UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Monday, 21 June, 2021 10:00am to 1:00pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

Let $\mathrm{Mat}_n(\mathbb{C})$ be the vector space of n by n complex matrices.

Given $A \in \mathrm{Mat}_n(\mathbb{C})$, define the linear map $\varphi_A : \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$,

$$X \mapsto AX - XA.$$

(i) Compute a basis of eigenvectors, and their associated eigenvalues, when A is the diagonal matrix

$$A = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{pmatrix}.$$

What is the rank of φ_A?

(ii) Now let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Write down the matrix of the linear transformation φ_A with respect to the standard basis of $\mathrm{Mat}_2(\mathbb{C})$.

What is its Jordan normal form?

2F Analysis and Topology

Let X be a topological space with an equivalence relation, $\tilde{X}$ the set of equivalence classes, $\pi : X \to \tilde{X}$, the quotient map taking a point in X to its equivalence class.

(a) Define the *quotient topology* on $\tilde{X}$ and check it is a topology.

(b) Prove that if Y is a topological space, a map $f : \tilde{X} \to Y$ is continuous if and only if $f \circ \pi$ is continuous.

(c) If X is Hausdorff, is it true that $\tilde{X}$ is also Hausdorff? Justify your answer.

3G Complex Analysis

Let f be a holomorphic function on a neighbourhood of $a \in \mathbb{C}$. Assume that f has a zero of order k at a with $k \geqslant 1$. Show that there exist $\varepsilon > 0$ and $\delta > 0$ such that for any b with $0 < |b| < \varepsilon$ there are exactly k distinct values of $z \in D(a, \delta)$ with $f(z) = b$.

4C Quantum Mechanics

Let $\Psi(x,t)$ be the wavefunction for a particle of mass m moving in one dimension in a potential $U(x)$. Show that, with suitable boundary conditions as $x \to \pm\infty$,

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx \; = \; 0 \, .$$

Why is this important for the interpretation of quantum mechanics?

Verify the result above by first calculating $|\Psi(x,t)|^2$ for the free particle solution

$$\Psi(x,t) \; = \; C f(t)^{1/2} \exp\Big(-\frac{1}{2} f(t) \, x^2 \Big) \quad \text{with} \quad f(t) = \Big(\alpha \, + \, \frac{i\hbar}{m} t \Big)^{-1} \, ,$$

where C and $\alpha > 0$ are real constants, and then considering the resulting integral.

5D Electromagnetism

Write down Maxwell's equations in a vacuum. Show that they admit wave solutions with

$$\mathbf{B}(\mathbf{x},t) = \mathrm{Re}\left[\mathbf{B}_0 \, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \right] ,$$

where $\mathbf{B}_0$, $\mathbf{k}$ and ω must obey certain conditions that you should determine. Find the corresponding electric field $\mathbf{E}(\mathbf{x},t)$.

A light wave, travelling in the x-direction and linearly polarised so that the magnetic field points in the z-direction, is incident upon a conductor that occupies the half-space $x > 0$. The electric and magnetic fields obey the boundary conditions $\mathbf{E} \times \mathbf{n} = \mathbf{0}$ and $\mathbf{B} \cdot \mathbf{n} = 0$ on the surface of the conductor, where $\mathbf{n}$ is the unit normal vector. Determine the contributions to the magnetic field from the incident and reflected waves in the region $x \leqslant 0$. Compute the magnetic field tangential to the surface of the conductor.

6B Numerical Analysis

(a) Given the data $f(0) = 0$, $f(1) = 4$, $f(2) = 2$, $f(3) = 8$, find the interpolating cubic polynomial $p_3 \in \mathbb{P}_3[x]$ in the Newton form.

(b) We add to the data one more value, $f(-2) = 10$. Find the interpolating quartic polynomial $p_4 \in \mathbb{P}_4[x]$ for the extended data in the Newton form.

7H Markov Chains

Show that the simple symmetric random walk on $\mathbb{Z}$ is recurrent.

Three particles perform independent simple symmetric random walks on $\mathbb{Z}$. What is the probability that they are all simultaneously at 0 infinitely often? Justify your answer.

[*You may assume without proof that there exist constants* $A, B > 0$ *such that* $A\sqrt{n}(n/e)^n \leqslant n! \leqslant B\sqrt{n}(n/e)^n$ *for all positive integers* n.]

SECTION II

8E Linear Algebra

(a) Let V be a complex vector space of dimension n.

What is a *Hermitian form* on V?

Given a Hermitian form, define the matrix A of the form with respect to the basis $v_1, \ldots, v_n$ of V, and describe in terms of A the value of the Hermitian form on two elements of V.

Now let $w_1, \ldots, w_n$ be another basis of V. Suppose $w_i = \sum_j p_{ij} v_j$, and let $P = (p_{ij})$. Write down the matrix of the form with respect to this new basis in terms of A and P.

Let $N = V^{\perp}$. Describe the dimension of N in terms of the matrix A.

(b) Write down the matrix of the real quadratic form

$$x^2 + y^2 + 2z^2 + 2xy + 2xz - 2yz.$$

Using the Gram–Schmidt algorithm, find a basis which diagonalises the form. What are its rank and signature?

(c) Let V be a real vector space, and $\langle , \rangle$ a symmetric bilinear form on it. Let A be the matrix of this form in some basis.

Prove that the signature of $\langle , \rangle$ is the number of positive eigenvalues of A minus the number of negative eigenvalues.

Explain, using an example, why the eigenvalues themselves depend on the choice of a basis.

9G Groups, Rings and Modules

Let H and P be subgroups of a finite group G. Show that the sets HxP, $x \in G$, partition G. By considering the action of H on the set of left cosets of P in G by left multiplication, or otherwise, show that

$$\frac{|HxP|}{|P|} = \frac{|H|}{|H \cap xPx^{-1}|}$$

for any $x \in G$. Deduce that if G has a Sylow p-subgroup, then so does H.

Let $p, n \in \mathbb{N}$ with p a prime. Write down the order of the group $GL_n(\mathbb{Z}/p\mathbb{Z})$. Identify in $GL_n(\mathbb{Z}/p\mathbb{Z})$ a Sylow p-subgroup and a subgroup isomorphic to the symmetric group S_n. Deduce that every finite group has a Sylow p-subgroup.

State Sylow's theorem on the number of Sylow p-subgroups of a finite group.

Let G be a group of order pq, where $p > q$ are prime numbers. Show that if G is non-abelian, then $q \mid p - 1$.

10F Analysis and Topology

(a) Let $g : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that for each $t \in [0,1]$, the partial derivatives $D_i g(t,x)$ $(i = 1, \ldots, n)$ of $x \mapsto g(t,x)$ exist and are continuous on $[0,1] \times \mathbb{R}^n$. Define $G : \mathbb{R}^n \to \mathbb{R}$ by

$$G(x) = \int_0^1 g(t,x)\,dt.$$

Show that G has continuous partial derivatives $D_i G$ given by

$$D_i G(x) = \int_0^1 D_i g(t,x)\,dt$$

for $i = 1, \ldots, n$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an infinitely differentiable function, that is, partial derivatives $D_{i_1} D_{i_2} \cdots D_{i_k} f$ exist and are continuous for all $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in \{1,2\}$. Show that for any $(x_1, x_2) \in \mathbb{R}^2$,

$$f(x_1,x_2) = f(x_1,0) + x_2\, D_2 f(x_1,0) + x_2^2\, h(x_1,x_2),$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is an infinitely differentiable function.

[*Hint: You may use the fact that if $u : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable, then*

$$u(1) = u(0) + u'(0) + \int_0^1 (1-t) u''(t)\,dt.]$$

11F Geometry

Define an *abstract smooth surface* and explain what it means for the surface to be *orientable*. Given two smooth surfaces S_1 and S_2 and a map $f : S_1 \to S_2$, explain what it means for f to be *smooth*.

For the cylinder

$$C = \{(x,y,z) \in \mathbb{R}^3 :\ x^2 + y^2 = 1\},$$

let $a : C \to C$ be the orientation reversing diffeomorphism $a(x,y,z) = (-x,-y,-z)$. Let S be the quotient of C by the equivalence relation $p \sim a(p)$ and let $\pi : C \to S$ be the canonical projection map. Show that S can be made into an abstract smooth surface so that π is smooth. Is S orientable? Justify your answer.

12B Complex Methods

Let $f(t)$ be defined for $t \geqslant 0$. Define the *Laplace transform* $\widehat{f}(s)$ of f. Find an expression for the Laplace transform of $\frac{df}{dt}$ in terms of $\widehat{f}$.

Three radioactive nuclei decay sequentially, so that the numbers $N_i(t)$ of the three types obey the equations

$$\begin{aligned} \frac{dN_1}{dt} &= -\lambda_1 N_1 \,, \\ \frac{dN_2}{dt} &= \lambda_1 N_1 - \lambda_2 N_2 \,, \\ \frac{dN_3}{dt} &= \lambda_2 N_2 - \lambda_3 N_3 \,, \end{aligned}$$

where $\lambda_3 > \lambda_2 > \lambda_1 > 0$ are constants. Initially, at $t = 0$, $N_1 = N$, $N_2 = 0$ and $N_3 = n$. Using Laplace transforms, find $N_3(t)$.

By taking an appropriate limit, find $N_3(t)$ when $\lambda_2 = \lambda_1 = \lambda > 0$ and $\lambda_3 > \lambda$.

13D Variational Principles

(a) Consider the functional

$$I[y] = \int_a^b L(y, y'; x)\, dx \,,$$

where $0 < a < b$, and $y(x)$ is subject to the requirement that $y(a)$ and $y(b)$ are some fixed constants. Derive the equation satisfied by $y(x)$ when $\delta I = 0$ for all variations δy that respect the boundary conditions.

(b) Consider the function

$$L(y, y'; x) = \frac{\sqrt{1 + y'^2}}{x} \,.$$

Verify that, if $y(x)$ describes an arc of a circle, with centre on the y-axis, then $\delta I = 0$.

(c) Consider the function

$$L(y, y'; x) = \frac{\sqrt{1 + y'^2}}{y} \,.$$

Find $y(x)$ such that $\delta I = 0$ subject to the requirement that $y(a) = a$ and $y(b) = \sqrt{2ab - b^2}$, with $b < 2a$. Sketch the curve $y(x)$.

14C Methods

The function $\theta(x,t)$ obeys the diffusion equation

$$\frac{\partial\theta}{\partial t} = D\frac{\partial^2\theta}{\partial x^2}\,. \qquad (*)$$

Verify that

$$\theta(x,t) = \frac{1}{\sqrt{t}}\, e^{-x^2/4Dt}$$

is a solution of $(*)$, and by considering $\int_{-\infty}^{\infty}\theta(x,t)\,dx$, find the solution having the initial form $\theta(x,0)=\delta(x)$ at $t=0$.

Find, in terms of the error function, the solution of $(*)$ having the initial form

$$\theta(x,0) = \begin{cases} 1\,, & |x| \leqslant 1\,, \\ 0\,, & |x| > 1\,. \end{cases}$$

Sketch a graph of this solution at various times $t \geqslant 0$.

[*The error function is*

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy\,.]$$

15C Quantum Mechanics

(a) Consider the angular momentum operators $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$ and $\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ where

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \quad \hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \text{ and } \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z.$$

Use the standard commutation relations for these operators to show that

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y \qquad \text{obeys} \qquad [\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm \quad \text{and} \quad [\hat{\mathbf{L}}^2, \hat{L}_\pm] = 0\,.$$

Deduce that if φ is a joint eigenstate of $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ with angular momentum quantum numbers m and ℓ respectively, then $\hat{L}_\pm\varphi$ are also joint eigenstates, provided they are non-zero, with quantum numbers $m \pm 1$ and ℓ.

(b) A harmonic oscillator of mass M in three dimensions has Hamiltonian

$$\hat{H} \;=\; \frac{1}{2M}(\,\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\,) \;+\; \frac{1}{2}\,M\omega^2(\,\hat{x}^2 + \hat{y}^2 + \hat{z}^2\,)\,.$$

Find eigenstates of $\hat{H}$ in terms of eigenstates ψ_n for an oscillator in one dimension with $n = 0, 1, 2, \ldots$ and eigenvalues $\hbar\omega(n + \frac{1}{2})$; hence determine the eigenvalues E of $\hat{H}$.

Verify that the ground state for $\hat{H}$ is a joint eigenstate of $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ with $\ell = m = 0$. At the first excited energy level, find an eigenstate of $\hat{L}_z$ with $m = 0$ and construct from this two eigenstates of $\hat{L}_z$ with $m = \pm 1$.

Why should you expect to find joint eigenstates of $\hat{L}_z$, $\hat{\mathbf{L}}^2$ and $\hat{H}$?

[*The first two eigenstates for an oscillator in one dimension are* $\psi_0(x) = C_0 \exp(-M\omega x^2/2\hbar)$ *and* $\psi_1(x) = C_1 x \exp(-M\omega x^2/2\hbar)$*, where* C_0 *and* C_1 *are normalisation constants.*]

16A Fluid Dynamics

Consider the spherically symmetric motion induced by the collapse of a spherical cavity of radius $a(t)$, centred on the origin. For $r < a$, there is a vacuum, while for $r > a$, there is an inviscid incompressible fluid with constant density ρ. At time $t = 0$, $a = a_0$, and the fluid is at rest and at constant pressure p_0.

(a) Consider the radial volume transport in the fluid $Q(R, t)$, defined as

$$Q(R,t) = \int_{r=R} u dS,$$

where u is the radial velocity, and dS is an infinitesimal element of the surface of a sphere of radius $R \geqslant a$. Use the incompressibility condition to establish that Q is a function of time alone.

(b) Using the expression for pressure in potential flow or otherwise, establish that

$$\frac{1}{4\pi a}\frac{dQ}{dt} - \frac{(\dot{a})^2}{2} = -\frac{p_0}{\rho},$$

where $\dot{a}(t)$ is the radial velocity of the cavity boundary.

(c) By expressing $Q(t)$ in terms of a and $\dot{a}$, show that

$$\dot{a} = -\sqrt{\frac{2p_0}{3\rho}\left(\frac{a_0^3}{a^3} - 1\right)}.$$

[*Hint: You may find it useful to assume* $\dot{a}(t)$ *is an explicit function of* a *from the outset.*]

(d) Hence write down an integral expression for the implosion time τ, i.e. the time for the radius of the cavity $a \to 0$. [Do not attempt to evaluate the integral.]

17H Statistics

Suppose we wish to estimate the probability $\theta \in (0,1)$ that a potentially biased coin lands heads up when tossed. After n independent tosses, we observe X heads.

(a) Write down the maximum likelihood estimator $\hat{\theta}$ of θ.

(b) Find the mean squared error $f(\theta)$ of $\hat{\theta}$ as a function of θ. Compute $\sup_{\theta \in (0,1)} f(\theta)$.

(c) Suppose a uniform prior is placed on θ. Find the Bayes estimator of θ under squared error loss $L(\theta, a) = (\theta - a)^2$.

(d) Now find the Bayes estimator $\tilde{\theta}$ under the loss $L(\theta, a) = \theta^{\alpha-1}(1-\theta)^{\beta-1}(\theta - a)^2$, where $\alpha, \beta \geqslant 1$. Show that

$$\tilde{\theta} = w\hat{\theta} + (1-w)\theta_0, \qquad (*)$$

where w and θ_0 depend on n, α and β.

(e) Determine the mean squared error $g_{w,\theta_0}(\theta)$ of $\tilde{\theta}$ as defined by (*).

(f) For what range of values of w do we have $\sup_{\theta \in (0,1)} g_{w,1/2}(\theta) \leqslant \sup_{\theta \in (0,1)} f(\theta)$?

[*Hint: The mean of a Beta(a,b) distribution is $a/(a+b)$ and its density $p(u)$ at $u \in [0,1]$ is $c_{a,b}u^{a-1}(1-u)^{b-1}$, where $c_{a,b}$ is a normalising constant.*]

18H Optimisation

(a) Consider the linear program

$$P: \qquad \text{maximise over } x \geqslant 0, \qquad c^T x$$
$$\text{subject to} \qquad Ax = b,$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. What is meant by a *basic feasible solution*?

(b) Prove that if P has a finite maximum, then there exists a solution that is a basic feasible solution.

(c) Now consider the optimisation problem

$$Q: \qquad \text{maximise over } x \geqslant 0, \qquad \frac{c^T x}{d^T x}$$
$$\text{subject to} \qquad Ax = b,$$
$$d^T x > 0,$$

where matrix A and vectors c, b are as in the problem P, and $d \in \mathbb{R}^n$. Suppose there exists a solution x^* to Q. Further consider the linear program

$$R: \qquad \text{maximise over } y \geqslant 0,\ t \geqslant 0, \qquad c^T y$$
$$\text{subject to} \qquad Ay = bt,$$
$$d^T y = 1.$$

(i) Suppose $d_i > 0$ for all $i = 1, \ldots, n$. Show that the maximum of R is finite and at least as large as that of Q.

(ii) Suppose, in addition to the condition in part (i), that the entries of A are strictly positive. Show that the maximum of R is equal to that of Q.

(iii) Let $\mathcal{B}$ be the set of basic feasible solutions of the linear program P. Assuming the conditions in parts (i) and (ii) above, show that

$$\frac{c^T x^*}{d^T x^*} = \max_{x \in \mathcal{B}} \frac{c^T x}{d^T x}.$$

[*Hint: Argue that if (y, t) is in the set $\mathcal{A}$ of basic feasible solutions to R, then $y/t \in \mathcal{B}$.*]

END OF PAPER