UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Tuesday, 8 June, 2021 10:00am to 1:00pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

State Euler's criterion.

Let p be an odd prime. Show that every primitive root modulo p is a quadratic non-residue modulo p.

Let p be a Fermat prime, that is, a prime of the form $2^{2^k}+1$ for some $k \geqslant 1$. By evaluating $\phi(p-1)$, or otherwise, show that every quadratic non-residue modulo p is a primitive root modulo p. Deduce that 3 is a primitive root modulo p for every Fermat prime p.

2H Topics in Analysis

Write

$$P = \{\mathbf{x} \in \mathbb{R}^n : x_j \geqslant 0 \text{ for all } 1 \leqslant j \leqslant n\}$$

and suppose that K is a non-empty, closed, convex and bounded subset of $\mathbb{R}^n$ with $K \cap \operatorname{Int} P \neq \emptyset$. By taking logarithms, or otherwise, show that there is a unique $\mathbf{x}^* \in K \cap P$ such that

$$\prod_{j=1}^{n} x_j \leqslant \prod_{j=1}^{n} x_j^*$$

for all $\mathbf{x} \in K \cap P$.

Show that $\displaystyle\sum_{j=1}^{n} \frac{x_j}{x_j^*} \leqslant n$ for all $\mathbf{x} \in K \cap P$.

Identify the point $\mathbf{x}^*$ in the case that K has the property

$$(x_1, x_2, \ldots, x_{n-1}, x_n) \in K \Rightarrow (x_2, x_3, \ldots, x_n, x_1) \in K\,,$$

and justify your answer.

Show that, given any $\mathbf{a} \in \operatorname{Int} P$, we can find a set K, as above, with $\mathbf{x}^* = \mathbf{a}$.

3K Coding and Cryptography

Let C be an $[n, m, d]$ code. Define the parameters n, m and d. In each of the following cases define the new code and give its parameters.

(i) C^+ is the parity extension of C.

(ii) C^- is the punctured code (assume $n \geqslant 2$).

(iii) $\overline{C}$ is the shortened code (assume $n \geqslant 2$).

Let $C = \{000, 100, 010, 001, 110, 101, 011, 111\}$. Suppose the parity extension of C is transmitted through a binary symmetric channel where p is the probability of a single-bit error in the channel. Calculate the probability that an error in the transmission of a single codeword is not noticed.

4F Automata and Formal Languages

Let $f_{n,k}$ be the partial function on k variables that is computed by the nth machine (or the empty function if n does not encode a machine).

Define the *halting set* $\mathbb{K}$.

Given $A, B \subseteq \mathbb{N}$, what is a *many–one reduction* $A \leqslant_m B$ of A to B?

State the $s - m - n$ theorem and use it to show that a subset X of $\mathbb{N}$ is recursively enumerable if and only if $X \leqslant_m \mathbb{K}$.

Give an example of a set $S \subseteq \mathbb{N}$ with $\mathbb{K} \leqslant_m S$ but $\mathbb{K} \neq S$.

[*You may assume that* $\mathbb{K}$ *is recursively enumerable and that* $0 \notin \mathbb{K}$.]

5J Statistical Modelling

Let $\mu > 0$. The probability density function of the inverse Gaussian distribution (with the shape parameter equal to 1) is given by

$$f(x;\mu) = \frac{1}{\sqrt{2\pi x^3}} \exp\left[-\frac{(x-\mu)^2}{2\mu^2 x}\right].$$

Show that this is a one-parameter exponential family. What is its natural parameter? Show that this distribution has mean μ and variance μ^3.

6E Mathematical Biology

(a) Consider a population of size $N(t)$ whose per capita rates of birth and death are be^{-aN} and d, respectively, where $b > d$ and all parameters are positive constants.

(i) Write down the equation for the rate of change of the population.

(ii) Show that a population of size $N^* = \frac{1}{a}\log\frac{b}{d}$ is stationary and that it is asymptotically stable.

(b) Consider now a disease introduced into this population, where the number of susceptibles and infectives, S and I, respectively, satisfy the equations

$$\frac{dS}{dt} = be^{-aS}S - \beta SI - dS\,,$$
$$\frac{dI}{dt} = \beta SI - (d+\delta)I\,.$$

(i) Interpret the biological meaning of each term in the above equations and comment on the reproductive capacity of the susceptible and infected individuals.

(ii) Show that the disease-free equilibrium, $S = N^*$ and $I = 0$, is linearly unstable if

$$N^* > \frac{d+\delta}{\beta}\,.$$

(iii) Show that when the disease-free equilibrium is unstable there exists an endemic equilibrium satisfying

$$\beta I + d = be^{-aS}$$

and that this equilibrium is linearly stable.

7E Further Complex Methods

Evaluate the integral

$$\mathcal{P}\int_0^\infty \frac{\sin x}{x(x^2-1)}\,dx\,,$$

stating clearly any standard results involving contour integrals that you use.

8D Classical Dynamics

Two equal masses m move along a straight line between two stationary walls. The mass on the left is connected to the wall on its left by a spring of spring constant k_1, and the mass on the right is connected to the wall on its right by a spring of spring constant k_2. The two masses are connected by a third spring of spring constant k_3.

(a) Show that the Lagrangian of the system can be written in the form

$$L = \frac{1}{2}T_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}V_{ij}x_i x_j \,,$$

where $x_i(t)$, for $i = 1, 2$, are the displacements of the two masses from their equilibrium positions, and T_{ij} and V_{ij} are symmetric 2×2 matrices that should be determined.

(b) Let

$$k_1 = k(1 + \epsilon\delta)\,, \qquad k_2 = k(1 - \epsilon\delta)\,, \qquad k_3 = k\epsilon\,,$$

where $k > 0$, $\epsilon > 0$ and $|\epsilon\delta| < 1$. Using Lagrange's equations of motion, show that the angular frequencies ω of the normal modes of the system are given by

$$\omega^2 = \lambda\,\frac{k}{m}\,,$$

where

$$\lambda = 1 + \epsilon\left(1 \pm \sqrt{1 + \delta^2}\right)\,.$$

9B Cosmology

The continuity, Euler and Poisson equations governing how non-relativistic fluids with energy density ρ, pressure P and velocity $\mathbf{v}$ propagate in an expanding universe take the form

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\boldsymbol{\nabla}\cdot(\rho \mathbf{v}) = 0\,,$$

$$\rho\, a\left(\frac{\partial}{\partial t} + \frac{\mathbf{v}}{a}\cdot\boldsymbol{\nabla}\right)\mathbf{u} = -\frac{1}{c^2}\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi\,,$$

$$\nabla^2\Phi = \frac{4\pi G}{c^2}\,\rho\, a^2\,,$$

where $\mathbf{u} = \mathbf{v} + a\,H\,\mathbf{x}$, $H = \dot{a}/a$ and $a(t)$ is the scale factor.

(a) Show that, for a homogeneous and isotropic flow with $P = \overline{P}(t)$, $\rho = \overline{\rho}(t)$, $\mathbf{v} = \mathbf{0}$ and $\Phi = \overline{\Phi}(t, \mathbf{x})$, consistency of the Euler equation with the Poisson equation implies Raychaudhuri's equation.

(b) Explain why this derivation of Raychaudhuri's equation is an improvement over the derivation of the Friedmann equation using only Newtonian gravity.

(c) Consider small perturbations about a homogeneous and isotropic flow,

$$\rho = \overline{\rho}(t) + \epsilon\,\delta\rho, \quad \mathbf{v} = \epsilon\,\delta\mathbf{v}, \quad P = \overline{P}(t) + \epsilon\,\delta P \quad \text{and} \quad \Phi = \overline{\Phi}(t, \mathbf{x}) + \epsilon\,\delta\Phi\,,$$

with $\epsilon \ll 1$. Show that, to first order in ϵ, the continuity equation can be written as

$$\frac{\partial}{\partial t}\left(\frac{\delta\rho}{\overline{\rho}}\right) = -\frac{1}{a}\boldsymbol{\nabla}\cdot\delta\mathbf{v}\,.$$

10D Quantum Information and Computation

Alice wishes to communicate to Bob a 1-bit message $m = 0$ or $m = 1$ chosen by her with equal prior probabilities 1/2. For $m = 0$ (respectively $m = 1$) she sends Bob the quantum state $|a_0\rangle$ (respectively $|a_1\rangle$). On receiving the state, Bob applies quantum operations to it, to try to determine Alice's message. The Helstrom–Holevo theorem asserts that the probability P_S for Bob to correctly determine Alice's message is bounded by $P_S \leqslant \frac{1}{2}(1 + \sin\theta)$, where $\theta = \cos^{-1}|\langle a_0|a_1\rangle|$, and that this bound is achievable.

(a) Suppose that $|a_0\rangle = |0\rangle$ and $|a_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and that Bob measures the received state in the basis $\{|b_0\rangle, |b_1\rangle\}$, where $|b_0\rangle = \cos\beta\,|0\rangle + \sin\beta\,|1\rangle$ and $|b_1\rangle = -\sin\beta\,|0\rangle + \cos\beta\,|1\rangle$, to produce his output 0 or 1, respectively. Calculate the probability P_S that Bob correctly determines Alice's message, and show that the maximum value of P_S over choices of $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ achieves the Helstrom–Holevo bound.

(b) State the no-cloning theorem as it applies to unitary processes and a set of two non-orthogonal states $\{|c_0\rangle, |c_1\rangle\}$. Show that the Helstrom–Holevo theorem implies the validity of the no-cloning theorem in this situation.

SECTION II

11K Coding and Cryptography

Let $\Sigma_1 = \{\mu_1, \ldots, \mu_N\}$ be a finite alphabet and X a random variable that takes each value μ_i with probability p_i. Define the *entropy* $H(X)$ of X.

Suppose $\Sigma_2 = \{0, 1\}$ and $c : \Sigma_1 \to \Sigma_2^*$ is a decipherable code. Write down an expression for the expected word length $E(S)$ of c.

Prove that the minimum expected word length S^* of a decipherable code $c : \Sigma_1 \to \Sigma_2^*$ satisfies

$$H(X) \leqslant S^* < H(X) + 1\,.$$

[You can use Kraft's and Gibbs' inequalities as long as they are clearly stated.]

Suppose a decipherable binary code has word lengths $s_1, \ldots, s_N$. Show that

$$N \log N \leqslant s_1 + \cdots + s_N\,.$$

Suppose X is a source that emits N sourcewords $a_1, \ldots, a_N$ and p_i is the probability that a_i is emitted, where $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_N$. Let $b_1 = 0$ and $b_i = \sum_{j=1}^{i-1} p_j$ for $2 \leqslant i \leqslant N$. Let $s_i = \lceil -\log p_i \rceil$ for $1 \leqslant i \leqslant N$. Now define a code c by $c(a_i) = b_i^*$ where b_i^* is the (fractional part of the) binary expansion of b_i to s_i decimal places. Prove that this defines a decipherable code.

What does it mean for a code to be *optimal*? Is the code c defined in the previous paragraph in terms of the b_i^* necessarily optimal? Justify your answer.

12F Automata and Formal Languages

For $k \geqslant 1$ give the definition of a *partial recursive* function $f : \mathbb{N}^k \to \mathbb{N}$ in terms of basic functions, composition, recursion and minimisation.

Show that the following partial functions from $\mathbb{N}$ to $\mathbb{N}$ are partial recursive:

(i) $s(n) = \begin{cases} 1 & n = 0 \\ 0 & n \geqslant 1\,, \end{cases}$

(ii) $r(n) = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even}\,, \end{cases}$

(iii) $p(n) = \begin{cases} \text{undefined if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even}\,. \end{cases}$

Which of these can be defined without using minimisation?

What is the class of functions $f : \mathbb{N}^k \to \mathbb{N}$ that can be defined using only basic functions and composition? [*Hint: See which functions you can obtain and then show that these form a class that is closed with respect to the above.*]

Show directly that every function in this class is computable.

13J Statistical Modelling

The following data were obtained in a randomised controlled trial for a drug. Due to a manufacturing error, a subset of trial participants received a low dose (LD) instead of a standard dose (SD) of the drug.

```
> data
  treatment  outcome count
1   Control   Better  5728
2   Control    Worse   101
3        LD   Better  1364
4        LD    Worse     3
5        SD   Better  4413
6        SD    Worse    27
```

(a) Below we analyse the data using Poisson regression:

```
> fit1 <- glm(count ~ treatment + outcome, family = poisson, data)
> fit2 <- glm(count ~ treatment * outcome, family = poisson, data)
> anova(fit1, fit2, test = "LRT")
Analysis of Deviance Table

Model 1: count ~ treatment + outcome
Model 2: count ~ treatment * outcome
  Resid. Df Resid. Dev Df Deviance  Pr(>Chi)
1         2      44.48
2         0       0.00  2    44.48 2.194e-10 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

(i) After introducing necessary notation, write down the Poisson models being fitted above.

(ii) Write down the corresponding multinomial models, then state the key theoretical result (the "Poisson trick") that allows you to fit the multinomial models using Poisson regression. [You do not need to prove this theoretical result.]

(iii) Explain why the number of degrees of freedom in the likelihood ratio test is 2 in the analysis of deviance table. What can you conclude about the drug?

(b) Below is the summary table of the second model:

[QUESTION CONTINUES ON THE NEXT PAGE]

```
> summary(fit2)
                          Estimate Std. Error z value Pr(>|z|)
(Intercept)                8.65312    0.01321 654.899  < 2e-16 ***
treatmentLD               -1.43494    0.03013 -47.628  < 2e-16 ***
treatmentSD               -0.26081    0.02003 -13.021  < 2e-16 ***
outcomeWorse              -4.03800    0.10038 -40.228  < 2e-16 ***
treatmentLD:outcomeWorse -2.08156    0.58664  -3.548 0.000388 ***
treatmentSD:outcomeWorse -1.05847    0.21758  -4.865 1.15e-06 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

(i) *Drug efficacy* is defined as one minus the ratio of the probability of worsening in the treated group to the probability of worsening in the control group. By using a more sophisticated method, a published analysis estimated that the drug efficacy is 90.0% for the LD treatment and 62.1% for the SD treatment. Are these numbers similar to what is obtained by Poisson regression? [*Hint:* $e^{-1} \approx 0.37$*,* $e^{-2} \approx 0.14$*, and* $e^{-3} \approx 0.05$*, where* e *is the base of the natural logarithm.*]

(ii) Explain why the information in the summary table is not enough to test the hypothesis that the LD drug and the SD drug have the same efficacy. Then describe how you can test this hypothesis using analysis of deviance in `R`.

14E Further Complex Methods

(a) Functions $g_1(z)$ and $g_2(z)$ are analytic in a connected open set $\mathcal{D} \subseteq \mathbb{C}$ with $g_1 = g_2$ in a non-empty open subset $\tilde{\mathcal{D}} \subset \mathcal{D}$. State the *identity theorem*.

(b) Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be connected open sets with $\mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset$. Functions $f_1(z)$ and $f_2(z)$ are analytic on $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively with $f_1 = f_2$ on $\mathcal{D}_1 \cap \mathcal{D}_2$. Explain briefly what is meant by *analytic continuation* of f_1 and use part (a) to prove that analytic continuation to $\mathcal{D}_2$ is unique.

(c) The function $F(z)$ is defined by

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{(t-z)^n} dt \,,$$

where $\operatorname{Im} z > 0$ and n is a positive integer. Use the method of contour deformation to construct the analytic continuation of $F(z)$ into $\operatorname{Im} z \leqslant 0$.

(d) The function $G(z)$ is defined by

$$G(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{(t-z)^n} dt \,,$$

where $\operatorname{Im} z \neq 0$ and n is a positive integer. Prove that $G(z)$ experiences a discontinuity when z crosses the real axis. Determine the value of this discontinuity. Hence, explain why $G(z)$ cannot be used as an analytic continuation of $F(z)$.

15B Cosmology

(a) Consider the following action for the inflaton field ϕ

$$S = \int \mathrm{d}^3x \, \mathrm{d}t \, a(t)^3 \left[\frac{1}{2}\dot{\phi}^2 - \frac{c^2}{2a(t)^2} \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi - V(\phi) \right] .$$

Use the principle of least action to derive the equation of motion for the inflaton ϕ,

$$\ddot{\phi} + 3H\dot{\phi} - \frac{c^2}{a(t)^2}\nabla^2\phi + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} = 0 \,, \qquad (*)$$

where $H = \dot{a}/a$. [In the derivation you may discard boundary terms.]

(b) Consider a regime where $V(\phi)$ is approximately constant so that the universe undergoes a period of exponential expansion during which $a = a_0 \, e^{H_{\rm inf}\, t}$. Show that $(*)$ can be written in terms of the spatial Fourier transform $\widehat{\phi}_{\mathbf{k}}(t)$ of $\phi(\mathbf{x}, t)$ as

$$\ddot{\widehat{\phi}}_{\mathbf{k}} + 3H_{\rm inf}\dot{\widehat{\phi}}_{\mathbf{k}} + \frac{c^2k^2}{a^2}\widehat{\phi}_{\mathbf{k}} = 0 \,. \qquad (**)$$

(c) Define *conformal time* τ and determine the range of τ when $a = a_0 \, e^{H_{\rm inf}\, t}$. Show that $(**)$ can be written in terms of the conformal time as

$$\frac{\mathrm{d}^2\widetilde{\phi}_{\mathbf{k}}}{\mathrm{d}\tau^2} + \left(c^2k^2 - \frac{2}{\tau^2} \right) \widetilde{\phi}_{\mathbf{k}} = 0 \,, \qquad \text{where} \qquad \widetilde{\phi}_{\mathbf{k}} = -\frac{1}{H_{\rm inf}\tau}\widehat{\phi}_{\mathbf{k}} \,.$$

(d) Let $|\mathrm{BD}\rangle$ denote the state that in the far past was in the ground state of the standard harmonic oscillator with frequency $\omega = c\,k$. Assuming that the quantum variance of $\widehat{\phi}_{\mathbf{k}}$ is given by

$$P_{\mathbf{k}} \equiv \langle \mathrm{BD} | \widehat{\phi}_{\mathbf{k}} \widehat{\phi}_{\mathbf{k}}^{\dagger} | \mathrm{BD} \rangle = \frac{\hbar \, H_{\rm inf}^2}{2c^3k^3} \left(1 + \tau^2 c^2 k^2 \right) \,,$$

explain in which sense inflation naturally generates a scale-invariant power spectrum. [You may use that $P_{\mathbf{k}}$ has dimensions of [length]^3.]

16G Logic and Set Theory

Let S and T be sets of propositional formulae.

(a) What does it mean to say that S is *deductively closed*? What does it mean to say that S is *consistent*? Explain briefly why if S is inconsistent then some finite subset of S is inconsistent.

(b) We write $S \vdash T$ to mean $S \vdash t$ for all $t \in T$. If $S \vdash T$ and $T \vdash S$ we say S and T are *equivalent.* If S is equivalent to a finite set F of formulae we say that S is *finitary.* Show that if S is finitary then there is a finite set $R \subset S$ with $R \vdash S$.

(c) Now let T_0, T_1, T_2, ... be deductively closed sets of formulae with

$$T_0 \subsetneq T_1 \subsetneq T_2 \subsetneq \cdots .$$

Show that each T_i is consistent.

Let $T = \bigcup_{i=0}^{\infty} T_i$. Show that T is consistent and deductively closed, but that it is not finitary.

17G Graph Theory

Define the *binomial random graph* $G(n,p)$, where $n \in \mathbb{N}$ and $p \in (0,1)$.

(a) Let $G_n \sim G(n,p)$ and let E_t be the event that G_n contains a copy of the complete graph K_t. Show that if $p = p(n)$ is such that $p \cdot n^{2/(t-1)} \to 0$ then $\mathbb{P}(E_t) \to 0$ as $n \to \infty$.

(b) State Chebyshev's inequality. Show that if $p \cdot n \to \infty$ then $\mathbb{P}(E_3) \to 1$.

(c) Let H be a triangle with an added leaf vertex, that is

$$H = (\{x_1, \dots, x_4\}, \{x_1x_2, x_2x_3, x_3x_1, x_1x_4\}),$$

where $x_1, \dots, x_4$ are distinct. Let F be the event that $G_n \sim G(n,p)$ contains a copy of H. Show that if $p = n^{-0.9}$ then $\mathbb{P}(F) \to 1$.

18I Galois Theory

(a) Let $K \subseteq L$ be fields, and $f(x) \in K[x]$ a polynomial.

Define what it means for L to be a *splitting field* for f over K.

Prove that splitting fields exist, and state precisely the theorem on uniqueness of splitting fields.

Let $f(x) = x^3 - 2 \in \mathbb{Q}[x]$. Find a subfield of $\mathbb{C}$ which is a splitting field for f over $\mathbb{Q}$. Is this subfield unique? Justify your answer.

(b) Let $L = \mathbb{Q}[\zeta_7]$, where ζ_7 is a primitive 7th root of unity.

Show that the extension $L/\mathbb{Q}$ is Galois. Determine all subfields $M \subseteq L$.

For each subfield M, find a primitive element for the extension $M/\mathbb{Q}$ explicitly in terms of ζ_7, find its minimal polynomial, and write down $\mathrm{Aut}(M/\mathbb{Q})$ and $\mathrm{Aut}(L/M)$.

Which of these subfields M are Galois over $\mathbb{Q}$?

[*You may assume the Galois correspondence, but should prove any results you need about cyclotomic extensions directly.*]

19I Representation Theory

(a) What does it mean to say that a representation of a group is *completely reducible*? State Maschke's theorem for representations of finite groups over fields of characteristic 0. State and prove Schur's lemma. Deduce that if there exists a faithful irreducible complex representation of G, then $Z(G)$ is cyclic.

(b) If G is any finite group, show that the regular representation $\mathbb{C}G$ is faithful. Show further that for every finite simple group G, there exists a faithful irreducible complex representation of G.

(c) Which of the following groups have a faithful irreducible representation? Give brief justification of your answers.

(i) the cyclic groups C_n (n a positive integer);

(ii) the dihedral group D_8;

(iii) the direct product $C_2 \times D_8$.

20G Number Fields

Let $K = \mathbb{Q}(\alpha)$, where $\alpha^3 = 5\alpha - 8$.

(a) Show that $[K : \mathbb{Q}] = 3$.

(b) Let $\beta = (\alpha + \alpha^2)/2$. By considering the matrix of β acting on K by multiplication, or otherwise, show that β is an algebraic integer, and that $(1, \alpha, \beta)$ is a $\mathbb{Z}$-basis for $\mathcal{O}_K$. [*The discriminant of* $T^3 - 5T + 8$ *is* $-4 \cdot 307$, *and* 307 *is prime.*]

(c) Compute the prime factorisation of the ideal (3) in $\mathcal{O}_K$. Is (2) a prime ideal of $\mathcal{O}_K$? Justify your answer.

21F Algebraic Topology

(a) What does it mean for two spaces X and Y to be *homotopy equivalent*?

(b) What does it mean for a subspace $Y \subseteq X$ to be a *retract* of a space X? What does it mean for a space X to be *contractible*? Show that a retract of a contractible space is contractible.

(c) Let X be a space and $A \subseteq X$ a subspace. We say the pair (X, A) has the *homotopy extension property* if, for any pair of maps $f : X \times \{0\} \to Y$ and $H' : A \times I \to Y$ with

$$f|_{A\times\{0\}} = H'|_{A\times\{0\}},$$

there exists a map $H : X \times I \to Y$ with

$$H|_{X\times\{0\}} = f, \qquad H|_{A\times I} = H'.$$

Now suppose that $A \subseteq X$ is contractible. Denote by X/A the quotient of X by the equivalence relation $x \sim x'$ if and only if $x = x'$ or $x, x' \in A$. Show that, if (X, A) satisfies the homotopy extension property, then X and X/A are homotopy equivalent.

22H Linear Analysis

Let H be a separable Hilbert space and $\{e_i\}$ be a Hilbertian (orthonormal) basis of H. Given a sequence (x_n) of elements of H and $x_\infty \in H$, we say that x_n weakly converges to x_∞, denoted $x_n \rightharpoonup x_\infty$, if $\forall h \in H$, $\lim_{n\to\infty}\langle x_n, h\rangle = \langle x_\infty, h\rangle$.

(a) Given a sequence (x_n) of elements of H, prove that the following two statements are equivalent:

(i) $\exists x_\infty \in H$ such that $x_n \rightharpoonup x_\infty$;

(ii) the sequence (x_n) is bounded in H and $\forall i \geqslant 1$, the sequence $(\langle x_n, e_i\rangle)$ is convergent.

(b) Let (x_n) be a bounded sequence of elements of H. Show that there exists $x_\infty \in H$ and a subsequence $(x_{\phi(n)})$ such that $x_{\phi(n)} \rightharpoonup x_\infty$ in H.

(c) Let (x_n) be a sequence of elements of H and $x_\infty \in H$ be such that $x_n \rightharpoonup x_\infty$. Show that the following three statements are equivalent:

(i) $\lim_{n\to\infty} \|x_n - x_\infty\| = 0$;

(ii) $\lim_{n\to\infty} \|x_n\| = \|x_\infty\|$;

(iii) $\forall \epsilon > 0$, $\exists I(\epsilon)$ such that $\forall n \geqslant 1$, $\sum_{i \geqslant I(\epsilon)} |\langle x_n, e_i\rangle|^2 < \epsilon$.

23H Analysis of Functions

Below, $\mathcal{M}$ is the σ-algebra of Lebesgue measurable sets and λ is Lebesgue measure.

(a) State the Lebesgue differentiation theorem for an integrable function $f : \mathbb{R}^n \to \mathbb{C}$. Let $g : \mathbb{R} \to \mathbb{C}$ be integrable and define $G : \mathbb{R} \to \mathbb{C}$ by $G(x) := \int_{[a,x]} g \, d\lambda$ for some $a \in \mathbb{R}$. Show that G is differentiable λ-almost everywhere.

(b) Suppose $h : \mathbb{R} \to \mathbb{R}$ is strictly increasing, continuous, and maps sets of λ-measure zero to sets of λ-measure zero. Show that we can define a measure ν on $\mathcal{M}$ by setting $\nu(A) := \lambda\big(h(A)\big)$ for $A \in \mathcal{M}$, and establish that $\nu \ll \lambda$. Deduce that h is differentiable λ-almost everywhere. Does the result continue to hold if h is assumed to be non-decreasing rather than strictly increasing?

[*You may assume without proof that a strictly increasing, continuous, function* $w : \mathbb{R} \to \mathbb{R}$ *is injective, and* $w^{-1} : w(\mathbb{R}) \to \mathbb{R}$ *is continuous.*]

24F Riemann Surfaces

(a) Consider an open disc $D \subseteq \mathbb{C}$. Prove that a real-valued function $u : D \to \mathbb{R}$ is harmonic if and only if

$$u = \mathrm{Re}(f)$$

for some analytic function f.

(b) Give an example of a domain D and a harmonic function $u : D \to \mathbb{R}$ that is not equal to the real part of an analytic function on D. Justify your answer carefully.

(c) Let u be a harmonic function on $\mathbb{C}_*$ such that $u(2z) = u(z)$ for every $z \in \mathbb{C}_*$. Prove that u is constant, justifying your answer carefully. Exhibit a countable subset $S \subseteq \mathbb{C}_*$ and a non-constant harmonic function u on $\mathbb{C}_* \setminus S$ such that for all $z \in \mathbb{C}_* \setminus S$ we have $2z \in \mathbb{C}_* \setminus S$ and $u(2z) = u(z)$.

(d) Prove that every non-constant harmonic function $u : \mathbb{C} \to \mathbb{R}$ is surjective.

25I Algebraic Geometry

Let k be an algebraically closed field and let $V \subset \mathbb{A}^n_k$ be a non-empty affine variety. Show that V is a finite union of irreducible subvarieties.

Let V_1 and V_2 be subvarieties of $\mathbb{A}^n_k$ given by the vanishing loci of ideals I_1 and I_2 respectively. Prove the following assertions.

(i) The variety $V_1 \cap V_2$ is equal to the vanishing locus of the ideal $I_1 + I_2$.

(ii) The variety $V_1 \cup V_2$ is equal to the vanishing locus of the ideal $I_1 \cap I_2$.

Decompose the vanishing locus

$$\mathbb{V}(X^2 + Y^2 - 1,\, X^2 - Z^2 - 1) \subset \mathbb{A}^3_{\mathbb{C}}.$$

into irreducible components.

Let $V \subset \mathbb{A}^3_k$ be the union of the three coordinate axes. Let W be the union of three distinct lines through the point $(0,0)$ in $\mathbb{A}^2_k$. Prove that W is not isomorphic to V.

26F Differential Geometry

(a) Let $S \subset \mathbb{R}^3$ be a surface. Give a parametrisation-free definition of the *first fundamental form* of S. Use this definition to derive a description of it in terms of the partial derivatives of a local parametrisation $\phi : U \subset \mathbb{R}^2 \to S$.

(b) Let a be a positive constant. Show that the half-cone

$$\Sigma = \{(x, y, z) \,|\, z^2 = a(x^2 + y^2),\ z > 0\}$$

is locally isometric to the Euclidean plane. [*Hint: Use polar coordinates on the plane.*]

(c) Define the *second fundamental form* and the *Gaussian curvature* of S. State Gauss' Theorema Egregium. Consider the set

$$V = \{(x, y, z) \,|\, x^2 + y^2 + z^2 - 2xy - 2yz = 0\} \backslash \{(0, 0, 0)\} \subset \mathbb{R}^3.$$

(i) Show that V is a surface.

(ii) Calculate the Gaussian curvature of V at each point. [*Hint: Complete the square.*]

27H Probability and Measure

(a) State and prove Fatou's lemma. [You may use the monotone convergence theorem without proof, provided it is clearly stated.]

(b) Show that the inequality in Fatou's lemma can be strict.

(c) Let $(X_n : n \in \mathbb{N})$ and X be non-negative random variables such that $X_n \to X$ almost surely as $n \to \infty$. Must we have $\mathbb{E}X \leqslant \sup_n \mathbb{E}X_n$?

28K Applied Probability

The particles of an *Ideal Gas* form a spatial Poisson process on $\mathbb{R}^3$ with constant intensity $z > 0$, called the *activity* of the gas.

(a) Prove that the independent mixture of two Ideal Gases with activities z_1 and z_2 is again an Ideal Gas. What is its activity? [You must prove any results about Poisson processes that you use. The independent mixture of two gases with particles $\Pi_1 \subset \mathbb{R}^3$ and $\Pi_2 \subset \mathbb{R}^3$ is given by $\Pi_1 \cup \Pi_2$.]

(b) For an Ideal Gas of activity $z > 0$, find the limiting distribution of

$$\frac{N(V_i) - \mathbb{E}N(V_i)}{\sqrt{|V_i|}}$$

as $i \to \infty$ for a given sequence of subsets $V_i \subset \mathbb{R}^3$ with $|V_i| \to \infty$.

(c) Let $g : \mathbb{R}^3 \to \mathbb{R}$ be a smooth non-negative function vanishing outside a bounded subset of $\mathbb{R}^3$. Find the mean and variance of $\sum_x g(x)$, where the sum runs over the particles $x \in \mathbb{R}^3$ of an ideal gas of activity $z > 0$. [You may use the properties of spatial Poisson processes established in the lectures.]

[Hint: recall that the characteristic function of a Poisson random variable with mean λ is $e^{(e^{it}-1)\lambda}$.]

29J Principles of Statistics

Let $X_1, \ldots, X_n$ be random variables with joint probability density function in a statistical model $\{f_\theta : \theta \in \mathbb{R}\}$.

(a) Define the *Fisher information* $I_n(\theta)$. What do we mean when we say that the Fisher information *tensorises*?

(b) Derive the relationship between the Fisher information and the derivative of the score function in a regular model.

(c) Consider the model defined by $X_1 = \theta + \varepsilon_1$ and

$$X_i = \theta(1 - \sqrt{\gamma}) + \sqrt{\gamma}\, X_{i-1} + \sqrt{1-\gamma}\; \varepsilon_i \quad \text{for } i = 2, \ldots, n,$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. $N(0,1)$ random variables, and $\gamma \in [0,1)$ is a known constant. Compute the Fisher information $I_n(\theta)$. For which values of γ does the Fisher information tensorise? State a lower bound on the variance of an unbiased estimator $\hat{\theta}$ in this model.

30K Stochastic Financial Models

(a) What does it mean to say that a stochastic process $(X_n)_{n\geqslant 0}$ is a *martingale* with respect to a filtration $(\mathcal{F}_n)_{n\geqslant 0}$?

(b) Let $(X_n)_{n\geqslant 0}$ be a martingale, and let $\xi_n = X_n - X_{n-1}$ for $n \geqslant 1$. Suppose ξ_n takes values in the set $\{-1, +1\}$ almost surely for all $n \geqslant 1$. Show that $(X_n)_{n\geqslant 0}$ is a simple symmetric random walk, i.e. that the sequence $(\xi_n)_{n\geqslant 1}$ is IID with $\mathbb{P}(\xi_1 = 1) = 1/2 = \mathbb{P}(\xi_1 = -1)$.

(c) Let $(X_n)_{n\geqslant 0}$ be a martingale and let the bounded process $(H_n)_{n\geqslant 1}$ be previsible. Let $\hat{X}_0 = 0$ and

$$\hat{X}_n = \sum_{k=1}^{n} H_k(X_k - X_{k-1}) \text{ for } n \geqslant 1.$$

Show that $(\hat{X}_n)_{n\geqslant 0}$ is a martingale.

(d) Let $(X_n)_{n\geqslant 0}$ be a simple symmetric random walk with $X_0 = 0$, and let

$$T_a = \inf\{n \geqslant 0 : X_n = a\},$$

where a is a positive integer. Let

$$\hat{X}_n = \begin{cases} X_n & \text{if } n \leqslant T_a \\ 2a - X_n & \text{if } n > T_a. \end{cases}$$

Show that $(\hat{X}_n)_{n\geqslant 0}$ is a simple symmetric random walk.

(e) Let $(X_n)_{n\geqslant 0}$ be a simple symmetric random walk with $X_0 = 0$, and let $M_n = \max_{0\leqslant k\leqslant n} X_k$. Compute $\mathbb{P}(M_n = a)$ for a positive integer a.

31J Mathematics of Machine Learning

Let $\mathcal{H}$ be a family of functions $h : \mathcal{X} \to \{0,1\}$ with $|\mathcal{H}| \geqslant 2$. Define the *shattering coefficient* $s(\mathcal{H}, n)$ and the *VC dimension* $\mathrm{VC}(\mathcal{H})$ of $\mathcal{H}$.

Briefly explain why if $\mathcal{H}' \subseteq \mathcal{H}$ and $|\mathcal{H}'| \geqslant 2$, then $\mathrm{VC}(\mathcal{H}') \leqslant \mathrm{VC}(\mathcal{H})$.

Prove that if $\mathcal{F}$ is a vector space of functions $f : \mathcal{X} \to \mathbb{R}$ with $\mathcal{F}' \subseteq \mathcal{F}$ and we define

$$\mathcal{H} = \{\mathbf{1}_{\{u: f(u) \leqslant 0\}} : f \in \mathcal{F}'\},$$

then $\mathrm{VC}(\mathcal{H}) \leqslant \dim(\mathcal{F})$.

Let $\mathcal{A} = \big\{\{x : \|x - c\|_2^2 \leqslant r^2\} \, : c \in \mathbb{R}^d, \, r \in [0, \infty)\big\}$ be the set of all spheres in $\mathbb{R}^d$. Suppose $\mathcal{H} = \{\mathbf{1}_A : A \in \mathcal{A}\}$. Show that

$$\mathrm{VC}(\mathcal{H}) \leqslant d + 2.$$

[*Hint: Consider the class of functions* $\mathcal{F}' = \big\{f_{c,r} : c \in \mathbb{R}^d, \, r \in [0, \infty)\big\}$, *where*

$$f_{c,r}(x) = \|x\|_2^2 - 2c^T x + \|c\|_2^2 - r^2 \, .$$]

32A Dynamical Systems

(a) State the properties defining a *Lyapunov function* for a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. State Lyapunov's first theorem and La Salle's invariance principle.

(b) Consider the system

$$\dot{x} = y \, ,$$
$$\dot{y} = -\frac{2x(1-x^2)}{(1+x^2)^3} - ky \, .$$

Show that for $k > 0$ the origin is asymptotically stable, stating clearly any arguments that you use.

[*Hint:* $\dfrac{d}{dx}\dfrac{x^2}{(1+x^2)^2} = \dfrac{2x(1-x^2)}{(1+x^2)^3}$.]

(c) Sketch the phase plane, (i) for $k = 0$ and (ii) for $0 < k \ll 1$, giving brief details of any reasoning and identifying the fixed points. Include the domain of stability of the origin in your sketch for case (ii).

(d) For $k > 0$ show that the trajectory $\mathbf{x}(t)$ with $\mathbf{x}(0) = (1, y_0)$, where $y_0 > 0$, satisfies $0 < y(t) < \sqrt{y_0^2 + \frac{1}{2}}$ for $t > 0$. Show also that, for any $\epsilon > 0$, the trajectory cannot remain outside the region $0 < y < \epsilon$.

33D Integrable Systems

(a) Let $U(z,\bar{z},\lambda)$ and $V(z,\bar{z},\lambda)$ be matrix-valued functions, whilst $\psi(z,\bar{z},\lambda)$ is a vector-valued function. Show that the linear system

$$\partial_z \psi = U\psi\,, \qquad \partial_{\bar{z}}\psi = V\psi$$

is over-determined and derive a consistency condition on U, V that is necessary for there to be non-trivial solutions.

(b) Suppose that

$$U = \frac{1}{2\lambda}\begin{pmatrix} \lambda\,\partial_z u & e^{-u} \\ e^{u} & -\lambda\,\partial_z u \end{pmatrix} \qquad \text{and} \qquad V = \frac{1}{2}\begin{pmatrix} -\partial_{\bar{z}} u & \lambda e^{u} \\ \lambda e^{-u} & \partial_{\bar{z}} u \end{pmatrix}\,,$$

where $u(z,\bar{z})$ is a scalar function. Obtain a partial differential equation for u that is equivalent to your consistency condition from part (a).

(c) Now let $z = x + iy$ and suppose u is independent of y. Show that the trace of $(U-V)^n$ is constant for all positive integers n. Hence, or otherwise, construct a non-trivial first integral of the equation

$$\frac{d^2\phi}{dx^2} = 4\sinh\phi\,, \qquad \text{where} \quad \phi = \phi(x)\,.$$

34B Principles of Quantum Mechanics

(a) A group G of transformations acts on a quantum system. Briefly explain why the Born rule implies that these transformations may be represented by operators $U(g) : \mathcal{H} \to \mathcal{H}$ obeying

$$U(g)^\dagger \, U(g) = 1_{\mathcal{H}}$$
$$U(g_1) \, U(g_2) = e^{i\phi(g_1,g_2)} \, U(g_1 \cdot g_2)$$

for all $g_1, g_2 \in G$, where $\phi(g_1, g_2) \in \mathbb{R}$.

What additional property does $U(g)$ have when G is a group of symmetries of the Hamiltonian? Show that symmetries correspond to conserved quantities.

(b) The Coulomb Hamiltonian describing the gross structure of the hydrogen atom is invariant under time reversal, $t \mapsto -t$. Suppose we try to represent time reversal by a unitary operator T obeying $U(t)T = TU(-t)$, where $U(t)$ is the time-evolution operator. Show that this would imply that hydrogen has no stable ground state.

An operator $A : \mathcal{H} \to \mathcal{H}$ is *anti*linear if

$$A\big(a|\alpha\rangle + b|\beta\rangle\big) = \bar{a} \, A|\alpha\rangle + \bar{b} \, A|\beta\rangle$$

for all $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$ and all $a, b \in \mathbb{C}$, and *anti*unitary if, in addition,

$$\langle\beta'|\alpha'\rangle = \overline{\langle\beta|\alpha\rangle}\,,$$

where $|\alpha'\rangle = A|\alpha\rangle$ and $|\beta'\rangle = A|\beta\rangle$. Show that if time reversal is instead represented by an antiunitary operator then the above instability of hydrogen is avoided.

35B Application of Quantum Mechanics

(a) Discuss the variational principle that allows one to derive an upper bound on the energy E_0 of the ground state for a particle in one dimension subject to a potential $V(x)$.

If $V(x) = V(-x)$, how could you adapt the variational principle to derive an upper bound on the energy E_1 of the first excited state?

(b) Consider a particle of mass $2m = \hbar^2$ (in certain units) subject to a potential

$$V(x) = -V_0 e^{-x^2} \quad \text{with} \quad V_0 > 0\,.$$

(i) Using the trial wavefunction

$$\psi(x) = e^{-\frac{1}{2}x^2 a}\,,$$

with $a > 0$, derive the upper bound $E_0 \leqslant E(a)$, where

$$E(a) = \frac{1}{2}a - V_0 \frac{\sqrt{a}}{\sqrt{1+a}}\,.$$

(ii) Find the zero of $E(a)$ in $a > 0$ and show that any extremum must obey

$$(1+a)^3 = \frac{V_0^2}{a}\,.$$

(iii) By sketching $E(a)$ or otherwise, deduce that there must always be a minimum in $a > 0$. Hence deduce the existence of a bound state.

(iv) Working perturbatively in $0 < V_0 \ll 1$, show that

$$-V_0 < E_0 \leqslant -\frac{1}{2}V_0^2 + \mathcal{O}(V_0^3)\,.$$

[*Hint: You may use that* $\int_{-\infty}^{\infty} e^{-bx^2} dx = \sqrt{\frac{\pi}{b}}$ *for* $b > 0$.]

36C Statistical Physics

Throughout this question you should consider a classical gas and assume that the number of particles is fixed.

(a) Write down the equation of state for an ideal gas. Write down an expression for the internal energy of an ideal gas in terms of the heat capacity at constant volume, C_V.

(b) Starting from the first law of thermodynamics, find a relation between C_V and the heat capacity at constant pressure, C_p, for an ideal gas. Hence give an expression for $\gamma = C_p/C_V$.

(c) Describe the meaning of an *adiabatic process*. Using the first law of thermodynamics, derive the equation for an adiabatic process in the (p, V)-plane for an ideal gas.

(d) Consider a simplified Otto cycle (an idealised petrol engine) involving an ideal gas and consisting of the following four reversible steps:

$A \to B$: Adiabatic compression from volume V_1 to volume $V_2 < V_1$;
$B \to C$: Heat Q_1 injected at constant volume;
$C \to D$: Adiabatic expansion from volume V_2 to volume V_1;
$D \to A$: Heat Q_2 extracted at constant volume.

Sketch the cycle in the (p, V)-plane and in the (T, S)-plane.

Derive an expression for the efficiency, $\eta = W/Q_1$, where W is the work out, in terms of the compression ratio $r = V_1/V_2$. How can the efficiency be maximized?

37C Electrodynamics

(a) An electromagnetic field is specified by a four-vector potential

$$A^\mu(\mathbf{x}, t) = \big(\phi(\mathbf{x}, t)/c\, ,\, \mathbf{A}(\mathbf{x}, t) \big)\, .$$

Define the corresponding *field-strength tensor* $F^{\mu\nu}$ and state its transformation property under a general Lorentz transformation.

(b) Write down two independent Lorentz scalars that are quadratic in the field strength and express them in terms of the electric and magnetic fields, $\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Show that both these scalars vanish when evaluated on an electromagnetic plane-wave solution of Maxwell's equations of arbitrary wavevector and polarisation.

(c) Find (non-zero) constant, homogeneous background fields $\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0$ and $\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0$ such that both the Lorentz scalars vanish. Show that, for any such background, the field-strength tensor obeys

$$F^{\mu}_{\ \ \rho}\, F^{\rho}_{\ \ \sigma} F^{\sigma}_{\ \ \nu} = 0\, .$$

(d) Hence find the trajectory of a relativistic particle of mass m and charge q in this background. You should work in an inertial frame where the particle is at rest at the origin at $t = 0$ and in which $\mathbf{B}_0 = (0, 0, B_0)$.

38C General Relativity

The Weyl tensor $C_{\alpha\beta\gamma\delta}$ may be defined (in $n=4$ spacetime dimensions) as

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{2}\left(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\delta}R_{\beta\gamma} - g_{\beta\gamma}R_{\alpha\delta}\right) + \frac{1}{6}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})R\,,$$

where $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor, $R_{\alpha\beta}$ is the Ricci tensor and R is the Ricci scalar.

(a) Show that $C^{\alpha}{}_{\beta\alpha\delta} = 0$ and deduce that all other contractions vanish.

(b) A conformally flat metric takes the form

$$g_{\alpha\beta} = e^{2\omega}\eta_{\alpha\beta}\,,$$

where $\eta_{\alpha\beta}$ is the Minkowski metric and ω is a scalar function. Calculate the Weyl tensor at a given point p. [You may assume that $\partial_\alpha\omega = 0$ at p.]

(c) The Schwarzschild metric outside a spherically symmetric mass (such as the Sun, Earth or Moon) is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2.$$

(i) Calculate the leading-order contribution to the Weyl component C_{trtr} valid at large distances, $r \gg 2M$, beyond the central spherical mass.

(ii) What physical phenomenon, known from ancient times, can be attributed to this component of the Weyl tensor at the location of the Earth? [This is after subtracting off the Earth's own gravitational field, and neglecting the Earth's motion within the solar system.] Briefly explain why your answer is consistent with the Einstein equivalence principle.

39A Fluid Dynamics II

(a) Write down the Stokes equations for the motion of an incompressible viscous fluid with negligible inertia (in the absence of body forces). What does it mean that Stokes flow is *linear* and *reversible*?

(b) The region $a < r < b$ between two concentric rigid spheres of radii a and b is filled with fluid of large viscosity μ. The outer sphere is held stationary, while the inner sphere is made to rotate with angular velocity $\mathbf{\Omega}$.

(i) Use symmetry and the properties of Stokes flow to deduce that $p = 0$, where p is the pressure due to the flow.

(ii) Verify that both solid-body rotation and $\mathbf{u}(\mathbf{x}) = \mathbf{\Omega} \wedge \mathbf{\nabla}(1/r)$ satisfy the Stokes equations with $p = 0$. Hence determine the fluid velocity between the spheres.

(iii) Calculate the stress tensor σ_{ij} in the flow.

(iv) Deduce that the couple $\mathbf{G}$ exerted by the fluid in $r < c$ on the fluid in $r > c$, where $a < c < b$, is given by

$$\mathbf{G} = \frac{8\pi\mu a^3 b^3 \mathbf{\Omega}}{b^3 - a^3},$$

independent of the value of c. [*Hint: Do not substitute the form of A and B in $A + Br^{-3}$ until the end of the calculation.*]

Comment on the form of this result for $a \ll b$ and for $b - a \ll a$.

$$\left[\textit{You may use } \int_{r=R} n_i n_j \, dS = \frac{4}{3}\pi R^2 \delta_{ij}, \textit{ where } \mathbf{n} \textit{ is the normal to } r = R. \right]$$

40A Waves

Compressible fluid of equilibrium density ρ_0, pressure p_0 and sound speed c_0 is contained in the region between an inner rigid sphere of radius R and an outer elastic sphere of equilibrium radius $2R$. The elastic sphere is made to oscillate radially in such a way that it exerts a spherically symmetric, perturbation pressure $\tilde{p} = \epsilon p_0 \cos \omega t$ on the fluid at $r = 2R$, where $\epsilon \ll 1$ and the frequency ω is sufficiently small that

$$\alpha \equiv \frac{\omega R}{c_0} \leqslant \frac{\pi}{2} .$$

You may assume that the acoustic velocity potential satisfies the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \nabla^2 \phi .$$

(a) Derive an expression for $\phi(r, t)$.

(b) Hence show that the net radial component of the acoustic intensity (wave-energy flux) $\mathbf{I} = \tilde{p}\mathbf{u}$ is zero when averaged appropriately in a way you should define. Interpret this result physically.

(c) Briefly discuss the possible behaviour of the system if the forcing frequency ω is allowed to increase to larger values.

$\left[\textit{For a spherically symmetric variable } \psi(r,t),\ \nabla^2 \psi = \frac{1}{r} \frac{\partial^2}{\partial r^2}(r\psi) \, . \right]$

41E Numerical Analysis

Let $A \in \mathbb{R}^{n \times n}$ with $n > 2$ and define $\operatorname{Spec}(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not invertible}\}$. The QR algorithm for computing $\operatorname{Spec}(A)$ is defined as follows. Set $A_0 = A$. For $k = 0, 1, \ldots$ compute the QR factorization $A_k = Q_k R_k$ and set $A_{k+1} = R_k Q_k$. (Here Q_k is an $n \times n$ orthogonal matrix and R_k is an $n \times n$ upper triangular matrix.)

(a) Show that A_{k+1} is related to the original matrix A by the similarity transformation $A_{k+1} = \bar{Q}_k^T A \bar{Q}_k$, where $\bar{Q}_k = Q_0 Q_1 \cdots Q_k$ is orthogonal and $\bar{Q}_k \bar{R}_k$ is the QR factorization of A^{k+1} with $\bar{R}_k = R_k R_{k-1} \cdots R_0$.

(b) Suppose that A is symmetric and that its eigenvalues satisfy

$$|\lambda_1| < |\lambda_2| < \cdots < |\lambda_{n-1}| = |\lambda_n| .$$

Suppose, in addition, that the first two canonical basis vectors are given by $\mathbf{e}_1 = \sum_{i=1}^n b_i \mathbf{w}_i$, $\mathbf{e}_2 = \sum_{i=1}^n c_i \mathbf{w}_i$, where $b_i \neq 0$, $c_i \neq 0$ for $i = 1, \ldots, n$ and $\{\mathbf{w}_i\}_{i=1}^n$ are the normalised eigenvectors of A.

Let $B_k \in \mathbb{R}^{2 \times 2}$ be the 2×2 upper left corner of A_k. Show that $d_{\mathrm{H}}\big(\operatorname{Spec}(B_k), S\big) \to 0$ as $k \to \infty$, where $S = \{\lambda_n\} \cup \{\lambda_{n-1}\}$ and d_{H} denotes the Hausdorff metric

$$d_{\mathrm{H}}(X, Y) = \max \left\{ \sup_{x \in X} \inf_{y \in Y} |x - y|, \sup_{y \in Y} \inf_{x \in X} |x - y| \right\}, \qquad X, Y \subset \mathbb{C} .$$

[*Hint: You may use the fact that for real symmetric matrices* U, V *we have* $d_{\mathrm{H}}\big(\operatorname{Spec}(U), \operatorname{Spec}(V)\big) \leqslant \|U - V\|_2$.]

END OF PAPER