UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 18 June, 2021 10:00am to 1:00pm

PAPER 3

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Groups, Rings and Modules

Let G be a finite group, and let H be a proper subgroup of G of index n.

Show that there is a normal subgroup K of G such that $|G/K|$ divides $n!$ and $|G/K| \geqslant n$.

Show that if G is non-abelian and simple, then G is isomorphic to a subgroup of A_n.

2E Geometry

State the local Gauss–Bonnet theorem for geodesic triangles on a surface. Deduce the Gauss–Bonnet theorem for closed surfaces. [Existence of a geodesic triangulation can be assumed.]

Let $S_r \subset \mathbb{R}^3$ denote the sphere with radius r centred at the origin. Show that the Gauss curvature of S_r is $1/r^2$. An octant is any of the eight regions in S_r bounded by arcs of great circles arising from the planes $x = 0$, $y = 0$, $z = 0$. Verify directly that the local Gauss–Bonnet theorem holds for an octant. [You may assume that the great circles on S_r are geodesics.]

3B Complex Methods

Find the value of A for which the function

$$\phi(x, y) = x \cosh y \sin x + Ay \sinh y \cos x$$

satisfies Laplace's equation. For this value of A, find a complex analytic function of which ϕ is the real part.

4D Variational Principles

Find the function $y(x)$ that gives a stationary value of the functional

$$I[y] = \int_0^1 \left(y'^2 + yy' + y' + y^2 + yx^2\right) dx\,,$$

subject to the boundary conditions $y(0) = -1$ and $y(1) = e - e^{-1} - \frac{3}{2}$.

5A Methods

Let $f(\theta)$ be a 2π-periodic function with Fourier expansion

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) .$$

Find the Fourier coefficients a_n and b_n for

$$f(\theta) = \begin{cases} 1, & 0 < \theta < \pi \\ -1, & \pi < \theta < 2\pi . \end{cases}$$

Hence, or otherwise, find the Fourier coefficients A_n and B_n for the 2π-periodic function F defined by

$$F(\theta) = \begin{cases} \theta, & 0 < \theta < \pi \\ 2\pi - \theta, & \pi < \theta < 2\pi . \end{cases}$$

Use your answers to evaluate

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \qquad \text{and} \qquad \sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} .$$

6C Quantum Mechanics

The electron in a hydrogen-like atom moves in a spherically symmetric potential $V(r) = -K/r$ where K is a positive constant and r is the radial coordinate of spherical polar coordinates. The two lowest energy spherically symmetric normalised states of the electron are given by

$$\chi_1(r) = \frac{1}{\sqrt{\pi}\, a^{3/2}} e^{-r/a} \quad \text{and} \quad \chi_2(r) = \frac{1}{4\sqrt{2\pi}\, a^{3/2}} \left(2 - \frac{r}{a}\right) e^{-r/2a}$$

where $a = \hbar^2/mK$ and m is the mass of the electron. For any spherically symmetric function $f(r)$, the Laplacian is given by $\nabla^2 f = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$.

(i) Suppose that the electron is in the state $\chi(r) = \frac{1}{2}\chi_1(r) + \frac{\sqrt{3}}{2}\chi_2(r)$ and its energy is measured. Find the expectation value of the result.

(ii) Suppose now that the electron is in state $\chi(r)$ (as above) at time $t = 0$. Let $R(t)$ be the expectation value of a measurement of the electron's radial position r at time t. Show that the value of $R(t)$ oscillates sinusoidally about a constant level and determine the frequency of the oscillation.

7A Fluid Dynamics

A two-dimensional flow $\mathbf{u} = (u, v)$ has a velocity field given by

$$u = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad \text{and} \quad v = \frac{2xy}{(x^2 + y^2)^2}.$$

(a) Show explicitly that this flow is incompressible and irrotational away from the origin.

(b) Find the stream function for this flow.

(c) Find the velocity potential for this flow.

8H Markov Chains

Consider a Markov chain $(X_n)_{n \geqslant 0}$ on a state space I.

(a) Define the notion of a *communicating class*. What does it mean for a communicating class to be *closed*?

(b) Taking $I = \{1, \ldots, 6\}$, find the communicating classes associated with the transition matrix P given by

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and identify which are closed.

(c) Find the expected time for the Markov chain with transition matrix P above to reach 6 starting from 1.

SECTION II

9E Linear Algebra

(a)(i) State the rank-nullity theorem.

Let U and W be vector spaces. Write down the definition of their direct sum $U \oplus W$ and the inclusions $i : U \to U \oplus W$, $j : W \to U \oplus W$.

Now let U and W be subspaces of a vector space V. Define $l : U \cap W \to U \oplus W$ by $l(x) = ix - jx$.

Describe the quotient space $(U \oplus W)/\mathrm{Im}(l)$ as a subspace of V.

(ii) Let $V = \mathbb{R}^5$, and let U be the subspace of V spanned by the vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 2 \\ 1 \\ -2 \end{pmatrix},$$

and W the subspace of V spanned by the vectors

$$\begin{pmatrix} 3 \\ 2 \\ -3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ -1 \\ -2 \\ 1 \end{pmatrix}.$$

Determine the dimension of $U \cap W$.

(b) Let A, B be complex n by n matrices with $\mathrm{rank}(B) = k$.

Show that $\det(A + tB)$ is a polynomial in t of degree at most k.

Show that if $k = n$ the polynomial is of degree precisely n.

Give an example where $k \geqslant 1$ but this polynomial is zero.

10G Groups, Rings and Modules

Let p be a non-zero element of a Principal Ideal Domain R. Show that the following are equivalent:

(i) p is prime;

(ii) p is irreducible;

(iii) (p) is a maximal ideal of R;

(iv) $R/(p)$ is a field;

(v) $R/(p)$ is an Integral Domain.

Let R be a Principal Ideal Domain, S an Integral Domain and $\phi\colon R \to S$ a surjective ring homomorphism. Show that either ϕ is an isomorphism or S is a field.

Show that if R is a commutative ring and $R[X]$ is a Principal Ideal Domain, then R is a field.

Let R be an Integral Domain in which every two non-zero elements have a highest common factor. Show that in R every irreducible element is prime.

11F Analysis and Topology

Define the terms *connected* and *path-connected* for a topological space. Prove that the interval $[0,1]$ is connected and that if a topological space is path-connected, then it is connected.

Let X be an open subset of Euclidean space $\mathbb{R}^n$. Show that X is connected if and only if X is path-connected.

Let X be a topological space with the property that every point has a neighbourhood homeomorphic to an open set in $\mathbb{R}^n$. Assume X is connected; must X be also path-connected? Briefly justify your answer.

Consider the following subsets of $\mathbb{R}^2$:

$$A = \{(x,0) : \ x \in (0,1]\}, \ \ B = \{(0,y) : \ y \in [1/2,1]\}, \text{ and}$$

$$C_n = \{(1/n, y) : \ y \in [0,1]\} \text{ for } n \geqslant 1.$$

Let

$$X = A \cup B \cup \bigcup_{n \geqslant 1} C_n$$

with the subspace topology. Is X path-connected? Is X connected? Justify your answers.

12E Geometry

Let $S \subset \mathbb{R}^3$ be an embedded smooth surface and $\gamma : [0,1] \to S$ a parameterised smooth curve on S. What is the *energy* of γ? By applying the Euler–Lagrange equations for stationary curves to the energy function, determine the differential equations for geodesics on S explicitly in terms of a parameterisation of S.

If S contains a straight line ℓ, prove from first principles that each segment $[P, Q] \subset \ell$ (with some parameterisation) is a geodesic on S.

Let $H \subset \mathbb{R}^3$ be the hyperboloid defined by the equation $x^2 + y^2 - z^2 = 1$ and let $P = (x_0, y_0, z_0) \in H$. By considering appropriate isometries, or otherwise, display explicitly *three* distinct (as subsets of H) geodesics $\gamma : \mathbb{R} \to H$ through P in the case when $z_0 \neq 0$ and *four* distinct geodesics through P in the case when $z_0 = 0$. Justify your answer.

Let $\gamma : \mathbb{R} \to H$ be a geodesic, with coordinates $\gamma(t) = (x(t), y(t), z(t))$. Clairaut's relation asserts $\rho(t) \sin \psi(t)$ is constant, where $\rho(t) = \sqrt{x(t)^2 + y(t)^2}$ and $\psi(t)$ is the angle between $\dot{\gamma}(t)$ and the plane through the point $\gamma(t)$ and the z-axis. Deduce from Clairaut's relation that there exist infinitely many geodesics $\gamma(t)$ on H which stay in the half-space $\{z > 0\}$ for all $t \in \mathbb{R}$.

[*You may assume that if $\gamma(t)$ satisfies the geodesic equations on H then γ is defined for all $t \in \mathbb{R}$ and the Euclidean norm $\|\dot{\gamma}(t)\|$ is constant. If you use a version of the geodesic equations for a surface of revolution, then that should be proved.*]

13G Complex Analysis

Let γ be a curve (not necessarily closed) in $\mathbb{C}$ and let $[\gamma]$ denote the image of γ. Let $\phi \colon [\gamma] \to \mathbb{C}$ be a continuous function and define

$$f(z) = \int_\gamma \frac{\phi(\lambda)}{\lambda - z}\, d\lambda$$

for $z \in \mathbb{C} \setminus [\gamma]$. Show that f has a power series expansion about every $a \notin [\gamma]$.

Using Cauchy's Integral Formula, show that a holomorphic function has complex derivatives of all orders. [Properties of power series may be assumed without proof.] Let f be a holomorphic function on an open set U that contains the closed disc $\overline{D}(a, r)$. Obtain an integral formula for the derivative of f on the open disc $D(a, r)$ in terms of the values of f on the boundary of the disc.

Show that if holomorphic functions f_n on an open set U converge locally uniformly to a holomorphic function f on U, then f_n' converges locally uniformly to f'.

Let D_1 and D_2 be two overlapping closed discs. Let f be a holomorphic function on some open neighbourhood of $D = D_1 \cap D_2$. Show that there exist open neighbourhoods U_j of D_j and holomorphic functions f_j on U_j, $j = 1, 2$, such that $f(z) = f_1(z) + f_2(z)$ on $U_1 \cap U_2$.

14A Methods

Let $P(x)$ be a solution of Legendre's equation with eigenvalue λ,

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \lambda P = 0\,,$$

such that P and its derivatives $P^{(k)}(x) = d^kP/dx^k$, $k = 0, 1, 2, \ldots$, are regular at all points x with $-1 \leqslant x \leqslant 1$.

(a) Show by induction that

$$(1-x^2)\frac{d^2}{dx^2}\left[P^{(k)}\right] - 2(k+1)x\frac{d}{dx}\left[P^{(k)}\right] + \lambda_k P^{(k)} = 0$$

for some constant λ_k. Find λ_k explicitly and show that its value is negative when k is sufficiently large, for a fixed value of λ.

(b) Write the equation for $P^{(k)}(x)$ in part (a) in self-adjoint form. Hence deduce that if $P^{(k)}(x)$ is not identically zero, then $\lambda_k \geqslant 0$.

[*Hint: Establish a relation between integrals of the form* $\int_{-1}^{1}[P^{(k+1)}(x)]^2 f(x)\,dx$ *and* $\int_{-1}^{1}[P^{(k)}(x)]^2 g(x)\,dx$ *for certain functions* $f(x)$ *and* $g(x)$.]

(c) Use the results of parts (a) and (b) to show that if $P(x)$ is a non-zero, regular solution of Legendre's equation on $-1 \leqslant x \leqslant 1$, then $P(x)$ is a polynomial of degree n and $\lambda = n(n+1)$ for some integer $n = 0, 1, 2, \ldots$.

15D Electromagnetism

(a) The energy density stored in the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ is given by

$$w = \frac{\epsilon_0}{2}\,\mathbf{E}\cdot\mathbf{E} + \frac{1}{2\mu_0}\,\mathbf{B}\cdot\mathbf{B}\,.$$

Show that, in regions where no electric current flows,

$$\frac{\partial w}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0$$

for some vector field $\mathbf{S}$ that you should determine.

(b) The coordinates $x'^{\mu} = (ct', \mathbf{x}')$ in an inertial frame $\mathcal{S}'$ are related to the coordinates $x^{\mu} = (ct, \mathbf{x})$ in an inertial frame $\mathcal{S}$ by a Lorentz transformation $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$, where

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$. Here v is the relative velocity of $\mathcal{S}'$ with respect to $\mathcal{S}$ in the x-direction.

In frame $\mathcal{S}'$, there is a static electric field $\mathbf{E}'(\mathbf{x}')$ with $\partial\mathbf{E}'/\partial t' = 0$, and no magnetic field. Calculate the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ in frame $\mathcal{S}$. Show that the energy density in frame $\mathcal{S}$ is given in terms of the components of $\mathbf{E}'$ by

$$w = \frac{\epsilon_0}{2}\left[E_x'^{\,2} + \left(\frac{c^2+v^2}{c^2-v^2}\right)\left(E_y'^{\,2} + E_z'^{\,2}\right)\right].$$

Use the fact that $\partial w/\partial t' = 0$ to show that

$$\frac{\partial w}{\partial t} + \boldsymbol{\nabla}\cdot(wv\,\mathbf{e}_x) = 0\,,$$

where $\mathbf{e}_x$ is the unit vector in the x-direction.

16A Fluid Dynamics

A two-dimensional layer of viscous fluid lies between two rigid boundaries at $y = \pm L_0$. The boundary at $y = L_0$ oscillates in its own plane with velocity $(U_0 \cos \omega t, 0)$, while the boundary at $y = -L_0$ oscillates in its own plane with velocity $(-U_0 \cos \omega t, 0)$. Assume that there is no pressure gradient and that the fluid flows parallel to the boundary with velocity $(u(y,t), 0)$, where $u(y,t)$ can be written as $u(y,t) = \mathrm{Re}[U_0 f(y) \exp(i\omega t)]$.

(a) By exploiting the symmetry of the system or otherwise, show that

$$f(y) \quad = \quad \frac{\sinh[(1+i)\Delta \hat{y}]}{\sinh[(1+i)\Delta]}, \text{ where } \hat{y} = \frac{y}{L_0} \text{ and } \Delta = \left(\frac{\omega L_0^2}{2\nu}\right)^{1/2}.$$

(b) Hence or otherwise, show that

$$\frac{u(y,t)}{U_0} = \frac{\cos \omega t\,[\cosh \Delta_+ \cos \Delta_- - \cosh \Delta_- \cos \Delta_+]}{(\cosh 2\Delta - \cos 2\Delta)} + \frac{\sin \omega t\,[\sinh \Delta_+ \sin \Delta_- - \sinh \Delta_- \sin \Delta_+]}{(\cosh 2\Delta - \cos 2\Delta)},$$

where $\Delta_\pm = \Delta(1 \pm \hat{y})$.

(c) Show that, for $\Delta \ll 1$,

$$u(y,t) \simeq \frac{U_0 y}{L_0} \cos \omega t,$$

and briefly interpret this result physically.

17B Numerical Analysis

The functions $p_0, p_1, p_2, \ldots$ are generated by the formula

$$p_n(x) = (-1)^n x^{-1/2} e^x \frac{d^n}{dx^n}\left(x^{n+1/2}e^{-x}\right), \qquad 0 \leqslant x < \infty\,.$$

(a) Show that $p_n(x)$ is a monic polynomial of degree n. Write down the explicit forms of $p_0(x)$, $p_1(x)$, $p_2(x)$.

(b) Demonstrate the orthogonality of these polynomials with respect to the scalar product

$$\langle f, g\rangle = \int_0^\infty x^{1/2} e^{-x} f(x) g(x)\, dx\,,$$

i.e. that $\langle p_n, p_m\rangle = 0$ for $m \neq n$, and show that

$$\langle p_n, p_n\rangle = n!\,\Gamma\left(n + \frac{3}{2}\right)\,,$$

where $\Gamma(y) = \int_0^\infty x^{y-1} e^{-x}\, dx$.

(c) Assuming that a three-term recurrence relation in the form

$$p_{n+1}(x) = (x - \alpha_n) p_n(x) - \beta_n p_{n-1}(x), \quad n = 1, 2, \ldots\,,$$

holds, find the explicit expressions for α_n and β_n as functions of n.

[*Hint: you may use the fact that* $\Gamma(y+1) = y\Gamma(y)$.]

18H Statistics

Consider the normal linear model $Y = X\beta + \varepsilon$ where X is a known $n \times p$ design matrix with $n-2 > p \geqslant 1$, $\beta \in \mathbb{R}^p$ is an unknown vector of parameters, and $\varepsilon \sim N_n(0, \sigma^2 I)$ is a vector of normal errors with each component having variance $\sigma^2 > 0$. Suppose X has full column rank.

(i) Write down the maximum likelihood estimators, $\hat{\beta}$ and $\hat{\sigma}^2$, for β and σ^2 respectively. [You need not derive these.]

(ii) Show that $\hat{\beta}$ is independent of $\hat{\sigma}^2$.

(iii) Find the distributions of $\hat{\beta}$ and $n\hat{\sigma}^2/\sigma^2$.

(iv) Consider the following test statistic for testing the null hypothesis $H_0 : \ \beta = 0$ against the alternative $\beta \neq 0$:

$$T := \frac{\|\hat{\beta}\|^2}{n\hat{\sigma}^2}.$$

Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p > 0$ be the eigenvalues of $X^T X$. Show that under H_0, T has the same distribution as

$$\frac{\sum_{j=1}^{p} \lambda_j^{-1} W_j}{Z}$$

where $Z \sim \chi^2_{n-p}$ and $W_1, \ldots, W_p$ are independent χ^2_1 random variables, independent of Z.

[*Hint: You may use the fact that* $X = UDV^T$ *where* $U \in \mathbb{R}^{n\times p}$ *has orthonormal columns,* $V \in \mathbb{R}^{p\times p}$ *is an orthogonal matrix and* $D \in \mathbb{R}^{p\times p}$ *is a diagonal matrix with* $D_{ii} = \sqrt{\lambda_i}$.]

(v) Find $\mathbb{E}T$ when $\beta \neq 0$. [*Hint: If* $R \sim \chi^2_\nu$ *with* $\nu > 2$, *then* $\mathbb{E}(1/R) = 1/(\nu - 2)$.]

19H Optimisation

Explain what is meant by a *two-person zero-sum game* with $m \times n$ payoff matrix A, and define what is meant by an *optimal strategy* for each player. What are the relationships between the optimal strategies and the value of the game?

Suppose now that

$$A = \begin{pmatrix} 0 & 1 & 1 & -4 \\ -1 & 0 & 2 & 2 \\ -1 & -2 & 0 & 3 \\ 4 & -2 & -3 & 0 \end{pmatrix}.$$

Show that if strategy $p = (p_1, p_2, p_3, p_4)^T$ is optimal for player I, it must also be optimal for player II. What is the value of the game in this case? Justify your answer.

Explain why we must have $(Ap)_i \leqslant 0$ for all i. Hence or otherwise, find the optimal strategy p and prove that it is unique.

END OF PAPER