\beta\,.]$$

Paper 2, Section II
15B Quantum Mechanics

A particle of mass m is confined to the region $0 \leqslant x \leqslant a$ by a potential that is zero inside the region and infinite outside.

(a) Find the energy eigenvalues E_n and the corresponding normalised energy eigenstates $\chi_n(x)$.

(b) At time $t = 0$ the wavefunction $\psi(x,t)$ of the particle is given by

$$\psi(x,0) = f(x)\,,$$

where $f(x)$ is not an energy eigenstate and satisfies the boundary conditions $f(0) = f(a) = 0$.

(i) Express $\psi(x,t)$ in terms of $\chi_n(x)$ and E_n.

(ii) Show that $T = 2ma^2/\pi\hbar$ is the earliest time at which $\psi(a-x,T)$ and $\psi(x,0)$ correspond to physically equivalent states. Thus, determine $\psi(x,2T)$.

Show that if $\psi(x,0) = 0$ for $a/2 \leqslant x \leqslant a$, then the probability of finding the particle in $0 \leqslant x \leqslant a/2$ at $t = T$ is zero.

(iii) For

$$f(x) = \begin{cases} \dfrac{2}{\sqrt{a}} \sin \dfrac{2\pi x}{a}\,, & 0 \leqslant x \leqslant \dfrac{a}{2}\,, \\[2ex] 0\,, & \dfrac{a}{2} \leqslant x \leqslant a\,, \end{cases}$$

find the probability that a measurement of the energy of the particle at time $t = 0$ will yield a value $2\pi^2\hbar^2/ma^2$.

What is the probability if, instead, the same measurement is carried out at time $t = 2T$? What is the probability at $t = T$?

Suppose that the result of the measurement of the energy was indeed $2\pi^2\hbar^2/ma^2$. What is the probability that a subsequent measurement of energy will yield the same result?

Paper 4, Section II
15B Quantum Mechanics

(a) Write down the time-dependent Schrödinger equation for the wavefunction $\psi(x,t)$ of a particle with Hamiltonian $\hat{H}$.

Suppose that A is an observable associated with the operator $\hat{A}$. Show that

$$i\hbar\frac{d\langle\hat{A}\rangle_\psi}{dt} = \langle[\hat{A},\hat{H}]\rangle_\psi + i\hbar\left\langle\frac{\partial\hat{A}}{\partial t}\right\rangle_\psi .$$

(b) Consider a particle of mass m subject to a constant gravitational field with potential energy $U(x) = mgx$.

[For the rest of the question you should assume that $\psi(x,t)$ is normalized.]

(i) Find the differential equation satisfied by the function $\Phi(x,t)$ defined by

$$\psi(x,t) = \Phi(x,t)\exp\left[-\frac{im}{\hbar}gt\left(x+\frac{1}{6}gt^2\right)\right] .$$

(ii) Show that $\Theta(X,T) = \Phi(x,t)$, with $X = x + \frac{1}{2}gt^2$ and $T = t$, satisfies the free-particle Schrödinger equation

$$i\hbar\frac{\partial\Theta}{\partial T} = -\frac{\hbar^2}{2m}\frac{\partial^2\Theta}{\partial X^2} .$$

Hence, show that

$$\frac{d\langle\hat{X}\rangle_\Theta}{dT} = \frac{1}{m}\langle\hat{P}\rangle_\Theta , \qquad \frac{d\langle\hat{P}\rangle_\Theta}{dT} = 0 ,$$

where $\hat{P} = -i\hbar\dfrac{\partial}{\partial X}$.

(iii) Express $\langle\hat{X}\rangle_\Theta$ in terms of $\langle\hat{x}\rangle_\psi$. Deduce that

$$\langle\hat{x}\rangle_\psi = a + vt - \tfrac{1}{2}gt^2 ,$$

for some constants a and v. Briefly comment on the physical significance of this result.

Paper 1, Section I

6H Statistics

State the Rao-Blackwell theorem.

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. Geometric(p) random variables; i.e., X_1 is distributed as the number of failures before the first success in a sequence of i.i.d. Bernoulli trials with probability of success p.

Let $\theta = p - p^2$, and consider the estimator $\hat{\theta} = 1_{\{X_1=1\}}$. Find an estimator for θ which is a function of the statistic $T = \sum_{i=1}^n X_i$ and which has variance strictly smaller than that of $\hat{\theta}$. [*Hint: Observe that T is a sufficient statistic for p.*]

Paper 2, Section I

6H Statistics

Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with probability density function

$$f_\theta(x) = \frac{1}{2} + \frac{1_{\{x<\theta\}}}{2\theta} \quad \text{for } x \in [0,1],$$

with parameter $\theta \in (0,1)$.

(a) Write down the likelihood function, and show that the maximum likelihood estimator coincides with one of the samples.

(b) Consider the estimator $\tilde{\theta} = 4\overline{X} - 1$ where $\overline{X} = n^{-1}\sum_{i=1}^n X_i$. Is $\tilde{\theta}$ unbiased? Construct an asymptotic $(1-\alpha)$-confidence interval for θ around this estimator.

Paper 1, Section II
18H Statistics

A clinical study follows n patients being treated for a disease for T months. Suppose we observe $X_1, \ldots, X_n$, where $X_i = t$ if patient i recovers at month t, and $X_i = T + 1$ if the patient does not recover at any point in the observation period. For $t = 1, \ldots, T$, the parameter $q_t \in [0, 1]$ is the probability that a patient recovers at month t, given that they have not already recovered.

We select a prior distribution which makes the parameters $q_1, \ldots, q_T$ i.i.d. and distributed as Beta$(T, 1)$.

(a) Write down the likelihood function. Compute the posterior distribution of $(q_1, \ldots, q_T)$.

(b) The parameter γ is the probability that a patient recovers at or before month M. Write down γ in terms of $q_1, \ldots, q_T$. Compute the Bayes estimator for γ under the quadratic loss.

(c) Suppose we wish to estimate γ, but our loss function is asymmetric; i.e., we prefer to underestimate rather than overestimate the parameter. In particular, the loss function is given by

$$L(\delta, \gamma) = \begin{cases} 2|\gamma - \delta| & \text{if } \delta \geqslant \gamma \\ |\gamma - \delta| & \text{if } \delta < \gamma. \end{cases}$$

Find an expression for the Bayes estimator of γ under this loss function, in terms of the posterior distribution function F of γ. [You need not derive F.]

Paper 3, Section II
18H Statistics

Consider a linear model $Y = X\beta + \varepsilon$, where $X \in \mathbb{R}^{n\times p}$ is a fixed design matrix of rank $p < n/2$, $\beta \in \mathbb{R}^p$, and $\varepsilon \sim N(0, \sigma^2 \Sigma_0)$, for some known positive definite matrix $\Sigma_0 \in \mathbb{R}^{n\times n}$ and an unknown scalar $\sigma^2 > 0$.

(a) Derive the maximum likelihood estimators $(\hat{\beta}, \hat{\sigma}^2)$ for the parameters (β, σ^2).

(b) Find the distribution of $\hat{\beta}$.

(c) Prove that $\hat{\beta}$ is the Best Linear Unbiased Estimator for β.

Now, suppose that $\varepsilon \sim N(0, \Sigma)$ where $\Sigma \in \mathbb{R}^{n\times n}$ is a diagonal matrix with

$$\Sigma_{ii} = \begin{cases} \sigma_1^2 & \text{if } i \leqslant n/2\,, \\ \sigma_2^2 & \text{if } i > n/2\,, \end{cases}$$

and where σ_1^2 and σ_2^2 are unknown parameters and n is even.

(d) Describe a test of size α for the null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ against the alternative $H_1 : \sigma_1^2 < \sigma_2^2$, using the test statistic

$$T = \frac{\|Y_1 - X_1(X_1^T X_1)^{-1} X_1^T Y_1\|^2}{\|Y_2 - X_2(X_2^T X_2)^{-1} X_2^T Y_2\|^2}$$

where,

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

with $Y_1, Y_2 \in \mathbb{R}^{n/2}$ and $X_1, X_2 \in \mathbb{R}^{n/2\times p}$. [You must specify the null distribution of T and the critical region, and you may quote any result from the lectures that you need without proof.]

Paper 4, Section II
17H Statistics

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. observations from a zero-inflated Poisson distribution with parameters $\pi \in [0,1]$ and $\lambda > 0$, which has probability mass function

$$f_{\pi,\lambda}(x) = \begin{cases} \pi + (1-\pi)e^{-\lambda} & \text{if } x = 0, \\ (1-\pi)\frac{\lambda^x e^{-\lambda}}{x!} & \text{if } x = 1, 2, \ldots. \end{cases}$$

Let $n_0 = \sum_{i=1}^n 1_{\{X_i = 0\}}$ and $S = \sum_{i=1}^n X_i$.

(a) What is meant by *sufficient statistic* and *minimal sufficient statistic*? Show that $T = (n_0, S)$ is a sufficient statistic. Is it minimal sufficient?

(b) Suppose the parameter λ is known to be equal to some value λ_0. We wish to test the null hypothesis $H_0 : \pi = 0$ against the alternative $H_1 : \pi = 1/2$. Suppose there exists a likelihood ratio test of size α for H_0 against H_1. Specify the test statistic and the critical region. Is this test uniformly most powerful for the alternative $H_1 : \pi > 0$?

(c) Now suppose that both π and λ are unknown. We wish to test the null hypothesis $H_0 : \pi = 1/2$ against the alternative $H_1 : \pi \in [0,1]$. State the asymptotic null distribution of the generalised likelihood ratio statistic:

$$W = 2 \log \frac{\max_{\lambda > 0, \pi \in [0,1]} L(\lambda, \pi; X)}{\max_{\lambda > 0} L(\lambda, 1/2; X)},$$

where $L(\lambda, \pi; X)$ is the likelihood function. Describe a test of size α using this statistic.

[*You may quote any result from the lectures that you need without proof.*]

Paper 1, Section I
4D Variational Principles

Write down the Euler-Lagrange equation for the functional

$$I[y] = \int_0^{\pi/2} \left[y'^{\,2} - y^2 - 2y\sin(x) \right] \mathrm{d}x \ .$$

Solve it subject to the boundary conditions $y'(0) = y'(\pi/2) = 0$.

Paper 3, Section I
4D Variational Principles

Explain the method of Lagrange multipliers for finding the stationary values of a function $F(x, y, z)$ subject to the constraint $G(x, y, z) = 0$.

Use the method of Lagrange multipliers to find the minimum of $x^2 + y^2 + z^2$ subject to the constraint $z - xy = 1$.

Find the maximum of $z - xy$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Paper 2, Section II
13D Variational Principles

(a) A functional $I[z]$ of $z(x)$ is given by

$$I[z] = \int_a^b f(z, z'; x)\ dx$$

where $z' = dz/dx$. State the Euler-Lagrange equation that governs the extrema of I.

If f does not depend explicitly on x, construct a non-constant quantity that, when evaluated on the extrema of I, does not depend on x.

Explain how to determine the extrema of I subject to the further functional constraint that $J[z]$ is constant.

(b) A heavy, uniform rope of fixed length L is suspended between two points $(x_1, z_1) = (-a, 0)$ and $(x_2, z_2) = (+a, 0)$ with $L > 2a$. In a gravitational potential $\Phi(z)$, the potential energy is given by

$$V[z] = \rho \int_{-a}^{a} \Phi(z) \sqrt{1 + z'^2}\, dx\ .$$

where ρ is the mass per unit length.

(i) Show that, in a gravitational potential $\Phi(z) = gz$, the shape adopted by the rope is

$$z - z_0 = -B \cosh\left(\frac{x}{B}\right)$$

where z_0 and B are two constants. Find implicit expressions for z_0 and B in terms of a and L.

(ii) What is the gravitational potential $\Phi(z)$ if, for $L = \pi a$, the rope hangs in a semi-circle?

Paper 4, Section II
13D Variational Principles

(a) Derive the Euler-Lagrange equation for the functional

$$\int_a^b f(y, y', y''; x)\, dx \ ,$$

where prime denotes differentiation with respect to x, and both y and y' are specified at $x = a, b$.

(b) If f does not depend explicitly on x show that, when evaluated on the extremum,

$$f - \left[\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\right] y' - \frac{\partial f}{\partial y''} y'' = \text{constant} \ .$$

(c) Find $y(x)$ that extremises the integral

$$\int_0^{\pi/2} \left(-\frac{1}{2} y''^2 + y'^2 - \frac{1}{2} y^2\right) dx$$

subject to $y(0) = y'(0) = 0$ and $y(\pi/2) = \pi/2$ and $y'(\pi/2) = 1$.

END OF PAPER