UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2022

List of Courses

Paper 1, Section I
3D Analysis I

State the alternating series test. Deduce that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges. Is this series absolutely convergent? Justify your answer.

Find a divergent series which has the same terms $\frac{(-1)^n}{\sqrt{n}}$ taken in a different order. You should justify the divergence.

[*You may use the comparison test, provided that you accurately state it.*]

Paper 1, Section I
4D Analysis I

Let $a \in \mathbb{R}$ and let f and g be continuous real-valued functions defined on $\mathbb{R}$ which are not identically zero on any interval containing a.

Must the function $F(x) = f(x) + g(x)$ be non-differentiable at $a \in \mathbb{R}$ if (a) f is differentiable at a and g is not differentiable at a; (b) both f and g are not differentiable at a?

Must the function $G(x) = f(x)g(x)$ be non-differentiable at $a \in \mathbb{R}$ if (a) f is differentiable at a and g is not differentiable at a; (b) both f and g are not differentiable at a?

Justify your answers.

Paper 1, Section II
9D Analysis I

(a) Let a_n be a sequence of real numbers. Show that if a_n converges, the sequence $\frac{1}{n}\sum_{k=1}^{n} a_k$ also converges and $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = \lim_{n\to\infty} a_n$.

If $\frac{1}{n}\sum_{k=1}^{n} a_k$ converges, must a_n converge too? Justify your answer.

(b) Let x_n be a sequence of real numbers with $x_n > 0$ for all n. By considering the sequence $\log x_n$, or otherwise, show that if x_n converges then $\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \ldots x_n} = \lim_{n\to\infty} x_n$. You may assume that exp and log are continuous functions.

Deduce that if the sequence $\dfrac{x_n}{x_{n-1}}$ converges, then $\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \dfrac{x_n}{x_{n-1}}$.

(c) What is a *Cauchy sequence*? State the general principle of convergence for real sequences.

Let a_n be a decreasing sequence of positive real numbers and suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim_{n\to\infty} n a_n = 0$.

Paper 1, Section II
10D Analysis I

Prove that every continuous real-valued function on a closed bounded interval is bounded and attains its bounds. [The Bolzano–Weierstrass theorem can be assumed provided it is accurately stated.]

Give an example of a continuous function $\phi : (0,1) \to \mathbb{R}$ that is bounded but does not attain its bounds and an example of a function $\psi : [0,1] \to \mathbb{R}$ that is not bounded on any interval $[a,b]$ such that $0 \leqslant a < b \leqslant 1$. Justify your examples.

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Prove that the functions

$$m(x) = \inf_{a \leqslant \xi \leqslant x} f(\xi) \qquad \text{and} \qquad M(x) = \sup_{a \leqslant \xi \leqslant x} f(\xi)$$

are also continuous on $[a,b]$.

Let a function $g : (0,\infty) \to \mathbb{R}$ be continuous and bounded. Show that for every $T > 0$ there exists a sequence x_n such that $x_n \to \infty$ and

$$\lim_{n\to\infty} (g(x_n + T) - g(x_n)) = 0.$$

[The intermediate value theorem can be assumed.]

Paper 1, Section II
11D Analysis I

In this question $a < b$ are real numbers.

(a) State and prove Rolle's theorem. State and prove the mean value theorem.

(b) Prove that if a continuous function $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b) and is not a linear function, then $f'(\xi) > \dfrac{f(b) - f(a)}{b - a}$ for some ξ with $a < \xi < b$.

(c) Let $f : [a,b] \to \mathbb{R}$ be a continuous function and let f be differentiable on (a,b). Must there exist, for every $\xi \in (a,b)$, two points x_1, x_2 with $a \leqslant x_1 < \xi < x_2 \leqslant b$ such that $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi)$? Give a proof or counterexample as appropriate.

(d) Let functions f and g be continuous on $[a,b]$ and differentiable on (a,b) with $g(a) \neq g(b)$ and suppose that $f'(x)$ and $g'(x)$ never vanish for the same value of x. By considering $\lambda f + \mu g + \nu$ for suitable real constants λ, μ, ν, or otherwise, prove that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)} \qquad \text{for some } \xi \text{ with } a < \xi < b.$$

Give an example to show that the condition that $f'(x)$ and $g'(x)$ never vanish for the same x cannot be omitted.

Paper 1, Section II
12D Analysis I

Let $f : [0,1] \to \mathbb{R}$ be a monotone function.

Show that for all dissections $\mathcal{D}$ and $\mathcal{D}'$ of $[0,1]$ one has $L_{\mathcal{D}}(f) \leqslant U_{\mathcal{D}'}(f)$, where $L_{\mathcal{D}}(f)$ and $U_{\mathcal{D}'}(f)$ are the lower and upper sums of f for the respective dissections. Show further that for each $\varepsilon > 0$ there is a dissection $\mathcal{D}$ such that $U_{\mathcal{D}}(f) - L_{\mathcal{D}}(f) < \varepsilon$. Deduce that f is integrable.

Show that

$$\left| \int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| < \frac{|f(1) - f(0)|}{n}$$

for all positive integers n.

Let a function F be continuous on some open interval containing $[0,1]$ and have a continuous derivative F' on $[0,1]$. Denote

$$\Delta_n = \int_0^1 F(x)dx - \frac{1}{n}\sum_{k=1}^{n} F\left(\frac{k}{n}\right).$$

Stating clearly any results from the course that you require, show that

$$\lim_{n\to\infty} n\Delta_n = (F(0) - F(1))/2.$$

[*Hint: it might be helpful to consider* $\int_{(k-1)/n}^{k/n}\left(F(x) - F\left(\frac{k}{n}\right)\right)dx$.]

Paper 2, Section I
1A Differential Equations

Consider the integral

$$I(x)=\int_0^\pi e^{x\cos\theta}d\theta\ .$$

Show, by differentiating under the integral sign, that

$$\frac{dI}{dx}=\int_0^\pi x\sin^2\theta\, e^{x\cos\theta}d\theta\ .$$

Hence, or otherwise, show that

$$\frac{d^2I}{dx^2}+\frac{1}{x}\frac{dI}{dx}-I=0\ .$$

Paper 2, Section I
2B Differential Equations

Solve the difference equation

$$x_{n+3}-6x_{n+2}+12x_{n+1}-8x_n=0\,,$$

given initial conditions $x_0=0, x_1=4, x_2=24$.

Paper 2, Section II
5C Differential Equations

(a) What is meant by an ordinary point and a regular singular point of a linear second-order ordinary differential equation?

Consider

$$x\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+\lambda y=0, \qquad (\dagger)$$

where λ is a real constant.

Find a solution to ($\dagger$) in the form of a series expansion around $x=0$. Obtain the general expression for the coefficients in the series.

For what values of λ do you obtain polynomial solutions?

(b) Determine the Wronskian of the equation ($\dagger$) as a function of x.

Let $\lambda=1$. Verify that $y_1=1-x$ is a solution to ($\dagger$). Using the Wronskian, calculate a second solution y_2 in the form

$$y_2=(1-x)\log x+b_1x+b_2x^2+\ldots,$$

where b_1 and b_2 are constants you need to find.

Paper 2, Section II
6A Differential Equations

(a) Let $f(x,y)$ be a real-valued function depending smoothly on real variables x and y, and $g(t) = f(\,a + t\cos\gamma,\ b + t\sin\gamma\,)$, where a, b and γ are constants. Express $g'(t)$ and $g''(t)$ in terms of partial derivatives of f.

Write down sufficient conditions for g to have a local minimum at $t = 0$ and deduce that a stationary point of f at $(x,y) = (a,b)$ is a local minimum if

$$\frac{\partial^2 f}{\partial y^2} > 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 .$$

(b) Now let

$$f(x,y) = x^4 - 3x^2 + 2xy + y^2 \ .$$

Find all stationary points of f and show that those at $(x,y) \neq (0,0)$ are local minima.

Show also that $g(t)$ with $a = b = 0$ has either (i) a local minimum or (ii) a local maximum at $t = 0$, depending on the value of γ. Determine carefully the ranges of values of $\tan\gamma$ for which cases (i) and (ii) occur and sketch the typical behaviour of $g(t)$ in each of these cases.

Paper 2, Section II
7C Differential Equations

Consider the system of linear differential equations

$$\frac{d\mathbf{z}}{dt} - A\mathbf{z} = \mathbf{f}, \quad \text{where} \quad A = \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}. \qquad (\dagger)$$

(a) Suppose $\mathbf{f} = \mathbf{0}$. Show that the general solution to $(\dagger)$ takes the form

$$\mathbf{z} = \alpha \mathbf{u}_1 e^{\lambda_1 t} + \beta \mathbf{u}_2 e^{\lambda_2 t}, \qquad (\star)$$

where α and β are arbitrary constants. Calculate $\mathbf{u}_1$, $\mathbf{u}_2$, λ_1, and λ_2.

(b) Suppose now that $\mathbf{f} = (1,\ a)^T$, where a is a constant parameter.

By writing $\mathbf{f}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, determine the value(s) of a for which the particular integral depends on time.

Using matrix methods, find the general solution to $(\dagger)$.

(c) Consider

$$\frac{d^n \mathbf{z}}{dt^n} - A\mathbf{z} = \mathbf{0},$$

where $n > 1$ is an integer.

Show that $(\star)$ is a solution to this system of equations. How many other linearly independent solutions must there be?

Paper 2, Section II

8B Differential Equations

(a) Consider the system

$$\dot{x} = 8x - 2x^2 - 2xy^2\,, \qquad \dot{y} = xy - y\,, \tag{$*$}$$

for $x(t) \geqslant 0\,, y(t) \geqslant 0$.

Find all the equilibrium points of $(*)$ and determine their type. Explain how solutions close to each equilibrium point will evolve, sketching their trajectories. [You may quote general results without proof.]

(b) Consider the system

$$\dot{x} = x(1-y)\,, \qquad \dot{y} = 3y(x-1)\,, \tag{$**$}$$

defined for $x > 0, y > 0$.

Show that it has precisely one equilibrium point in the given range. Obtain an equation for dy/dx. Show that this equation is separable and hence obtain a solution in the form $E(x,y) = C$, where C is a constant and $E(x,y)$ is a nontrivial conserved quantity for solutions of $(**)$. Show that $E(x,y)$ has a single stationary point in the quadrant $x > 0, y > 0$, and identify what type of stationary point it is. Hence show that solutions close to the equilibrium point at time $t = 0$ remain close at all times.

Paper 4, Section I
3C Dynamics and Relativity

A particle of mass m, charge q, and position vector $\mathbf{x}$ moves in a constant non-zero electric field $\mathbf{E}$ and a constant non-zero magnetic field $\mathbf{B}$, with $\mathbf{E}$ perpendicular to $\mathbf{B}$. The particle's motion is described by $m\ddot{\mathbf{x}} = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B})$. At time $t = 0$ the particle is located at $\mathbf{x} = \mathbf{x}_0$ and has velocity $\dot{\mathbf{x}} = \mathbf{v}$, where $\mathbf{v}$ is perpendicular to both $\mathbf{E}$ and $\mathbf{B}$.

(a) Using vector methods, show that the motion lies in a plane and give the vector equation of that plane.

(b) Adopt a Cartesian coordinate system centred on $\mathbf{x}_0$ with the x-axis directed along $\mathbf{E}$ and the y-axis along $\mathbf{B}$. Assume $\mathbf{v} = \mathbf{0}$. Find an expression for $\mathbf{x}$ as a function of t.

Paper 4, Section I
4C Dynamics and Relativity

Consider space-time with only one spatial dimension, and two inertial frames S and S'. Frame S' moves relative to frame S with speed u, and their origins coincide when clocks in the two frames read $t = t' = 0$.

According to an observer at the origin of frame S, an event has coordinates (ct, x). According to an observer at the origin of frame S', its coordinates are (ct', x'), which are given by

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = A \begin{pmatrix} ct \\ x \end{pmatrix},$$

where c is the speed of light and A is a 2×2 matrix.

(a) Write down the matrix A in terms of $\beta = u/c$ when working in:

(i) Newtonian dynamics;

(ii) special relativity.

Show that the two transformations agree in an appropriate limit, assuming $|x| < c|t|$.

(b) Calculate the eigenvalues and eigenvectors of A in special relativity, and interpret the eigenvectors.

Paper 4, Section II

9C Dynamics and Relativity

Consider two particles of masses m_1 and m_2, and locations $\mathbf{x}_1$ and $\mathbf{x}_2$, that exert forces $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$ upon each other. There are no external forces. The particles' equations of motion are $m_1\ddot{\mathbf{x}}_1 = \mathbf{F}_{12}$, and $m_2\ddot{\mathbf{x}}_2 = \mathbf{F}_{21}$.

(a) Define the centre of mass $\mathbf{R}$. Prove that the centre of mass moves at a constant velocity. If $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, show that $\mu\ddot{\mathbf{r}} = \mathbf{F}_{12}$, where you must give an expression for μ.

(b) For the remainder of the question, assume the force law

$$\mathbf{F}_{ij} = -km_i m_j \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3},$$

with k a positive constant.

Let $m_1 = m_2 = m$. In a Cartesian coordinate system whose origin is at the centre of mass, verify that

$$\mathbf{x}_1 = a(\cos\omega t,\, \sin\omega t\,, 0), \qquad \mathbf{x}_2 = -a(\cos\omega t,\, \sin\omega t,\, 0), \tag{$\dagger$}$$

is a solution to the equations of motion, where a is a fixed constant and ω is a frequency that you should find.

(c) A third particle is now placed upon the z-axis. Its mass m_3 is negligible compared to m, and its position vector $\mathbf{x}_3$ obeys $m_3\ddot{\mathbf{x}}_3 = \mathbf{F}_{31} + \mathbf{F}_{32}$, while the motion of particles 1 and 2 is given by ($\dagger$).

(i) If the initial velocity of the third particle is parallel to the z-axis, show that it remains on that axis and that its location $z(t)$ obeys

$$\ddot{z} = -\frac{2mkz}{(z^2 + a^2)^{3/2}}.$$

(ii) What is the effective potential governing the particle's motion? Describe the different kinds of behaviour possible.

(iii) Assume that at $t = 0$, $z = z_0$, and $\dot{z} = 0$. If z_0 is very small, show that the motion is oscillatory and find the period of the oscillations.

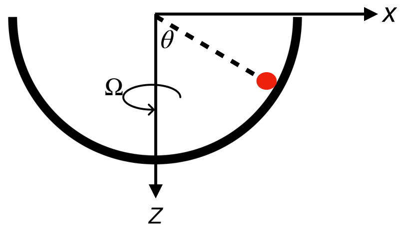
(iv) Now assume that at $t = 0$, $z = 0$, and $\dot{z} = u$. What is the criterion for the particle to escape to infinity?

Paper 4, Section II

10C Dynamics and Relativity

Consider an infinitely long ramp with semi-circular cross-section of radius R, as shown in the figure. Adopt a Cartesian coordinate system with the y-axis directed along the ramp, pointing out of the page, and the z-axis directed vertically downwards. The ramp rotates about the z-axis with constant angular velocity $\mathbf{\Omega} = -\Omega\hat{\mathbf{z}}$ and the coordinate system rotates with the ramp.

A ball of mass m and negligible size slides along the surface of the ramp without any friction but experiences a constant gravitational acceleration $\mathbf{g} = g\hat{\mathbf{z}}$. A line from the ball to the origin projected on to the xz-plane makes an angle θ with the z-axis, as shown in the figure.



(a) If $\mathbf{x} = (x, y, z)$ is the ball's position vector, its equation of motion is

$$\ddot{\mathbf{x}} = -2\mathbf{\Omega} \times \dot{\mathbf{x}} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}) + \mathbf{g} + \frac{1}{m}\mathbf{N},$$

where $\mathbf{N}$ is the normal force due to the ramp. What do the first two terms on the right side correspond to? Write down the equation's three Cartesian components.

(b) Using your results from part (a), or otherwise, show that

$$R\ddot{\theta} = -g\sin\theta + \Omega^2 R\cos\theta\sin\theta - 2\Omega\dot{y}\cos\theta,$$
$$\ddot{y} = \Omega^2 y + 2\Omega R\dot{\theta}\cos\theta.$$

Find all the solutions for which the ball is at rest in the rotating frame.

(c) Suppose that Ω is sufficiently small so that terms of order Ω^2 may be neglected and that at time $t = 0$, $\theta = \theta_0$, $\dot{\theta} = 0$, and $y = \dot{y} = 0$.

To linear order in small θ, show that the ball undergoes oscillations in θ and find their frequency. Determine the associated motion in y.

Paper 4, Section II
11C Dynamics and Relativity

(a) Define the moment of inertia of a rigid body V, of density ρ, rotating about a given axis.

A thin circular disc has radius r, thickness $\delta \ll r$, and uniform density ρ. Its centre of mass is at the origin of a Cartesian coordinate system whose z-axis is perpendicular to the disc's circular face. To leading order in small δ, find the disc's moment of inertia when it is rotating about:

(i) the x-axis,

(ii) a line of the form $y = 0$, $z = h$.

(b) Consider a cone with circular cross-section, base of radius R, height H, and uniform density ρ. The cone rotates about an axis that passes through its apex and which is perpendicular to its axis of symmetry.

(i) Using part (a)(ii), or otherwise, show that the cone's moment of inertia is $I = M(\alpha R^2 + \beta H^2)$, where M is the cone's mass, and α and β are constants you need to find.

[You may assume that the volume of the cone is $\frac{1}{3}\pi R^2 H$.]

(ii) If there is no friction and initially the cone has kinetic energy K_0, how long does it take to execute one full rotation?

(iii) Now suppose there is friction so that if the cone rotates by an angle $\Delta\theta$ the work done by the friction is equal to $W\Delta\theta$, where W is a constant.

If the cone initially has kinetic energy K_0, show that it comes to rest after a time

$$t = \sqrt{\frac{2K_0 I}{W^2}}.$$

Paper 4, Section II
12C Dynamics and Relativity

(a) State the definition of a four-vector U. Prove that $U \cdot U$ is the same in all inertial frames.

(b) Relative to an inertial reference frame S, a second inertial frame S' moves with constant three-velocity $\mathbf{V} = (V,\, 0,\, 0)$, and the two frames coincide when $t = t' = 0$.

A particle is travelling with a constant three-velocity $\mathbf{u} = (u_x,\, u_y,\, 0)$, as measured in frame S, and passes through the origin of S at $t = 0$.

(i) By considering the transformation of the particle's position vector in space-time, calculate $\mathbf{u}'$, the particle's three-velocity in S'.

(ii) Suppose that V/c is small. To leading order in V/c, show that

$$\mathbf{u}' = \mathbf{u} - \mathbf{V} + \frac{(\mathbf{V} \cdot \mathbf{u})}{c^2}\mathbf{u}.$$

(iii) A light source at the origin of frame S emits photons at an angle θ relative to the x-axis. According to an observer in frame S', the photons are emitted at an angle θ' relative to the x'-axis. Show

$$\theta' - \theta = \frac{V}{c} \sin\theta,$$

to leading order in small V/c.

(c) In the laboratory frame, a photon of wavelength λ collides with an electron of mass m, initially at rest. After the collision, the three-momenta of the photon and electron are collinear. Find the wavelength of the photon after the collision.

Paper 3, Section I
1E Groups

Let G and H be finite groups and $g \in G$.

Define the *order* of g.

Show that if $\phi\colon G \to H$ is a homomorphism then the order of $\phi(g)$ divides the order of g.

Show that if ϕ is surjective and H has an element of order m then G has an element of order m.

How many homomorphisms $C_9 \to S_4$ are there?

Paper 3, Section I
2E Groups

What does it mean to say a group is *abelian*? What does it mean to say a group is *cyclic*?

Show that every cyclic group is abelian. Show that not every abelian group is cyclic.

Recall that the *proper subgroups* of a group G are the subgroups of G not equal to G. If every proper subgroup of a group G is cyclic then must G be abelian? Justify your answer.

Paper 3, Section II
5E Groups

What does it mean for a group G to *act* on a set X. Given such an action and $x \in X$ define the *orbit* and *stabiliser* of x. State and prove the orbit–stabiliser theorem for a finite group.

State and prove Cauchy's theorem.

Suppose that G is a group of order 33. By considering the conjugation action of a subgroup of G on G, show that G must be cyclic.

Paper 3, Section II
6E Groups

What is a *Möbius transformation*?

Show carefully that if (z_1, z_2, z_3) and (w_1, w_2, w_3) are two ordered subsets of the extended complex plane $\widehat{\mathbb{C}}$, each consisting of three distinct points, then there is a unique Möbius transformation f such that $f(z_i) = w_i$ for $i = 1, 2, 3$. [You may assume that the Möbius transformations form a group under composition.]

Define the *cross-ratio* $[z_1, z_2, z_3, z_4]$ of four distinct points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$. Show that a bijection $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a Möbius transformation if and only if f preserves the cross-ratio of any four distinct points in $\widehat{\mathbb{C}}$; that is, if and only if

$$[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)]$$

for any four distinct points z_1, z_2, z_3, z_4 in $\widehat{\mathbb{C}}$.

Are there complex numbers a and b such that the map that sends z to $a\bar{z} + b$ for $z \in \mathbb{C}$ and fixes ∞ is Möbius? Justify your answer. [Here $\bar{z}$ denotes the complex conjugate of z.]

Paper 3, Section II
7E Groups

Suppose G is a group. What does it mean to say that a subset K of G is a *normal subgroup* of G? For N a normal subgroup of G explain how to define the *quotient group* G/N. Briefly explain why G/N is a group.

Define the *kernel* and the *image* of a group homomorphism. Show that a subset K of G is a normal subgroup of G if and only if there is a group H and a group homomorphism $\theta : G \to H$ such that K is the kernel of θ. Show moreover that in this case the image of θ is a subgroup of H and G/K is isomorphic to the image of θ.

By defining a suitable group homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{C}\backslash\{0\}, \cdot)$, show that $\mathbb{R}/\mathbb{Z}$ is isomorphic to the subgroup of $(\mathbb{C}\backslash\{0\}, \cdot)$ consisting of complex numbers of modulus 1. What characterises the elements of the image of $\mathbb{Q}/\mathbb{Z}$ under this isomorphism?

Paper 3, Section II

8E Groups

Show that the set $S(\mathbb{N})$ of invertible functions $\tau\colon \mathbb{N} \to \mathbb{N}$ is a group under composition. Show that the subset $S^{\text{fin}}(\mathbb{N})$ of invertible functions $\tau\colon \mathbb{N} \to \mathbb{N}$ such that there is some $n \geqslant 1$ with $\tau(m) = m$ for all $m > n$ is a subgroup of $S(\mathbb{N})$.

A *cycle* is a non-identity element σ of $S^{\text{fin}}(\mathbb{N})$ such that for every $m, n \in \mathbb{N}$ either $\sigma(m) = m$ or $\sigma(n) = n$ or there is an integer a such that $\sigma^a(m) = n$. Show that if σ is a cycle and $n \in \mathbb{N}$ such that $\sigma(n) \neq n$ then the order of σ is the least positive integer l such that $\sigma^l(n) = n$. Show in particular that the order of σ is always finite.

Show that every element τ of $S^{\text{fin}}(\mathbb{N})$ can be written as a product of cycles $\sigma_1 \cdots \sigma_k$ such that for every $1 \leqslant i < j \leqslant k$ and every $n \in \mathbb{N}$ either $\sigma_i(n) = n$ or $\sigma_j(n) = n$ (or both). Show moreover that $\sigma_i \sigma_j = \sigma_j \sigma_i$ for all $1 \leqslant i < j \leqslant k$. What is the relationship between the order of τ and the orders of $\sigma_1, \ldots, \sigma_k$? Justify your answer.

Paper 4, Section I
1E Numbers and Sets

By considering numbers of the form $3p_1 \ldots p_k - 1$, show that there are infinitely many primes of the form $3n+2$ with $n \in \mathbb{N}$.

For which primes p is the number $2p^2+1$ also prime? Justify your answer.

Paper 4, Section I
2D Numbers and Sets

Prove that $\sqrt[3]{2} + \sqrt[3]{3}$ is irrational.

Using the fact that the number $e - e^{-1}$ can be represented by a convergent series $2\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$, prove that $e - e^{-1}$ is irrational.

What is a *transcendental* number? Given that e is transcendental, show that $ae + be^{-1}$ is also transcendental for any integers a, b that are not both zero.

Paper 4, Section II
5E Numbers and Sets

State Bezout's theorem. Suppose that $p \in \mathbb{N}$ is prime and $a, b \in \mathbb{N}$. Show that if p divides ab then p divides a or p divides b.

Show that if $m, n \in \mathbb{N}$ are coprime then any pair of congruences of the form

$$x \equiv a \mod m \quad \text{and} \quad x \equiv b \mod n$$

has a unique simultaneous solution modulo mn.

Show that if p is an odd prime and $d \in \mathbb{N}$ then there are precisely 2 solutions of $x^2 \equiv 1$ modulo p^d. Deduce that if $n \geqslant 3$ is odd, then the number of solutions of $x^2 \equiv 1$ modulo n is equal to 2^k, where k denotes the number of distinct prime factors of n.

How many solutions of $x^2 \equiv 1$ modulo n are there when $n = 2^d$?

Paper 4, Section II

6D Numbers and Sets

(a) Define the binomial coefficient $\binom{n}{k}$ for $0 \leqslant k \leqslant n$. Show from your definition that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ holds when both sides are well-defined.

(b) Prove the following special case of the binomial theorem: $(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k$ for any real number t. By integrating this expression over a suitable range, or otherwise, evaluate $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$ and $\sum_{k=0}^{n} \frac{(-1)^k}{k+1}\binom{n}{k}$.

Deduce that $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}\binom{n}{k} = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$.

(c) The Fibonacci numbers are defined by

$$F_1 = 1, \qquad F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n \quad \text{for } n \geqslant 1.$$

By using induction, or otherwise, prove that

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}$$

for all $n \geqslant 1$, where $\lfloor \frac{n-1}{2} \rfloor$ denotes the largest integer less than or equal to $\frac{n-1}{2}$.

Paper 4, Section II
7F Numbers and Sets

For a given natural number $n \geqslant 2$, let S be the set of ordered real n-tuples $x = (x_1, \ldots, x_n)$ where $x_i \geqslant 0$ for $1 \leqslant i \leqslant n$. For $x \in S$, let

$$P(x) = \{i : x_i > 0\}.$$

Define the relation $\preceq$ by

$$x \preceq y \text{ if and only if } P(x) \subseteq P(y).$$

(a) Is the relation $\preceq$ reflexive? Is it transitive? Is it symmetric? Justify your answers.

(b) Show that $x \preceq y$ if and only if there exists $z \in S$ such that $x_i = y_i z_i$ for all $1 \leqslant i \leqslant n$.

(c) Define the relation $\sim$ by

$$x \sim y \text{ if and only if } x \preceq y \text{ and } y \preceq x.$$

Show that $\sim$ defines an equivalence relation on S. Into how many equivalence classes does $\sim$ partition S?

(d) Define the relation $\perp$ by

$$x \perp y \text{ if and only if } P(x) \cap P(y) = \emptyset.$$

Given $s \in S$, show that for every $x \in S$ there exist unique $y, z \in S$ such that $x = y + z$ where $y \preceq s$ and $z \perp s$.

Paper 4, Section II
8F Numbers and Sets

(a) What does it mean to say a set is *countable*?

(b) Show from first principles that the following sets are countable:

(i) the Cartesian product $\mathbb{N} \times \mathbb{N}$, where $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers,

(ii) the rational numbers,

(iii) the points of discontinuity of an increasing function $F : \mathbb{R} \to \mathbb{R}$.

(c) Let $A_1, A_2, \ldots$ be a collection of non-empty countable sets and consider the Cartesian product

$$B = A_1 \times A_2 \times \cdots .$$

Show from first principles that B is countable if and only if there exists a natural number N such that $|A_n| = 1$ for all $n > N$.

Paper 2, Section I

3F Probability

What does it mean to say a function is *convex*? State Jensen's inequality for a convex function f and an integrable random variable X.

Let $x_1, \ldots, x_n$ be positive real numbers. Show that

$$\frac{\sum_{i=1}^n x_i \log x_i}{\sum_{i=1}^n x_i} \geqslant \log\left(\frac{\sum_{i=1}^n x_i}{n}\right).$$

[You may use without proof a standard sufficient condition for convexity if it is stated carefully.]

Paper 2, Section I

4F Probability

Let X be a random variable with mean μ and variance σ^2. Let

$$G(a) = \mathbb{E}\big[(X-a)^2\big].$$

Show that $G(a) \geqslant \sigma^2$ for all a. For what value of a is there equality?

Let

$$H(a) = \mathbb{E}\big[|X-a|\big].$$

Supposing that X is a continuous random variable with probability density function f, express $H(a)$ in terms of f. Show that H is minimised for a such that $\int_{-\infty}^{a} f(x)dx = 1/2$.

Paper 2, Section II
9F Probability

(a) Let U and V be two *bounded* random variables such that $\mathbb{E}[U^k] = \mathbb{E}[V^k]$ for all non-negative integers k. Show that U and V have the same moment generating function.

(b) Let X be a continuous random variable with probability density function

$$f(x) = Ae^{-x^2/2}$$

for all real x, where A is a normalising constant. Compute the moment generating function of X.

(c) Let Y be a discrete random variable with probability mass function

$$\mathbb{P}(Y = n) = Be^{-n^2/2}$$

for all integers n, where B is a normalising constant. Show that

$$\mathbb{E}[e^{kY}] = \mathbb{E}[e^{kX}]$$

for all integers k, where X is a standard normal random variable.

(d) Let U and V be *unbounded* random variables such that U^k and V^k are integrable and $\mathbb{E}[U^k] = \mathbb{E}[V^k]$ for all non-negative integers k. Does it follow that U and V have the same distribution?

Paper 2, Section II
10F Probability

(a) Let X be a random variable valued in $\{1, 2, \ldots\}$ and let G_X be its probability generating function. Show that

$$\mathbb{P}(X = n) = \frac{G_X^{(n)}(0)}{n!}$$

where $G_X^{(n)}$ denotes the nth derivative of G_X.

(b) Let Y be another random variable valued in $\{1, 2, \ldots\}$, independent of X. Prove that $G_{X+Y}(s) = G_X(s)G_Y(s)$ for all $0 \leqslant s \leqslant 1$.

(c) Compute G_X in the case where X is a geometric random variable taking values in $\{1, 2, \ldots\}$ with $\mathbb{P}(X = 1) = p$ for a given constant $0 < p \leqslant 1$.

(d) A jar contains n marbles. Initially, all of the marbles are red. Every minute, a marble is drawn at random from the jar, and then replaced with a blue marble. Let T be the number of minutes until the jar contains only blue marbles. Compute the probability generating function G_T.

Paper 2, Section II
11F Probability

Consider a coin that is biased such that when tossed the probability of heads is p and tails is $1-p$.

(a) Suppose that the coin was tossed n times. What is the probability that the coin came up heads exactly k times?

(b) Suppose that the coin was tossed n times. Given that the coin came up heads exactly k times, what is the probability that the coin came up heads k times in a row?

(c) Suppose that the coin was tossed repeatedly until heads came up k times. What is the probability that the total number of tosses was n?

(d) Suppose that the coin was tossed repeatedly until heads came up k times in a row. Find the expected number of tosses.

Paper 2, Section II
12F Probability

Let $A_1, A_2, \ldots$ be a collection of events. Let $N = \sum_{n \geqslant 1} \mathbf{1}_{A_n}$ be the random variable that counts how many of these events occur. Note that N takes values in $\{0, 1, \ldots\} \cup \{\infty\}$.

(a) By considering the quantity $\mathbb{E}(N)$, show that if $\sum_{n \geqslant 1} \mathbb{P}(A_n) < \infty$ then $\mathbb{P}(\text{an infinite number of the events occur}) = 0$.

(b) Suppose now that the events are independent. Show the inequality $\mathbb{E}(2^{-N}) \leqslant e^{-\frac{1}{2}\mathbb{E}(N)}$, with the convention that $2^{-\infty} = 0$. [*Hint: use the inequality* $1 - x \leqslant e^{-x}$ *for all* x.]

(c) Again suppose that the events are independent. Show that if $\sum_{n \geqslant 1} \mathbb{P}(A_n) = \infty$ then $\mathbb{P}(\text{an infinite number of the events occur}) = 1$.

(d) A monkey types by randomly striking keys on a 26-letter keyboard, with each letter of the alphabet equally likely to be struck and the keystrokes independent. Show that with probability one, the word HELLO appears infinitely often.

Paper 3, Section I
3A Vector Calculus

Let D be the region in the positive quadrant of the xy plane defined by

$$y \leqslant x \leqslant \alpha y\,, \qquad \frac{1}{y} \leqslant x \leqslant \frac{\alpha}{y}\,,$$

where $\alpha > 1$ is a constant. By using the change of variables $u = x/y$, $v = xy$, or otherwise, evaluate

$$\int_D x^2 \, dx \, dy\ .$$

Paper 3, Section I
4A Vector Calculus

Consider the curve in $\mathbb{R}^3$ defined by $y = \log x$, $z = 0$. Using a parametrization of your choice, find an expression for the unit tangent vector $\mathbf{t}$ at a general point on the curve. Calculate the curvature κ as a function of your chosen parameter. Hence find the maximum value of κ and the point on the curve at which it is attained.

[*You may assume that* $\kappa = |\,\mathbf{t} \times (d\mathbf{t}/ds)\,|$ *where* s *is the arc-length.*]

Paper 3, Section II
9A Vector Calculus

(a) Using Cartesian coordinates x_i in $\mathbb{R}^3$, write down an expression for $\partial r/\partial x_i$, where r is the radial coordinate ($r^2 = x_i x_i$), and deduce that

$$\nabla \cdot (\, g(r)\mathbf{x}\,) \;=\; r g'(r) + 3g(r)$$

for any differentiable function $g(r)$.

(b) For spherical polar coordinates r, θ, ϕ satisfying

$$x_1 = r\sin\theta\cos\phi\,, \quad x_2 = r\sin\theta\sin\phi\,, \quad x_3 = r\cos\theta\,,$$

find scalars h_r, h_θ, h_ϕ and unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ such that

$$d\mathbf{x} \;=\; h_r\,\mathbf{e}_r\,dr \;+\; h_\theta\,\mathbf{e}_\theta\,d\theta \;+\; h_\phi\,\mathbf{e}_\phi\,d\phi\;.$$

Hence, using the relation $df = d\mathbf{x}\cdot\nabla f$, find an expression for ∇f in spherical polars for any differentiable function $f(\mathbf{x})$.

(c) Consider the vector fields

$$\mathbf{A}^+ \;=\; \frac{1}{r}\tan\frac{\theta}{2}\,\mathbf{e}_\phi \quad (r \neq 0,\, \theta \neq \pi)\,, \qquad \mathbf{A}^- \;=\; -\frac{1}{r}\cot\frac{\theta}{2}\,\mathbf{e}_\phi \quad (r \neq 0,\, \theta \neq 0)\,.$$

Compute $\nabla\times\mathbf{A}^+$ and $\nabla\times\mathbf{A}^-$ and use the result in part (a) to check explicitly that your answers have zero divergence.

[*You may use without proof the formula* $\nabla\times\mathbf{A} \;=\; \frac{1}{h_r h_\theta h_\phi}\begin{vmatrix} h_r\mathbf{e}_r & h_\theta\mathbf{e}_\theta & h_\phi\mathbf{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ h_r A_r & h_\theta A_\theta & h_\phi A_\phi \end{vmatrix}\,.$]

(d) From your answers in part (c), explain briefly on general grounds why

$$\mathbf{A}^+ \;-\; \mathbf{A}^- \;=\; \nabla f$$

for some function $f(\mathbf{x})$. Find a solution for f that is defined on the region $x_1 > 0$.

Paper 3, Section II
10A Vector Calculus

Let H be the unbounded surface defined by $x^2 + y^2 = z^2 + 1$, and S the bounded surface defined as the subset of H with $1 \leqslant z \leqslant \sqrt{2}$. Calculate the vector area element $d\mathbf{S}$ on S in terms of ρ and ϕ, where $x = \rho \cos\phi$ and $y = \rho \sin\phi$. Sketch the surface and indicate the sense of the corresponding normal.

Compute directly

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

where $\mathbf{A} = (-yz^2, xz^2, 0)$. Now verify your answer using Stokes' Theorem.

What is the value of

$$\int_{S'} \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

where S' is defined as the subset of H with $-1 \leqslant z \leqslant \sqrt{2}$? Justify your answer.

Paper 3, Section II
11A Vector Calculus

Let V be a region in $\mathbb{R}^3$ with boundary a closed surface S. Consider a function ϕ defined in V that satisfies

$$\nabla^2\phi - m^2\phi = 0$$

for some constant $m \geqslant 0$.

(i) If $\partial\phi/\partial n = g$ on S, for some given function g, show that ϕ is unique provided that $m > 0$. Does this conclusion change if $m = 0$?

[*Recall:* $\partial/\partial n = \mathbf{n} \cdot \nabla$, *where* $\mathbf{n}$ *is the outward pointing unit normal on* S.]

(ii) Now suppose instead that $\phi = f$ on S, for some given function f. Show that for any function ψ with $\psi = f$ on S,

$$\int_V (\, |\nabla\psi|^2 + m^2\psi^2 \,)\, dV \;\geqslant\; \int_V (\, |\nabla\phi|^2 + m^2\phi^2 \,)\, dV \;.$$

What is the condition for equality to be achieved, and is this result sufficient to deduce that ϕ is unique? Justify your answers, distinguishing carefully between the cases $m > 0$ and $m = 0$.

Paper 3, Section II
12A Vector Calculus

Consider a rigid body B of uniform density ρ and total mass M rotating with constant angular velocity $\boldsymbol{\omega}$ relative to a point $\mathbf{a}$. The angular momentum $\mathbf{L}$ about $\mathbf{a}$ is defined by

$$\mathbf{L} = \int_B (\mathbf{x}-\mathbf{a}) \times [\boldsymbol{\omega} \times (\mathbf{x}-\mathbf{a})]\, \rho\, dV\,,$$

and the inertia tensor $I_{ij}(\mathbf{a})$ about $\mathbf{a}$ is defined by the relation

$$L_i = I_{ij}(\mathbf{a})\, \omega_j\,.$$

(a) Given that $\mathbf{L}$ is a vector for any choice of the vector $\boldsymbol{\omega}$, show from first principles that $I_{ij}(\mathbf{a})$ is indeed a tensor, of rank 2.

Assuming that the centre of mass of B is located at the origin $\mathbf{0}$, so that

$$\int_B x_i\, dV \;=\; 0\,,$$

show that

$$I_{ij}(\mathbf{a}) \;=\; I_{ij}(\mathbf{0}) \;+\; M(\, a_k a_k \delta_{ij} - a_i a_j\,)\,,$$

and find an explicit integral expression for $I_{ij}(\mathbf{0})$.

(b) Now suppose that B is a cube centred at $\mathbf{0}$ with edges of length ℓ parallel to the coordinate axes, *i.e.* B occupies the region $-\frac{1}{2}\ell \leqslant x_i \leqslant \frac{1}{2}\ell$. Using symmetry, explain in outline why $I_{ij}(\mathbf{0}) \;=\; \lambda\, \delta_{ij}$ for some constant λ.

Given that $\lambda = M\ell^2/6$, find $I_{ij}(\mathbf{a})$ when $\mathbf{a} = \frac{1}{2}\ell(1,1,0)$, writing the result in matrix form. Hence, or otherwise, show that if the cube is rotating relative to $\mathbf{a}$ with $|\boldsymbol{\omega}| = 1$ then, depending on the direction of the angular velocity, $|\mathbf{L}|$ has a maximum value that is four times larger than its minimum value.

Paper 1, Section I
1B Vectors and Matrices

(a) Consider the equation

$$|z-a|+|z-b|=c\,,$$

for $z \in \mathbb{C}$, where $a, b \in \mathbb{C}$, $a \neq b$, and $c \in \mathbb{R}$, $c > 0$.

For each of the following cases, either show the equation has no solutions for z or give a rough sketch of the set of solutions:

(i) $c < |a-b|$,

(ii) $c = |a-b|$,

(iii) $c > |a-b|$.

(b) Let ω be the solution to $\omega^3 = 1$ with $\mathrm{Im}(\omega) > 0$. Calculate the following:

(i) $(1+\omega)^{10^6}$,

(ii) all values of $(1+\omega)^{1+\omega}$.

Paper 1, Section I
2B Vectors and Matrices

Consider the equation

$$M^T J M = J\,, \tag{$*$}$$

where M is a 2×2 real matrix and

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) (i) What are the possible values of $\det(M)$ if M satisfies $(*)$? Justify your answer.

(ii) Suppose that $(*)$ holds when $M = M_1$ and when $M = M_2$. Show that $(*)$ also holds when $M = M_1 M_2$ and when $M = (M_1)^{-1}$.

(b) Show that if M satisfies $(*)$ and its first entry satisfies $M_{11} > 0$ then M takes one of the forms

$$\begin{pmatrix} a(u) & b(u) \\ c(u) & d(u) \end{pmatrix}, \qquad \begin{pmatrix} a(u) & -b(u) \\ c(u) & -d(u) \end{pmatrix},$$

where $u \in \mathbb{R}$ and $a(u), b(u), c(u), d(u)$ are hyperbolic functions whose form you should determine.

Paper 1, Section II
5B Vectors and Matrices

Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ be vectors in $\mathbb{R}^3$.

(a) (i) Define the scalar product $\mathbf{A} \cdot \mathbf{B}$ and the vector product $\mathbf{A} \times \mathbf{B}$, expressing the products in terms of vector components.

(ii) Obtain expressions for $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ as linear combinations of $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$.

(iii) Now suppose that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent. By considering the expression $(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D})$, obtain an expression for $\mathbf{D}$ as a linear combination of $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

(b) Again suppose that the vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are linearly independent, and that they are position vectors of points on a sphere S that passes through the origin $\mathbf{O}$. By writing the position vector of the centre of S in the form

$$\mathbf{P} = \alpha \mathbf{A} + \beta \mathbf{B} + \gamma \mathbf{C} \,,$$

obtain three linear equations for α, β, γ in terms of $\mathbf{A}, \mathbf{B}, \mathbf{C}$. Hence find $\mathbf{P}$ when $\mathbf{A} = (1,0,0), \mathbf{B} = (1,1,0), \mathbf{C} = (0,1,2)$ in Cartesian coordinates.

Paper 1, Section II
6B Vectors and Matrices

(a) (i) Find, with brief justification, the 2×2 matrix R representing an anticlockwise rotation through angle θ in the xy-plane and the 2×2 matrix M representing reflection in the x-axis in the xy-plane.

(ii) Show that $MRM = R^a$, where a is an integer that you should determine.

(iii) Can $R^a = MR^b$ for some integers a, b? Justify your answer.

(b) Now let $n \geqslant 3$ be an integer and $\theta = \frac{2\pi}{n}$. Consider matrices of the form

$$M^{m_1} R^{n_1} M^{m_2} R^{n_2} \ldots M^{m_k} R^{n_k} \,,$$

where $k \geqslant 1$ is an integer and $m_i \geqslant 0, n_i \geqslant 0$ are integers for $i = 1, \ldots, k$.

Show that there are precisely $2n$ distinct matrices of this form, and give explicit expressions for them as 2×2 matrices.

Paper 1, Section II
7B Vectors and Matrices

An $n\times n$ complex matrix P is called an *orthogonal projection matrix* if $P^2 = P = P^\dagger$, where † denotes the Hermitian conjugate. An $n \times n$ complex matrix A is *positive semidefinite* if $(\mathbf{x}, A\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in \mathbb{C}^n$. [Recall that the standard inner product on $\mathbb{C}^n$ is defined by $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\dagger \mathbf{y}$.]

(a) Show that the eigenvalues of any $n \times n$ Hermitian matrix are real.

(b) Show that every $n \times n$ orthogonal projection matrix is positive semidefinite.

(c) If P is an $n \times n$ orthogonal projection matrix, show that every vector $\mathbf{v} \in \mathbb{C}^n$ can be written in the form $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$, where $\mathbf{v}_0$ is in the kernel of P, $P\mathbf{v}_1 = \mathbf{v}_1$ and $(\mathbf{v}_0, \mathbf{v}_1) = 0$.

(d) If A and B are distinct $n\times n$ Hermitian matrices, show that there is an orthogonal projection matrix P such that $\mathrm{Tr}(PA) \neq \mathrm{Tr}(PB)$.

(e) If P and Q are $n\times n$ orthogonal projection matrices, is PQ necessarily a positive semidefinite matrix? Justify your answer.

Paper 1, Section II
8B Vectors and Matrices

Let M be an $n \times n$ complex matrix with columns $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbb{C}^n$. Verify that $\mathbf{c}_1 = M\mathbf{e}_1, \ldots, \mathbf{c}_n = M\mathbf{e}_n$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors for $\mathbb{C}^n$. If P is also an $n \times n$ complex matrix, show that PM has columns $P\mathbf{c}_1, \ldots, P\mathbf{c}_n$.

For an $n \times n$ complex matrix A with characteristic polynomial

$$\chi_A(x) = (-1)^n (x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0)\,,$$

consider the matrix

$$C = \begin{pmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & \ldots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -a_{n-1} \end{pmatrix}.$$

Show that there exists an invertible matrix S such that

$$S^{-1}AS = C$$

if and only if there is a vector $\mathbf{v} \in \mathbb{C}^n$ such that $\mathbf{v}, A\mathbf{v}, \ldots, A^{n-1}\mathbf{v}$ are linearly independent. You may assume that $\chi_A(A) = 0$ (the Cayley-Hamilton theorem).

[*Hint: consider the columns of* S.]

END OF PAPER