UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday, 10 June, 2021 10:00am to 1:00pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Define the *Möbius function* μ, and explain what it means for it to be *multiplicative.*

Show that for every positive integer n

$$\sum_{d|n} \frac{\mu(d)^2}{\phi(d)} = \frac{n}{\phi(n)},$$

where ϕ is the Euler totient function.

Fix an integer $k \geqslant 1$. Use the Chinese remainder theorem to show that there are infinitely many positive integers n for which

$$\mu(n) = \mu(n+1) = \cdots = \mu(n+k).$$

2H Topics in Analysis

Let Ω be a non-empty bounded open set in $\mathbb{R}^2$ with closure $\overline{\Omega}$ and boundary $\partial\Omega$ and let $\phi : \overline{\Omega} \to \mathbb{R}$ be a continuous function. Give a proof or a counterexample for each of the following assertions.

(i) If ϕ is twice differentiable on Ω with $\nabla^2\phi(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Omega$, then there exists an $\mathbf{x}_0 \in \partial\Omega$ with $\phi(\mathbf{x}_0) \geqslant \phi(\mathbf{x})$ for all $\mathbf{x} \in \overline{\Omega}$.

(ii) If ϕ is twice differentiable on Ω with $\nabla^2\phi(\mathbf{x}) < 0$ for all $\mathbf{x} \in \Omega$, then there exists an $\mathbf{x}_0 \in \partial\Omega$ with $\phi(\mathbf{x}_0) \geqslant \phi(\mathbf{x})$ for all $\mathbf{x} \in \overline{\Omega}$.

(iii) If ϕ is four times differentiable on Ω with

$$\frac{\partial^4\phi}{\partial x^4}(\mathbf{x}) + \frac{\partial^4\phi}{\partial y^4}(\mathbf{x}) > 0$$

for all $\mathbf{x} \in \Omega$, then there exists an $\mathbf{x}_0 \in \partial\Omega$ with $\phi(\mathbf{x}_0) \geqslant \phi(\mathbf{x})$ for all $\mathbf{x} \in \overline{\Omega}$.

(iv) If ϕ is twice differentiable on Ω with $\nabla^2\phi(\mathbf{x}) = 0$ for all $\mathbf{x} \in \Omega$, then there exists an $\mathbf{x}_0 \in \partial\Omega$ with $\phi(\mathbf{x}_0) \geqslant \phi(\mathbf{x})$ for all $\mathbf{x} \in \overline{\Omega}$.

3K Coding and Cryptography

State Shannon's noisy coding theorem for a binary symmetric channel, defining the terms involved.

Suppose a channel matrix, with output alphabet of size n, is such that the entries in each row are the elements of the set $\{p_1, \ldots, p_n\}$ in some order. Further suppose that all columns are permutations of one another. Show that the channel's information capacity C is given by

$$C = \log n + \sum_{i=1}^{n} p_i \log p_i \, .$$

Show that the information capacity of the channel matrix

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

is given by $C = \frac{5}{3} - \log 3$.

4F Automata and Formal Languages

Assuming the definition of a deterministic finite-state automaton (DFA) $D = (Q, \Sigma, \delta, q_0, F)$, what is the *extended transition function* $\hat{\delta}$ for D? Also assuming the definition of a nondeterministic finite-state automaton (NFA) N, what is $\hat{\delta}$ in this case?

Define the *languages* accepted by D and N, respectively, in terms of $\hat{\delta}$.

Given an NFA N as above, describe the *subset construction* and show that the resulting DFA $\overline{N}$ accepts the same language as N. If N has one accept state then how many does $\overline{N}$ have?

5J Statistical Modelling

Define a *generalised linear model* for a sample $Y_1, \ldots, Y_n$ of independent random variables. Define further the concept of the *link function*. Define the *binomial regression model* (without the dispersion parameter) with logistic and probit link functions. Which of these is the canonical link function?

6E Mathematical Biology

Consider a stochastic birth–death process in a population of size $n(t)$, where deaths occur in pairs for $n \geqslant 2$. The probability per unit time of a birth, $n \to n+1$ for $n \geqslant 0$, is b, that of a pair of deaths, $n \to n-2$ for $n \geqslant 2$, is dn, and that of the death of a lonely singleton, $1 \to 0$, is D.

(a) Write down the master equation for $p_n(t)$, the probability of a population of size n at time t, distinguishing between the cases $n \geqslant 2$, $n = 0$ and $n = 1$.

(b) For a function $f(n)$, $n \geqslant 0$, show carefully that

$$\frac{d}{dt}\langle f(n)\rangle = b\sum_{n=0}^{\infty}(f_{n+1} - f_n)p_n - d\sum_{n=2}^{\infty}(f_n - f_{n-2})np_n - D(f_1 - f_0)p_1 \ ,$$

where $f_n = f(n)$.

(c) Deduce the evolution equation for the mean $\mu(t) = \langle n\rangle$, and simplify it for the case $D = 2d$.

(d) For the same value of D, show that

$$\frac{d}{dt}\langle n^2\rangle = b(2\mu + 1) - 4d\big(\langle n^2\rangle - \mu\big) - 2dp_1$$

Deduce that the variance σ^2 in the stationary state for $b, d > 0$ satisfies

$$\frac{3b}{4d} - \frac{1}{2} < \sigma^2 < \frac{3b}{4d} \, .$$

7E Further Complex Methods

The function $w(z)$ satisfies the differential equation

$$\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0\,, \qquad (\dagger)$$

where $p(z)$ and $q(z)$ are complex analytic functions except, possibly, for isolated singularities in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ (the extended complex plane).

(a) Given equation ($\dagger$), state the conditions for a point $z_0 \in \mathbb{C}$ to be

(i) an *ordinary point*,

(ii) a *regular singular point*,

(iii) an *irregular singular point*.

(b) Now consider $z_0 = \infty$ and use a suitable change of variables $z \to t$, with $y(t) = w(z)$, to rewrite ($\dagger$) as a differential equation that is satisfied by $y(t)$. Hence, deduce the conditions for $z_0 = \infty$ to be

(i) an ordinary point,

(ii) a regular singular point,

(iii) an irregular singular point.

[In each case, you should express your answer in terms of the functions p and q.]

(c) Use the results above to prove that any equation of the form ($\dagger$) must have at least one singular point in $\overline{\mathbb{C}}$.

8D Classical Dynamics

Show that, in a uniform gravitational field, the net gravitational torque on a system of particles, about its centre of mass, is zero.

Let S be an inertial frame of reference, and let S' be the frame of reference with the same origin and rotating with angular velocity $\boldsymbol{\omega}(t)$ with respect to S. You may assume that the rates of change of a vector $\mathbf{v}$ observed in the two frames are related by

$$\left(\frac{d\mathbf{v}}{dt}\right)_S = \left(\frac{d\mathbf{v}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{v}\,.$$

Derive Euler's equations for the torque-free motion of a rigid body.

Show that the general torque-free motion of a symmetric top involves precession of the angular-velocity vector about the symmetry axis of the body. Determine how the direction and rate of precession depend on the moments of inertia of the body and its angular velocity.

9B Cosmology

(a) The generalised Boltzmann distribution $P(\mathbf{p})$ is given by

$$P(\mathbf{p}) = \frac{e^{-\beta(E_{\mathbf{p}}\, n_{\mathbf{p}} - \mu\, n_{\mathbf{p}})}}{\mathcal{Z}_{\mathbf{p}}}\,,$$

where $\beta = (k_B T)^{-1}$, μ is the chemical potential,

$$\mathcal{Z}_{\mathbf{p}} = \sum_{n_{\mathbf{p}}} e^{-\beta(E_{\mathbf{p}}\, n_{\mathbf{p}} - \mu\, n_{\mathbf{p}})}, \quad E_{\mathbf{p}} = \sqrt{m^2c^4 + p^2c^2} \quad \text{and} \quad p = |\mathbf{p}|\,.$$

Find the average particle number $\langle N(\mathbf{p})\rangle$ with momentum $\mathbf{p}$, assuming that all particles have rest mass m and are either

(i) bosons, or

(ii) fermions.

(b) The photon total number density n_γ is given by

$$n_\gamma = \frac{2\zeta(3)}{\pi^2\hbar^3c^3}(k_B\, T)^3\,,$$

where $\zeta(3) \approx 1.2$. Consider now the fractional ionisation of hydrogen

$$X_e = \frac{n_e}{n_e + n_H}\,.$$

In our universe $n_e + n_H = n_p + n_H \approx \eta\, n_\gamma$, where $\eta \sim 10^{-9}$ is the baryon-to-photon number density. Find an expression for the ratio

$$\frac{1 - X_e}{X_e^2}$$

in terms of η, $(k_B\, T)$, the electron mass m_e, the speed of light c and the ionisation energy of hydrogen $I \approx 13.6\,\text{eV}$.

One might expect neutral hydrogen to form at a temperature $k_B\, T \sim I$, but instead in our universe it happens at the much lower temperature $k_B\, T \approx 0.3\,\text{eV}$. Briefly explain why this happens.

[*You may use without proof the Saha equation*

$$\frac{n_H}{n_e^2} = \left(\frac{2\pi\hbar^2}{m_e\, k_B\, T}\right)^{3/2} e^{\beta I}\,,$$

for chemical equilibrium in the reaction $e^- + p^+ \leftrightarrow H + \gamma$.]

10D Quantum Information and Computation

Let $\mathcal{B}_n$ denote the set of all n-bit strings and let $f : \mathcal{B}_n \to \mathcal{B}_1$ be a Boolean function which obeys either

(I) $f(x) = 0$ for all $x \in \mathcal{B}_n$, or
(II) $f(x) = 0$ for exactly half of all $x \in \mathcal{B}_n$.

Suppose we are given the n-qubit state

$$|\,\xi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \mathcal{B}_n} (-1)^{f(x)} \, |x\rangle \ .$$

Show how we may determine with certainty whether f is of case (I) or case (II).

Suppose now that Alice and Bob are separated in space. Alice possesses a quantum oracle for a Boolean function $f_A : \mathcal{B}_n \to \mathcal{B}_1$ and Bob similarly possess a quantum oracle for a Boolean function $f_B : \mathcal{B}_n \to \mathcal{B}_1$. These functions are arbitrary, except that either

(1) $f_A(x) = f_B(x)$ for all $x \in \mathcal{B}_n$, or
(2) $f_A(x) = f_B(x)$ for exactly half of all $x \in \mathcal{B}_n$.

Alice and Bob each have available a supply of qubits in state $|0\rangle$ and each can apply local quantum operations (including their own function oracle) to any qubits in their possession. Additionally, they can send qubits to each other.

Show how Bob may decide with certainty which case applies, after he has received n qubits from Alice. [*Hint: You may find it helpful to consider the function* $h(x) = f_A(x) \oplus f_B(x)$, *where* $\oplus$ *denotes addition mod 2.*]

SECTION II

11H Topics in Analysis

Let $r : [-1,1] \to \mathbb{R}$ be a continuous function with $r(x) > 0$ for all but finitely many values of x.

(a) Show that

$$\langle u, v \rangle = \int_{-1}^{1} u(x)v(x)r(x)\,dx \qquad (*)$$

defines an inner product on $C([-1,1])$.

(b) Show that for each n there exists a polynomial P_n of degree exactly n which is orthogonal, with respect to the inner product $(*)$, to all polynomials of lower degree.

(c) Show that P_n has n simple zeros $\omega_1(n)$, $\omega_2(n)$, $\ldots$, $\omega_n(n)$ on $[-1,1]$.

(d) Show that for each n there exist unique real numbers $A_j(n)$, $1 \leqslant j \leqslant n$, such that whenever Q is a polynomial of degree at most $2n-1$,

$$\int_{-1}^{1} Q(x)r(x)\,dx = \sum_{j=1}^{n} A_j(n) Q\big(\omega_j(n)\big).$$

(e) Show that

$$\sum_{j=1}^{n} A_j(n) f\big(\omega_j(n)\big) \to \int_{-1}^{1} f(x)r(x)\,dx$$

as $n \to \infty$ for all $f \in C([-1,1])$.

(f) If $R > 1$, $K > 0$, a_m is real with $|a_m| \leqslant KR^{-m}$ and $f(x) = \displaystyle\sum_{m=1}^{\infty} a_m x^m$, show that

$$\left| \int_{-1}^{1} f(x)r(x)\,dx - \sum_{j=1}^{n} A_j(n) f\big(\omega_j(n)\big) \right| \leqslant \frac{2KR^{-2n+1}}{R-1} \int_{-1}^{1} r(x)\,dx.$$

(g) If $r(x) = (1-x^2)^{1/2}$ and $P_n(0) = 1$, identify P_n (giving brief reasons) and the $\omega_j(n)$. [*Hint: A change of variable may be useful.*]

12K Coding and Cryptography

(a) Define what it means to say that C is a *binary cyclic code*. Explain the bijection between the set of binary cyclic codes of length n and the factors of $X^n - 1$ in $\mathbb{F}_2[X]$.

(b) What is a *linear feedback shift register*?

Suppose that $M : \mathbb{F}_2^d \to \mathbb{F}_2^d$ is a linear feedback shift register. Further suppose $\mathbf{0} \neq \mathbf{x} \in \mathbb{F}_2^d$ and k is a positive integer such that $M^k \mathbf{x} = \mathbf{x}$. Let H be the $d \times k$ matrix $(\mathbf{x}, M\mathbf{x}, \ldots, M^{k-1}\mathbf{x})$. Considering H as a parity check matrix of a code C, show that C is a binary cyclic code.

(c) Suppose that C is a binary cyclic code. Prove that, if C does not contain the codeword $11\ldots1$, then all codewords in C have even weight.

13E Further Complex Methods

The temperature $T(x,t)$ in a semi-infinite bar ($0 \leqslant x < \infty$) satisfies the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad \text{for } x > 0 \text{ and } t > 0\,,$$

where κ is a positive constant.

For $t < 0$, the bar is at zero temperature. For $t \geqslant 0$, the temperature is subject to the boundary conditions

$$T(0,t) = a(1 - e^{-bt}),$$

where a and b are positive constants, and $T(x,t) \to 0$ as $x \to \infty$.

(a) Show that the Laplace transform of $T(x,t)$ with respect to t takes the form

$$\hat{T}(x,p) = \hat{f}(p) e^{-x\sqrt{p/\kappa}}\,,$$

and find $\hat{f}(p)$. Hence write $\hat{T}(x,p)$ in terms of a, b, κ, p and x.

(b) By performing the inverse Laplace transform using contour integration, show that for $t \geqslant 0$

$$T(x,t) = a\left[1 - e^{-bt}\cos\left(\sqrt{\frac{b}{\kappa}}\,x\right)\right] + \frac{2ab}{\pi}\mathcal{P}\int_0^\infty \frac{e^{-v^2 t}\sin(xv/\sqrt{\kappa})}{v(v^2 - b)}dv\,.$$

14D Classical Dynamics

(a) Show that the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \,,$$

where ω is a positive constant, describes a simple harmonic oscillator with angular frequency ω. Show that the energy E and the action I of the oscillator are related by $E = \omega I$.

(b) Let $0 < \epsilon < 2$ be a constant. Verify that the differential equation

$$\ddot{x} + \frac{x}{(\epsilon t)^2} = 0 \qquad \text{subject to} \qquad x(1) = 0\,, \quad \dot{x}(1) = 1$$

is solved by

$$x(t) = \frac{\sqrt{t}}{k}\sin(k\log t)$$

when $t > 1$, where k is a constant you should determine in terms of ϵ.

(c) Show that the solution in part (b) obeys

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}\frac{x^2}{(\epsilon t)^2} = \frac{1 - \cos(2k\log t) + 2k\sin(2k\log t) + 4k^2}{8k^2 t} \,.$$

Hence show that the fractional variation of the action in the limit $\epsilon \ll 1$ is $O(\epsilon)$, but that these variations do not accumulate. Comment on this behaviour in relation to the theory of adiabatic invariance.

15D Quantum Information and Computation

Alice and Bob are separated in space and can communicate only over a noiseless public classical channel, i.e. they can exchange bit string messages perfectly, but the messages can be read by anyone. An eavesdropper Eve constantly monitors the channel, but cannot alter any passing messages. Alice wishes to communicate an m-bit string message to Bob whilst keeping it secret from Eve.

(a) Explain how Alice can do this by the one-time pad method, specifying clearly any additional resource that Alice and Bob need. Explain why in this method, Alice's message does, in fact, remain secure against eavesdropping.

(b) Suppose now that Alice and Bob do not possess the additional resource needed in part (a) for the one-time pad, but that they instead possess n pairs of qubits, where $n \gg 1$, with each pair being in the state

$$|\psi\rangle_{AB} = t\,|00\rangle_{AB} + s\,|11\rangle_{AB}\,,$$

where the real parameters (t, s) are known to Alice and Bob and obey $t > s > 0$ and $t^2 + s^2 = 1$. For each qubit pair in state $|\psi\rangle_{AB}$, Alice possesses qubit A and Bob possesses qubit B. They each also have available a supply of ancilla qubits, each in state $|0\rangle$, and they can each perform local quantum operations on qubits in their possession.

Show how Alice, using only local quantum operations, can convert each $|\psi\rangle_{AB}$ state into $|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$ by a process that succeeds with non-zero probability. [*Hint: It may be useful for Alice to start by adjoining an ancilla qubit* $|0\rangle_{A'}$ *and work locally on her two qubits in* $|0\rangle_{A'}\,|\psi\rangle_{AB}$.]

Hence, or otherwise, show how Alice can communicate a bit string of expected length $(2s^2)n$ to Bob in a way that keeps it secure against eavesdropping by Eve.

16G Logic and Set Theory

Write down the inductive definition of ordinal exponentiation. Show that $\omega^\alpha \geqslant \alpha$ for every ordinal α. Deduce that, for every ordinal α, there is a least ordinal α^* with $\omega^{\alpha^*} > \alpha$. Show that, if $\alpha \neq 0$, then α^* must be a successor ordinal.

Now let α be a non-zero ordinal. Show that there exist ordinals β and γ, where $\gamma < \alpha$, and a positive integer n such that $\alpha = \omega^\beta n + \gamma$. Hence, or otherwise, show that α can be written in the form

$$\alpha = \omega^{\beta_1} n_1 + \omega^{\beta_2} n_2 + \cdots + \omega^{\beta_k} n_k \, ,$$

where k, n_1, n_2, $\ldots$, n_k are positive integers and $\beta_1 > \beta_2 > \cdots > \beta_k$ are ordinals. [We call this the *Cantor normal form* of α, and you may henceforth assume that it is unique.]

Given ordinals δ_1, δ_2 and positive integers m_1, m_2 find the Cantor normal form of $\omega^{\delta_1} m_1 + \omega^{\delta_2} m_2$. Hence, or otherwise, given non-zero ordinals α and α', find the Cantor normal form of $\alpha + \alpha'$ in terms of the Cantor normal forms

$$\alpha = \omega^{\beta_1} n_1 + \omega^{\beta_2} n_2 + \cdots + \omega^{\beta_k} n_k$$

and

$$\alpha' = \omega^{\beta'_1} n'_1 + \omega^{\beta'_2} n'_2 + \cdots + \omega^{\beta'_{k'}} n'_{k'}$$

of α and α'.

17G Graph Theory

(a) Define a *tree* and what it means for a graph to be *acyclic*. Show that if G is an acyclic graph on n vertices then $e(G) \leqslant n-1$. [You may use the fact that a spanning tree on n vertices has $n-1$ edges.]

(b) Show that any 3-regular graph on n vertices contains a cycle of length $\leqslant 100 \log n$. Hence show that there exists n_0 such that every 3-regular graph on more than n_0 vertices must contain two cycles C_1, C_2 with disjoint vertex sets.

(c) An *unfriendly partition* of a graph $G = (V, E)$ is a partition $V = A \cup B$, where $A, B \neq \emptyset$, such that every vertex $v \in A$ has $|N(v) \cap B| \geqslant |N(v) \cap A|$ and every $v \in B$ has $|N(v) \cap A| \geqslant |N(v) \cap B|$. Show that every graph G with $|G| \geqslant 2$ has an unfriendly partition.

(d) A *friendly partition* of a graph $G = (V, E)$ is a partition $V = S \cup T$, where $S, T \neq \emptyset$, such that every vertex $v \in S$ has $|N(v) \cap S| \geqslant |N(v) \cap T|$ and every $v \in T$ has $|N(v) \cap T| \geqslant |N(v) \cap S|$. Give an example of a 3-regular graph (on at least 1 vertex) that does not have a friendly partition. Using part (b), show that for large enough n_0 every 3-regular graph G with $|G| \geqslant n_0$ has a friendly partition.

18I Galois Theory

(a) Let $f(x) \in \mathbb{F}_q[x]$ be a polynomial of degree n, and let L be its splitting field.

(i) Suppose that f is irreducible. Compute $\mathrm{Gal}(f)$, carefully stating any theorems you use.

(ii) Now suppose that $f(x)$ factors as $f = h_1 \cdots h_r$ in $\mathbb{F}_q[x]$, with each h_i irreducible, and $h_i \neq h_j$ if $i \neq j$. Compute $\mathrm{Gal}(f)$, carefully stating any theorems you use.

(iii) Explain why $L/\mathbb{F}_q$ is a cyclotomic extension. Define the corresponding homomorphism $\mathrm{Gal}(L/\mathbb{F}_q) \hookrightarrow (\mathbb{Z}/m\mathbb{Z})^*$ for this extension (for a suitable integer m), and compute its image.

(b) Compute $\mathrm{Gal}(f)$ for the polynomial $f = x^4 + 8x + 12 \in \mathbb{Q}[x]$. [You may assume that f is irreducible and that its discriminant is 576^2.]

19I Representation Theory

Let G be a finite group and work over $\mathbb{C}$.

(a) Let χ be a faithful character of G, and suppose that $\chi(g)$ takes precisely r different values as g varies over all the elements of G. Show that every irreducible character of G is a constituent of one of the powers $\chi^0, \chi^1, \ldots, \chi^{r-1}$. [Standard properties of the Vandermonde matrix may be assumed if stated correctly.]

(b) Assuming that the number of irreducible characters of G is equal to the number of conjugacy classes of G, show that the irreducible characters of G form a basis of the complex vector space of all class functions on G. Deduce that $g, h \in G$ are conjugate if and only if $\chi(g) = \chi(h)$ for all characters χ of G.

(c) Let χ be a character of G which is not faithful. Show that there is some irreducible character ψ of G such that $\langle \chi^n, \psi \rangle = 0$ for all integers $n \geqslant 0$.

20G Number Fields

Let K be a field containing $\mathbb{Q}$. What does it mean to say that an element of K is *algebraic*? Show that if $\alpha \in K$ is algebraic and non-zero, then there exists $\beta \in \mathbb{Z}[\alpha]$ such that $\alpha\beta$ is a non-zero (rational) integer.

Now let K be a number field, with ring of integers $\mathcal{O}_K$. Let R be a subring of $\mathcal{O}_K$ whose field of fractions equals K. Show that every element of K can be written as r/m, where $r \in R$ and m is a positive integer.

Prove that R is a free abelian group of rank $[K:\mathbb{Q}]$, and that R has finite index in $\mathcal{O}_K$. Show also that for every nonzero ideal I of R, the index $(R:I)$ of I in R is finite, and that for some positive integer m, $m\mathcal{O}_K$ is an ideal of R.

Suppose that for every pair of non-zero ideals I, $J \subset R$, we have

$$(R:IJ) = (R:I)(R:J)\,.$$

Show that $R = \mathcal{O}_K$.

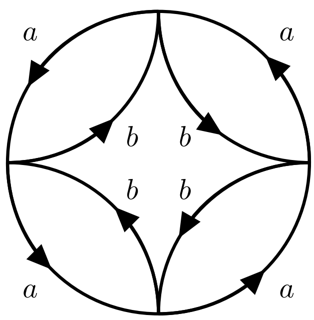
[*You may assume without proof that $\mathcal{O}_K$ is a free abelian group of rank* $[K:\mathbb{Q}]$.]

21F Algebraic Topology

(a) State a suitable version of the Seifert–van Kampen theorem and use it to calculate the fundamental groups of the torus $T^2 := S^1 \times S^1$ and of the real projective plane $\mathbb{RP}^2$.

(b) Show that there are no covering maps $T^2 \to \mathbb{RP}^2$ or $\mathbb{RP}^2 \to T^2$.

(c) Consider the following covering space of $S^1 \vee S^1$:



Here the line segments labelled a and b are mapped to the two different copies of S^1 contained in $S^1 \vee S^1$, with orientations as indicated.

Using the Galois correspondence with basepoints, identify a subgroup of

$$\pi_1(S^1 \vee S^1, x_0) = F_2$$

(where x_0 is the wedge point) that corresponds to this covering space.

22H Linear Analysis

(a) Let V be a real normed vector space. Show that any proper subspace of V has empty interior.

Assuming V to be infinite-dimensional and complete, prove that any algebraic basis of V is uncountable. [The Baire category theorem can be used if stated properly.] Deduce that the vector space of polynomials with real coefficients cannot be equipped with a complete norm, i.e. a norm that makes it complete.

(b) Suppose that $\|\cdot\|_1$ and $\|\cdot\|_2$ are norms on a vector space V such that $(V, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ are both complete. Prove that if there exists $C_1 > 0$ such that $\|x\|_2 \leqslant C_1\|x\|_1$ for all $x \in V$, then there exists $C_2 > 0$ such that $\|x\|_1 \leqslant C_2\|x\|_2$ for all $x \in V$. Is this still true without the assumption that $(V, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ are both complete? Justify your answer.

(c) Let V be a real normed vector space (not necessarily complete) and V^* be the set of linear continuous forms $f : V \to \mathbb{R}$. Let $(x_n)_{n\geqslant 1}$ be a sequence in V such that $\sum_{n\geqslant 1} |f(x_n)| < \infty$ for all $f \in V^*$. Prove that

$$\sup_{\|f\|_{V^*}\leqslant 1} \sum_{n\geqslant 1} |f(x_n)| < \infty .$$

23H Analysis of Functions

Define the *Schwartz space*, $\mathscr{S}(\mathbb{R}^n)$, and the *space of tempered distributions*, $\mathscr{S}'(\mathbb{R}^n)$, stating what it means for a sequence to converge in each space.

For a C^k function $f : \mathbb{R}^n \to \mathbb{C}$, and non-negative integers N, k, we say $f \in X_{N,k}$ if

$$\|f\|_{N,k} := \sup_{x\in\mathbb{R}^n; |\alpha|\leqslant k} \left| \left(1 + |x|^2\right)^{\frac{N}{2}} D^\alpha f(x) \right| < \infty.$$

You may assume that $X_{N,k}$ equipped with $\|\cdot\|_{N,k}$ is a Banach space in which $\mathscr{S}(\mathbb{R}^n)$ is dense.

(a) Show that if $u \in \mathscr{S}'(\mathbb{R}^n)$ there exist $N, k \in \mathbb{Z}_{\geqslant 0}$ and $C > 0$ such that

$$|u[\phi]| \leqslant C\|\phi\|_{N,k} \text{ for all } \phi \in \mathscr{S}(\mathbb{R}^n) .$$

Deduce that there exists a unique $\tilde{u} \in X'_{N,k}$ such that $\tilde{u}[\phi] = u[\phi]$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

(b) Recall that $v \in \mathscr{S}'(\mathbb{R}^n)$ is *positive* if $v[\phi] \geqslant 0$ for all $\phi \in \mathscr{S}(\mathbb{R}^n)$ satisfying $\phi \geqslant 0$. Show that if $v \in \mathscr{S}'(\mathbb{R}^n)$ is positive, then there exist $M \in \mathbb{Z}_{\geqslant 0}$ and $K > 0$ such that

$$|v[\phi]| \leqslant K\|\phi\|_{M,0}, \quad \text{for all } \phi \in \mathscr{S}(\mathbb{R}^n).$$

$\Big[$*Hint: Note that* $|\phi(x)| \leqslant \|\phi\|_{M,0}\left(1 + |x|^2\right)^{-\frac{M}{2}}$. $\Big]$

24F Riemann Surfaces

Let $D \subseteq \mathbb{C}$ be a domain, let (f, U) be a function element in D, and let $\alpha : [0,1] \to D$ be a path with $\alpha(0) \in U$. Define what it means for a function element (g, V) to be an *analytic continuation of* (f, U) *along* α.

Suppose that $\beta : [0,1] \to D$ is a path homotopic to α and that (h, V) is an analytic continuation of (f, U) along β. Suppose, furthermore, that (f, U) can be analytically continued along any path in D. Stating carefully any theorems that you use, prove that $g\big(\alpha(1)\big) = h\big(\beta(1)\big)$.

Give an example of a function element (f, U) that can be analytically continued to every point of $\mathbb{C}_*$ and a pair of homotopic paths α, β in $\mathbb{C}_*$ starting in U such that the analytic continuations of (f, U) along α and β take different values at $\alpha(1) = \beta(1)$.

25I Algebraic Geometry

Let k be an algebraically closed field and $n \geqslant 1$. Exhibit $GL(n, k)$ as an open subset of affine space $\mathbb{A}_k^{n^2}$. Deduce that $GL(n, k)$ is smooth. Prove that it is also irreducible.

Prove that $GL(n, k)$ is isomorphic to a closed subvariety in an affine space.

Show that the matrix multiplication map

$$GL(n, k) \times GL(n, k) \to GL(n, k)$$

that sends a pair of matrices to their product is a morphism.

Prove that any morphism from $\mathbb{A}_k^n$ to $\mathbb{A}_k^1 \smallsetminus \{0\}$ is constant.

Prove that for $n \geqslant 2$ any morphism from $\mathbb{P}_k^n$ to $\mathbb{P}_k^1$ is constant.

26F Differential Geometry

Let U be a domain in $\mathbb{R}^2$, and let $\phi : U \to \mathbb{R}^3$ be a smooth map. Define what it means for ϕ to be an *immersion*. What does it mean for an immersion to be *isothermal*?

Write down a formula for the mean curvature of an immersion in terms of the first and second fundamental forms. What does it mean for an immersed surface to be *minimal*? Assume that $\phi(u, v) = \big(x(u, v), y(u, v), z(u, v)\big)$ is an isothermal immersion. Prove that it is minimal if and only if x, y, z are harmonic functions of u, v.

For $u \in \mathbb{R}, v \in [0, 2\pi]$, and smooth functions $f, g : \mathbb{R} \to \mathbb{R}$, assume that

$$\phi(u, v) = \big(f(u) \cos v, f(u) \sin v, g(u)\big)$$

is an isothermal immersion. Find all possible pairs (f, g) such that this immersion is minimal.

27H Probability and Measure

Let $(E, \mathcal{E}, \mu)$ be a measure space. A function f is *simple* if it is of the form $f = \sum_{i=1}^{N} a_i 1_{A_i}$, where $a_i \in \mathbb{R}, N \in \mathbb{N}$ and $A_i \in \mathcal{E}$.

Now let $f : (E, \mathcal{E}, \mu) \to [0, \infty]$ be a Borel-measurable map. Show that there exists a sequence f_n of simple functions such that $f_n(x) \to f(x)$ for all $x \in E$ as $n \to \infty$.

Next suppose f is also μ-integrable. Construct a sequence f_n of simple μ-integrable functions such that $\int_E |f_n - f| d\mu \to 0$ as $n \to \infty$.

Finally, suppose f is also bounded. Show that there exists a sequence f_n of simple functions such that $f_n \to f$ uniformly on E as $n \to \infty$.

28K Applied Probability

Let X be an irreducible, non-explosive, continuous-time Markov process on the state space $\mathbb{Z}$ with generator $Q = (q_{x,y})_{x,y \in \mathbb{Z}}$.

(a) Define its *jump chain* Y and prove that it is a discrete-time Markov chain.

(b) Define what it means for X to be *recurrent* and prove that X is recurrent if and only if its jump chain Y is recurrent. Prove also that this is the case if the transition semigroup $(p_{x,y}(t))$ satisfies

$$\int_0^\infty p_{0,0}(t)\, dt = \infty.$$

(c) Show that X is recurrent for at least one of the following generators:

$$q_{x,y} = (1 + |x|)^{-2} e^{-|x-y|^2} \qquad (x \neq y),$$
$$q_{x,y} = (1 + |x - y|)^{-2} e^{-|x|^2} \qquad (x \neq y).$$

[*Hint: You may use that the semigroup associated with a Q-matrix on $\mathbb{Z}$ such that $q_{x,y}$ depends only on $x - y$ (and has sufficient decay) can be written as*

$$p_{x,y}(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-t\lambda(k)} e^{ik(x-y)}\, dk,$$

where $\lambda(k) = \sum_y q_{0,y}(1 - e^{iky})$. You may also find the bound $1 - \cos x \leqslant x^2/2$ useful.]

29J Principles of Statistics

Let $X_1, \ldots, X_n$ be i.i.d. random observations taking values in $[0,1]$ with a continuous distribution function F. Let $\hat{F}_n(x) = n^{-1}\sum_{i=1}^n \mathbf{1}_{\{X_i \leqslant x\}}$ for each $x \in [0,1]$.

(a) State the Kolmogorov–Smirnov theorem. Explain how this theorem may be used in a goodness-of-fit test for the null hypothesis $H_0 : F = F_0$, with F_0 continuous.

(b) Suppose you do not have access to the quantiles of the sampling distribution of the Kolmogorov–Smirnov test statistic. However, you are given i.i.d. samples $Z_1, \ldots, Z_{nm}$ with distribution function F_0. Describe a test of $H_0 : F = F_0$ with size exactly $1/(m+1)$.

(c) Now suppose that $X_1, \ldots, X_n$ are i.i.d. taking values in $[0, \infty)$ with probability density function f, with $\sup_{x \geqslant 0} \big(|f(x)| + |f'(x)|\big) < 1$. Define the density estimator

$$\hat{f}_n(x) = n^{-2/3} \sum_{i=1}^n \mathbf{1}_{\left\{X_i - \frac{1}{2n^{1/3}} \leqslant x \leqslant X_i + \frac{1}{2n^{1/3}}\right\}}, \quad x \geqslant 0.$$

Show that for all $x \geqslant 0$ and all $n \geqslant 1$,

$$\mathbb{E}\big[\big(\hat{f}_n(x) - f(x)\big)^2\big] \leqslant \frac{2}{n^{2/3}}.$$

30K Stochastic Financial Models

Consider a one-period market model with d risky assets and one risk-free asset. Let S_t denote the vector of prices of the risky assets at time $t \in \{0,1\}$ and let r be the interest rate.

(a) What does it mean to say a portfolio $\varphi \in \mathbb{R}^d$ is an *arbitrage* for this market?

(b) An investor wishes to maximise their expected utility of time-1 wealth X_1 attainable by investing in the market with their time-0 wealth $X_0 = x$. The investor's utility function U is increasing and concave. Show that, if there exists an optimal solution X_1^* to the investor's expected utility maximisation problem, then the market has no arbitrage. [Assume that $U(X_1)$ is integrable for any attainable time-1 wealth X_1.]

(c) Now introduce a contingent claim with time-1 bounded payout Y. How does the investor in part (b) calculate an *indifference bid price* $\pi(Y)$ for the claim? Assuming each such claim has a unique indifference price, show that the map $Y \mapsto \pi(Y)$ is concave. [Assume that any relevant utility maximisation problem that you consider has an optimal solution.]

(d) Consider a contingent claim with time-1 bounded payout Y. Let $I \subseteq \mathbb{R}$ be the set of initial no-arbitrage prices for the claim; that is, the set I consists of all p such that the market augmented with the contingent claim with time-0 price p has no arbitrage. Show that $\pi(Y) \leqslant \sup\{p \in I\}$. [Assume that any relevant utility maximisation problem that you consider has an optimal solution. You may use results from lectures without proof, such as the fundamental theorem of asset pricing or the existence of marginal utility prices, as long as they are clearly stated.]

31J Mathematics of Machine Learning

(a) What is meant by the *subdifferential* $\partial f(x)$ of a convex function $f : \mathbb{R}^d \to \mathbb{R}$ at $x \in \mathbb{R}^d$? Write down the subdifferential $\partial f(x)$ of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \gamma|x|$, where $\gamma > 0$.

Show that x minimises f if and only if $0 \in \partial f(x)$.

What does it mean for a function $f : \mathbb{R}^d \to \mathbb{R}$ to be *strictly convex*? Show that any minimiser of a strictly convex function must be unique.

(b) Suppose we have input–output pairs $(x_1, y_1), \ldots, (x_n, y_n) \in \{-1, 1\}^p \times \{-1, 1\}$ with $p \geqslant 2$. Consider the objective function

$$f(\beta) = \frac{1}{n} \sum_{i=1}^{n} \exp(-y_i x_i^T \beta) + \gamma \|\beta\|_1,$$

where $\beta = (\beta_1, \ldots, \beta_p)^T$ and $\gamma > 0$. Assume that $(y_i)_{i=1}^n \neq (x_{i1})_{i=1}^n$. Fix $\beta_2, \ldots, \beta_p$ and define

$$\kappa_1 = \sum_{\substack{1 \leqslant i \leqslant n : \\ x_{i1} \neq y_i}} \exp(-y_i \eta_i) \quad \text{and} \quad \kappa_2 = \sum_{i=1}^{n} \exp(-y_i \eta_i),$$

where $\eta_i = \sum_{j=2}^{p} x_{ij} \beta_j$ for $i = 1, \ldots, n$. Show that if $|2\kappa_1 - \kappa_2| \leqslant \gamma$, then

$$\operatorname{argmin}_{\beta_1 \in \mathbb{R}} f(\beta_1, \beta_2, \ldots, \beta_p) = 0.$$

[*You may use any results from the course without proof, other than those whose proof is asked for directly.*]

32A Asymptotic Methods

(a) Let $x(t)$ and $\phi_n(t)$, for $n = 0, 1, 2, \ldots$, be real-valued functions on $\mathbb{R}$.

(i) Define what it means for the sequence $\left\{\phi_n(t)\right\}_{n=0}^{\infty}$ to be an *asymptotic sequence* as $t \to \infty$.

(ii) Define what it means for $x(t)$ to have the *asymptotic expansion*

$$x(t) \sim \sum_{n=0}^{\infty} a_n \phi_n(t) \quad \text{as} \quad t \to \infty .$$

(b) Use the method of stationary phase to calculate the leading-order asymptotic approximation as $x \to \infty$ of

$$I(x) = \int_0^1 \sin\left(x(2t^4 - t^2)\right) dt .$$

[*You may assume that* $\displaystyle\int_{-\infty}^{\infty} e^{iu^2} du = \sqrt{\pi}\, e^{i\pi/4}$.]

(c) Use Laplace's method to calculate the leading-order asymptotic approximation as $x \to \infty$ of

$$J(x) = \int_0^1 \sinh\left(x(2t^4 - t^2)\right) dt .$$

[*In parts (b) and (c) you should include brief qualitative reasons for the origin of the leading-order contributions, but you do not need to give a formal justification.*]

33A Dynamical Systems

Consider a modified van der Pol system defined by

$$\begin{aligned} \dot{x} &= y - \mu(\tfrac{1}{3}x^3 - x), \\ \dot{y} &= -x + F, \end{aligned}$$

where $\mu > 0$ and F are constants.

(a) A parallelogram $PQRS$ of width $2L$ is defined by

$$\begin{aligned} P &= \big(L,\, \mu f(L)\big)\,, & Q &= \big(L,\, 2L - \mu f(L)\big)\,, \\ R &= \big(-L,\, -\mu f(L)\big)\,, & S &= \big(-L,\, \mu f(L) - 2L\big), \end{aligned}$$

where $f(L) = \frac{1}{3}L^3 - L$. Show that if L is sufficiently large then trajectories never leave the region inside the parallelogram.

Hence show that if $F^2 < 1$ there must be a periodic orbit. Explain your reasoning carefully.

(b) Use the energy-balance method to analyse the behaviour of the system for $\mu \ll 1$, identifying the difference in behaviours between $F^2 < 1$ and $F^2 > 1$.

(c) Describe the behaviour of the system for $\mu \gg 1$, using sketches of the phase plane to illustrate your arguments for the cases $0 < F < 1$ and $F > 1$.

34D Integrable Systems

(a) Explain briefly how the linear operators $L = -\partial_x^2 + u(x,t)$ and $A = 4\partial_x^3 - 3u\partial_x - 3\partial_x u$ can be used to give a Lax-pair formulation of the KdV equation $u_t + u_{xxx} - 6uu_x = 0$.

(b) Give a brief definition of the *scattering data*

$$\mathcal{S}_{u(t)} = \big\{ \{R(k,t)\}_{k\in\mathbb{R}}\,,\ \{-\kappa_n(t)^2, c_n(t)\}_{n=1}^N \big\}$$

attached to a smooth solution $u = u(x,t)$ of the KdV equation at time t. [You may assume $u(x,t)$ to be rapidly decreasing in x.] State the time dependence of $\kappa_n(t)$ and $c_n(t)$, and derive the time dependence of $R(k,t)$ from the Lax-pair formulation.

(c) Show that

$$F(x,t) = \sum_{n=1}^N c_n(t)^2\, e^{-\kappa_n(t)x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k,t)\, e^{ikx}\, dk$$

satisfies $\partial_t F + 8\partial_x^3 F = 0$. Now let $K(x,y,t)$ be the solution of the equation

$$K(x,y,t) + F(x+y,t) + \int_x^\infty K(x,z,t)F(z+y,t)\, dz = 0$$

and let $u(x,t) = -2\partial_x\phi(x,t)$, where $\phi(x,t) = K(x,x,t)$. Defining $G(x,y,t)$ by $G = \big(\partial_x^2 - \partial_y^2 - u(x,t)\big)K(x,y,t)$, show that

$$G(x,y,t) + \int_x^\infty G(x,z,t)F(z+y,t)\, dz = 0\,.$$

(d) Given that $K(x,y,t)$ obeys the equations

$$\begin{aligned} (\partial_x^2 - \partial_y^2)K - uK &= 0\,, \\ (\partial_t + 4\partial_x^3 + 4\partial_y^3)K - 3(\partial_x u)K - 6u\,\partial_x K &= 0\,, \end{aligned}$$

where $u = u(x,t)$, deduce that

$$\partial_t K + (\partial_x + \partial_y)^3 K - 3u\,(\partial_x + \partial_y)K = 0\,,$$

and hence that u solves the KdV equation.

35B Principles of Quantum Mechanics

(a) Let $\{|n\rangle\}$ be a basis of eigenstates of a non-degenerate Hamiltonian H, with corresponding eigenvalues $\{E_n\}$. Write down an expression for the energy levels of the perturbed Hamiltonian $H + \lambda\,\Delta H$, correct to second order in the dimensionless constant $\lambda \ll 1$.

(b) A particle travels in one dimension under the influence of the potential

$$V(X) = \frac{1}{2}m\omega^2 X^2 + \lambda\,\hbar\omega\,\frac{X^3}{L^3}\,,$$

where m is the mass, ω a frequency and $L = \sqrt{\hbar/2m\omega}$ a length scale. Show that, to first order in λ, all energy levels coincide with those of the harmonic oscillator. Calculate the energy of the ground state to second order in λ.

Does perturbation theory in λ converge for this potential? Briefly explain your answer.

36B Applications of Quantum Mechanics

(a) The s-wave solution ψ_0 for the scattering problem of a particle of mass m and momentum $\hbar k$ has the asymptotic form

$$\psi_0(r) \sim \frac{A}{r}\big[\sin(kr) + g(k)\cos(kr)\big]\,.$$

Define the *phase shift* δ_0 and verify that $\tan\delta_0 = g(k)$.

(b) Define the *scattering amplitude* f. For a spherically symmetric potential of finite range, starting from $\sigma_T = \int |f|^2 d\Omega$, derive the expression

$$\sigma_T = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

giving the cross-section σ_T in terms of the phase shifts δ_l of the partial waves.

(c) For $g(k) = -k/K$ with $K > 0$, show that a bound state exists and compute its energy. Neglecting the contributions from partial waves with $l > 0$, show that

$$\sigma_T \approx \frac{4\pi}{K^2 + k^2}\,.$$

(d) For $g(k) = \gamma/(K_0 - k)$ with $K_0 > 0$, $\gamma > 0$ compute the s-wave contribution to σ_T. Working to leading order in $\gamma \ll K_0$, show that σ_T has a local maximum at $k = K_0$. Interpret this fact in terms of a resonance and compute its energy and decay width.

37C Statistical Physics

(a) What systems are described by *microcanonical*, *canonical* and *grand canonical ensembles*? Under what conditions is the choice of ensemble irrelevant?

(b) In a simple model a meson consists of two quarks bound in a linear potential, $U(\mathbf{r}) = \alpha|\mathbf{r}|$, where $\mathbf{r}$ is the relative displacement of the two quarks and α is a positive constant. You are given that the classical (non-relativistic) Hamiltonian for the meson is

$$H(\mathbf{P}, \mathbf{R}, \mathbf{p}, \mathbf{r}) = \frac{|\mathbf{P}|^2}{2M} + \frac{|\mathbf{p}|^2}{2\mu} + \alpha|\mathbf{r}|\,,$$

where $M = 2m$ is the total mass, $\mu = m/2$ is the reduced mass, $\mathbf{P}$ is the total momentum, $\mathbf{p} = \mu\, d\mathbf{r}/dt$ is the internal momentum, and $\mathbf{R}$ is the centre of mass position.

(i) Show that the partition function for a single meson in thermal equilibrium at temperature T in a three-dimensional volume V can be written as $Z_1 = Z_{\text{trans}} Z_{\text{int}}$, where

$$Z_{\text{trans}} = \frac{V}{(2\pi\hbar)^3} \int d^3P\, e^{-\beta|\mathbf{P}|^2/(2M)}\,, \qquad Z_{\text{int}} = \frac{1}{(2\pi\hbar)^3} \int d^3r\, d^3p\, e^{-\beta|\mathbf{p}|^2/(2\mu)} e^{-\beta\alpha|\mathbf{r}|}$$

and $\beta = 1/(k_{\text{B}}T)$.

Evaluate Z_{trans} and evaluate Z_{int} in the large-volume limit ($\beta\alpha V^{1/3} \gg 1$).

What is the average separation of the quarks within the meson at temperature T?

$\left[\textit{You may assume that } \int_{-\infty}^{\infty} e^{-cx^2} dx = \sqrt{\pi/c} \text{ for } c > 0.\right]$

(ii) Now consider an ideal gas of N such mesons in a three-dimensional volume V. Calculate the total partition function of the gas.

What is the heat capacity C_V?

38C General Relativity

Consider the following metric for a 3-dimensional, static and rotationally symmetric Lorentzian manifold:

$$ds^2 = r^{-2}(-dt^2 + dr^2) + r^2 d\theta^2 \,.$$

(a) Write down a Lagrangian $\mathcal{L}$ for arbitrary geodesics in this metric, if the geodesic is affinely parameterized with respect to λ. What condition may be imposed to distinguish spacelike, timelike, and null geodesics?

(b) Find the three constants of motion for any geodesic.

(c) Two observation stations are sitting at radii $r = R$ and $r = 2R$ respectively, and at the same angular coordinate. Each is accelerating so as to remain stationary with respect to time translations. At $t = 0$ a photon is emitted from the naked singularity at $r = 0$.

(i) At what time t_1 does the photon reach the inner station?

(ii) Express the frequency ν_2 of the photon at the outer station in terms of the frequency ν_1 at the inner station. Explain whether the photon is redshifted or blueshifted as it travels.

(d) Consider a complete (i.e. infinite in both directions) spacelike geodesic on a constant-t slice with impact parameter $b = r_{\rm min} > 0$. What is the angle $\Delta\theta$ between the two asymptotes of the geodesic at $r = \infty$? [You need not be concerned with the sign of $\Delta\theta$ or the periodicity of the θ coordinate.]

[*Hint: You may find integration by substitution useful.*]

39A Fluid Dynamics II

(a) Incompressible fluid of viscosity μ fills the thin, slowly varying gap between rigid boundaries at $z = 0$ and $z = h(x, y) > 0$. The boundary at $z = 0$ translates in its own plane with a constant velocity $\mathbf{U} = (U, 0, 0)$, while the other boundary is stationary. If h has typical magnitude H and varies on a lengthscale L, state conditions for the lubrication approximation to be appropriate.

Write down the lubrication equations for this problem and show that the horizontal volume flux $\mathbf{q} = (q_x, q_y, 0)$ is given by

$$\mathbf{q} = \frac{\mathbf{U}h}{2} - \frac{h^3}{12\mu}\boldsymbol{\nabla} p,$$

where $p(x, y)$ is the pressure.

Explain why $\mathbf{q} = \boldsymbol{\nabla} \wedge (0, 0, \psi)$ for some function $\psi(x, y)$. Deduce that ψ satisfies the equation

$$\boldsymbol{\nabla} \cdot \left(\frac{1}{h^3}\boldsymbol{\nabla}\psi\right) = -\frac{U}{h^3}\frac{\partial h}{\partial y}\,.$$

(b) Now consider the case $\mathbf{U} = \mathbf{0}$, $h = h_0$ for $r > a$ and $h = h_1$ for $r < a$, where h_0, h_1 and a are constants, and (r, θ) are polar coordinates. A uniform pressure gradient $\boldsymbol{\nabla} p = -G\mathbf{e}_x$ is applied at infinity. Show that $\psi \sim Ar \sin\theta$ as $r \to \infty$, where the constant A is to be determined.

Given that $a \gg h_0, h_1$, you may assume that the equations of part (a) apply for $r < a$ and $r > a$, and are subject to conditions that the radial component q_r of the volume flux and the pressure p are both continuous across $r = a$. Show that these continuity conditions imply that

$$\left[\frac{\partial\psi}{\partial\theta}\right]_-^+ = 0 \quad \text{and} \quad \left[\frac{1}{h^3}\frac{\partial\psi}{\partial r}\right]_-^+ = 0\,,$$

respectively, where $[\;]_-^+$ denotes the jump across $r = a$.

Hence determine $\psi(r, \theta)$ and deduce that the total flux through $r = a$ is given by

$$\frac{4Aah_1^3}{h_0^3 + h_1^3}\,.$$

40A Waves

A semi-infinite elastic medium with shear modulus μ and shear-wave speed c_s lies in $z \leqslant 0$. Above it, there is a layer $0 \leqslant z \leqslant h$ of a second elastic medium with shear modulus $\overline{\mu}$ and shear-wave speed $\overline{c}_s < c_s$. The top boundary is stress-free. Consider a monochromatic SH-wave propagating in the x-direction at speed c with wavenumber $k > 0$.

(a) Derive the dispersion relation

$$\tan\left[kh\sqrt{c^2/\overline{c}_s^2 - 1}\right] = \frac{\mu}{\overline{\mu}}\frac{\sqrt{1 - c^2/c_s^2}}{\sqrt{c^2/\overline{c}_s^2 - 1}}$$

for trapped modes with no disturbance as $z \to -\infty$.

(b) Show graphically that there is always a zeroth mode, and show that the other modes have cut-off frequencies

$$\omega_c^{(n)} = \frac{n\pi\overline{c}_s c_s}{h\sqrt{c_s^2 - \overline{c}_s^2}},$$

where n is a positive integer. Sketch a graph of frequency ω against k for the $n = 1$ mode showing the behaviour near cut-off and for large k.

41E Numerical Analysis

(a) Let $\mathbf{x} \in \mathbb{R}^N$ and define $\mathbf{y} \in \mathbb{R}^{2N}$ by

$$y_n = \begin{cases} x_n, & 0 \leqslant n \leqslant N-1 \\ x_{2N-n-1}, & N \leqslant n \leqslant 2N-1. \end{cases}$$

Let $\mathbf{Y} \in \mathbb{C}^{2N}$ be defined as the discrete Fourier transform (DFT) of $\mathbf{y}$, i.e.

$$Y_k = \sum_{n=0}^{2N-1} y_n \omega_{2N}^{nk}, \quad \omega_{2N} = \exp\left(-\pi i/N\right), \quad 0 \leqslant k \leqslant 2N-1.$$

Show that

$$Y_k = 2\omega_{2N}^{-k/2} \sum_{n=0}^{N-1} x_n \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right], \quad 0 \leqslant k \leqslant 2N-1.$$

(b) Define the discrete cosine transform (DCT) $\mathcal{C}_N : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\mathbf{z} = \mathcal{C}_N \mathbf{x}, \text{ where } z_k = \sum_{n=0}^{N-1} x_n \cos\left[\frac{\pi}{N}\left(n+\frac{1}{2}\right)k\right], \quad k = 0, \ldots, N-1\,.$$

For $N = 2^p$ with $p \in \mathbb{N}$, show that, similar to the Fast Fourier Transform (FFT), there exists an algorithm that computes the DCT of a vector of length N, where the number of multiplications required is bounded by $CN \log N$, where C is some constant independent of N.

[*You may not assume that the FFT algorithm requires $\mathcal{O}(N \log N)$ multiplications to compute the DFT of a vector of length N. If you use this, you must prove it.*]

END OF PAPER