UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 17 June, 2021 10:00am to 1:00pm

PAPER 2

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Groups, Rings and Modules

Let M be a module over a Principal Ideal Domain R and let N be a submodule of M. Show that M is finitely generated if and only if N and M/N are finitely generated.

2F Analysis and Topology

Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and let $C([0,1])$ denote the set of continuous real-valued functions on $[0,1]$. Given $f \in C([0,1])$, define the function Tf by the expression

$$Tf(x) = \int_0^1 K(x,y)f(y)\,dy.$$

(a) Prove that T is a continuous map $C([0,1]) \to C([0,1])$ with the uniform metric on $C([0,1])$.

(b) Let d_1 be the metric on $C([0,1])$ given by

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\,dx.$$

Is T continuous with respect to d_1?

3C Methods

Consider the differential operator

$$\mathcal{L}y = \frac{d^2y}{dx^2} + 2\frac{dy}{dx}$$

acting on real functions $y(x)$ with $0 \leqslant x \leqslant 1$.

(i) Recast the eigenvalue equation $\mathcal{L}y = -\lambda y$ in Sturm-Liouville form $\tilde{\mathcal{L}}y = -\lambda w y$, identifying $\tilde{\mathcal{L}}$ and w.

(ii) If boundary conditions $y(0) = y(1) = 0$ are imposed, show that the eigenvalues form an infinite discrete set $\lambda_1 < \lambda_2 < \ldots$ and find the corresponding eigenfunctions $y_n(x)$ for $n = 1, 2, \ldots$. If $f(x) = x - x^2$ on $0 \leqslant x \leqslant 1$ is expanded in terms of your eigenfunctions i.e. $f(x) = \sum_{n=1}^{\infty} A_n\, y_n(x)$, give an expression for A_n. The expression can be given in terms of integrals that you need not evaluate.

4D Electromagnetism

State Gauss's Law in the context of electrostatics.

A simple coaxial cable consists of an inner conductor in the form of a perfectly conducting, solid cylinder of radius a, surrounded by an outer conductor in the form of a perfectly conducting, cylindrical shell of inner radius $b > a$ and outer radius $c > b$. The cylinders are coaxial and the gap between them is filled with a perfectly insulating material. The cable may be assumed to be straight and arbitrarily long.

In a steady state, the inner conductor carries an electric charge $+Q$ per unit length, and the outer conductor carries an electric charge $-Q$ per unit length. The charges are distributed in a cylindrically symmetric way and no current flows through the cable.

Determine the electrostatic potential and the electric field as functions of the cylindrical radius r, for $0 < r < \infty$. Calculate the capacitance C of the cable per unit length and the electrostatic energy U per unit length, and verify that these are related by

$$U = \frac{Q^2}{2C}\,.$$

5A Fluid Dynamics

Consider an axisymmetric container, initially filled with water to a depth h_I. A small circular hole of radius r_0 is opened in the base of the container at $z = 0$.

(a) Determine how the radius r of the container should vary with $z < h_I$ so that the depth of the water will decrease at a constant rate.

(b) For such a container, determine how the cross-sectional area A of the free surface should decrease with time.

[*You may assume that the flow rate through the opening is sufficiently small that Bernoulli's theorem for steady flows can be applied.*]

6H Statistics

The efficacy of a new drug was tested as follows. Fifty patients were given the drug, and another fifty patients were given a placebo. A week later, the numbers of patients whose symptoms had gone entirely, improved, stayed the same and got worse were recorded, as summarised in the following table.

	Drug	Placebo
symptoms gone	14	6
improved	21	19
same	10	10
worse	5	15

Conduct a 5% significance level test of the null hypothesis that the medicine and placebo have the same effect, against the alternative that their effects differ.

[*Hint: You may find some of the following values relevant:*

Distribution	χ^2_1	χ^2_2	χ^2_3	χ^2_4	χ^2_6	χ^2_8
95th percentile	3.84	5.99	7.81	9.48	12.59	15.51

]

7H Optimisation

Find the solution to the following optimisation problem using the simplex algorithm:

$$\begin{aligned} \text{maximise} \quad & 3x_1 + 6x_2 + 4x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + x_3 \leqslant 7, \\ & 4x_1 + 2x_2 + 2x_3 \leqslant 5, \\ & x_1 + x_2 + 2x_3 \leqslant 2, \qquad x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

Write down the dual problem and give its solution.

SECTION II

8E Linear Algebra

(a) Compute the characteristic polynomial and minimal polynomial of

$$A = \begin{pmatrix} -2 & -6 & -9 \\ 3 & 7 & 9 \\ -1 & -2 & -2 \end{pmatrix}.$$

Write down the Jordan normal form for A.

(b) Let V be a finite-dimensional vector space over $\mathbb{C}$, $f : V \to V$ be a linear map, and for $\alpha \in \mathbb{C}$, $n \geqslant 1$, write

$$W_{\alpha,n} := \{v \in V \mid (f - \alpha I)^n v = 0\}.$$

(i) Given $v \in W_{\alpha,n}$, $v \neq 0$, construct a non-zero eigenvector for f in terms of v.

(ii) Show that if $w_1, \ldots, w_d$ are non-zero eigenvectors for f with eigenvalues $\alpha_1, \ldots, \alpha_d$, and $\alpha_i \neq \alpha_j$ for all $i \neq j$, then $w_1, \ldots, w_d$ are linearly independent.

(iii) Show that if $v_1 \in W_{\alpha_1,n}, \ldots, v_d \in W_{\alpha_d,n}$ are all non-zero, and $\alpha_i \neq \alpha_j$ for all $i \neq j$, then $v_1, \ldots, v_d$ are linearly independent.

9G Groups, Rings and Modules

Let M be a module over a ring R and let $S \subset M$. Define what it means that S *freely generates* M. Show that this happens if and only if for every R-module N, every function $f \colon S \to N$ extends uniquely to a homomorphism $\phi \colon M \to N$.

Let M be a free module over a (non-trivial) ring R that is generated (not necessarily freely) by a subset $T \subset M$ of size m. Show that if S is a basis of M, then S is finite with $|S| \leqslant m$. Hence, or otherwise, deduce that any two bases of M have the same number of elements. Denoting this number $\mathrm{rk} M$ and by quoting any result you need, show that if R is a Euclidean Domain and N is a submodule of M, then N is free with $\mathrm{rk} N \leqslant \mathrm{rk} M$.

State the Primary Decomposition Theorem for a finitely generated module M over a Euclidean Domain R. Deduce that any finite subgroup of the multiplicative group of a field is cyclic.

10F Analysis and Topology

Let $k_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions satisfying the following properties:

1. $k_n(x) \geqslant 0$ for all n and $x \in \mathbb{R}$ and there is $R > 0$ such that k_n vanishes outside $[-R, R]$ for all n;
2. each k_n is continuous and
$$\int_{-\infty}^{\infty} k_n(t)\,dt = 1;$$
3. given $\varepsilon > 0$ and $\delta > 0$, there exists a positive integer N such that if $n \geqslant N$, then
$$\int_{-\infty}^{-\delta} k_n(t)\,dt + \int_{\delta}^{\infty} k_n(t)\,dt < \varepsilon.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and set

$$f_n(x) := \int_{-\infty}^{\infty} k_n(t) f(x-t)\,dt.$$

Show that f_n converges uniformly to f on any compact subset of $\mathbb{R}$.

Let $g : [0,1] \to \mathbb{R}$ be a continuous function with $g(0) = g(1) = 0$. Show that there is a sequence of polynomials p_n such that p_n converges uniformly to g on $[0,1]$. [*Hint: consider the functions*

$$k_n(t) = \begin{cases} (1-t^2)^n / c_n & t \in [-1,1] \\ 0 & \text{otherwise,} \end{cases}$$

where c_n is a suitably chosen constant.]

11E Geometry

Define $\mathbb{H}$, the *upper half plane model* for the hyperbolic plane, and show that $PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by isometries, and that these isometries preserve the orientation of $\mathbb{H}$.

Show that every orientation preserving isometry of $\mathbb{H}$ is in $PSL_2(\mathbb{R})$, and hence the full group of isometries of $\mathbb{H}$ is $G = PSL_2(\mathbb{R}) \ \cup \ PSL_2(\mathbb{R})\tau$, where $\tau z = -\bar{z}$.

Let ℓ be a hyperbolic line. Define the reflection σ_ℓ in ℓ. Now let ℓ, ℓ' be two hyperbolic lines which meet at a point $A \in \mathbb{H}$ at an angle θ. What are the possibilities for the group G generated by σ_ℓ and $\sigma_{\ell'}$? Carefully justify your answer.

12B Complex Analysis or Complex Methods

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let $a > 0$, $b > 0$ be constants. Show that if

$$|f(z)| \leqslant a|z|^{n/2} + b$$

for all $z \in \mathbb{C}$, where n is a positive odd integer, then f must be a polynomial with degree not exceeding $\lfloor n/2 \rfloor$ (closest integer part rounding down).

Does there exist a function f, analytic in $\mathbb{C} \setminus \{0\}$, such that $|f(z)| \geqslant 1/\sqrt{|z|}$ for all nonzero z? Justify your answer.

(b) State Liouville's Theorem and use it to show the following.

(i) If u is a positive harmonic function on $\mathbb{R}^2$, then u is a constant function.

(ii) Let $L = \{z \,|\, z = ax + b, x \in \mathbb{R}\}$ be a line in $\mathbb{C}$ where $a, b \in \mathbb{C}$, $a \neq 0$. If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f(\mathbb{C}) \cap L = \emptyset$, then f is a constant function.

13D Variational Principles

A particle of unit mass moves in a smooth one-dimensional potential $V(x)$. Its path $x(t)$ is such that the action integral

$$S[x] = \int_a^b L(x, \dot{x})\, dt$$

has a stationary value, where a and $b > a$ are constants, a dot denotes differentiation with respect to time t,

$$L(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - V(x)$$

is the Lagrangian function and the initial and final positions $x(a)$ and $x(b)$ are fixed.

By considering $S[x + \epsilon\xi]$ for suitably restricted functions $\xi(t)$, derive the differential equation governing the motion of the particle and obtain an integral expression for the second variation $\delta^2 S$.

If $x(t)$ is a solution of the equation of motion and $x(t) + \epsilon u(t) + O(\epsilon^2)$ is also a solution of the equation of motion in the limit $\epsilon \to 0$, show that $u(t)$ satisfies the equation

$$\ddot{u} + V''(x)\, u = 0\,.$$

If $u(t)$ satisfies this equation and is non-vanishing for $a \leqslant t \leqslant b$, show that

$$\delta^2 S = \frac{1}{2}\int_a^b \left(\dot{\xi} - \frac{\dot{u}\xi}{u}\right)^2 dt\,.$$

Consider the simple harmonic oscillator, for which

$$V(x) = \frac{1}{2}\omega^2 x^2\,,$$

where $2\pi/\omega$ is the oscillation period. Show that the solution of the equation of motion is a local minimum of the action integral, provided that the time difference $b - a$ is less than half an oscillation period.

14A Methods

The Fourier transform $\tilde{f}(k)$ of a function $f(x)$ and its inverse are given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx, \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}dk.$$

(a) Calculate the Fourier transform of the function $f(x)$ defined by:

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ -1 & \text{for } -1 < x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Show that the inverse Fourier transform of $\tilde{g}(k) = e^{-\lambda|k|}$, for λ a positive real constant, is given by

$$g(x) = \frac{\lambda}{\pi(x^2 + \lambda^2)}.$$

(c) Consider the problem in the quarter plane $0 \leqslant x$, $0 \leqslant y$:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0; \\ u(x,0) &= \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise;} \end{cases} \\ u(0,y) = \lim_{x\to\infty} u(x,y) = \lim_{y\to\infty} u(x,y) &= 0. \end{aligned}$$

Use the answers from parts (a) and (b) to show that

$$u(x,y) = \frac{4xy}{\pi}\int_0^1 \frac{vdv}{[(x-v)^2+y^2][(x+v)^2+y^2]}.$$

(d) Hence solve the problem in the quarter plane $0 \leqslant x$, $0 \leqslant y$:

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= 0; \\ w(x,0) &= \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise;} \end{cases} \\ w(0,y) &= \begin{cases} 1 & \text{for } 0 < y < 1, \\ 0 & \text{otherwise;} \end{cases} \\ \lim_{x\to\infty} w(x,y) = \lim_{y\to\infty} w(x,y) &= 0. \end{aligned}$$

[*You may quote without proof any property of Fourier transforms.*]

15C Quantum Mechanics

(a) Write down the expressions for the probability density ρ and associated current density j of a quantum particle in one dimension with wavefunction $\psi(x,t)$. Show that if ψ is a stationary state then the function j is constant.

For the non-normalisable free particle wavefunction $\psi(x,t) = Ae^{ikx-iEt/\hbar}$ (where E and k are real constants and A is a complex constant) compute the functions ρ and j, and briefly give a physical interpretation of the functions ψ, ρ and j in this case.

(b) A quantum particle of mass m and energy $E > 0$ moving in one dimension is incident from the left in the potential $V(x)$ given by

$$V(x) = \begin{cases} -V_0 & 0 \leqslant x \leqslant a \\ 0 & x < 0 \text{ or } x > a \end{cases}$$

where a and V_0 are positive constants. Write down the form of the wavefunction in the regions $x < 0$, $0 \leqslant x \leqslant a$ and $x > a$.

Suppose now that $V_0 = 3E$. Show that the probability T of transmission of the particle into the region $x > a$ is given by

$$T = \frac{16}{16 + 9\sin^2\left(\frac{a\sqrt{8mE}}{\hbar}\right)}.$$

16D Electromagnetism

(a) Show that, for $|\mathbf{x}| \gg |\mathbf{y}|$,

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} = \frac{1}{|\mathbf{x}|}\left[1 + \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2} + \frac{3(\mathbf{x}\cdot\mathbf{y})^2 - |\mathbf{x}|^2|\mathbf{y}|^2}{2|\mathbf{x}|^4} + O\left(\frac{|\mathbf{y}|^3}{|\mathbf{x}|^3}\right)\right].$$

(b) A particle with electric charge $q > 0$ has position vector $(a,0,0)$, where $a > 0$. An earthed conductor (held at zero potential) occupies the plane $x = 0$. Explain why the boundary conditions can be satisfied by introducing a fictitious 'image' particle of appropriate charge and position. Hence determine the electrostatic potential and the electric field in the region $x > 0$. Find the leading-order approximation to the potential for $|\mathbf{x}| \gg a$ and compare with that of an electric dipole. Directly calculate the total flux of the electric field through the plane $x = 0$ and comment on the result. Find the induced charge distribution on the surface of the conductor, and the total induced surface charge. Sketch the electric field lines in the plane $z = 0$.

(c) Now consider instead a particle with charge q at position $(a,b,0)$, where $a > 0$ and $b > 0$, with earthed conductors occupying the planes $x = 0$ and $y = 0$. Find the leading-order approximation to the potential in the region $x, y > 0$ for $|\mathbf{x}| \gg a, b$ and state what type of multipole potential this is.

17B Numerical Analysis

(a) Define *Householder reflections* and show that a real Householder reflection is symmetric and orthogonal. Moreover, show that if $H, A \in \mathbb{R}^{n\times n}$, where H is a Householder reflection and A is a full matrix, then the computational cost (number of arithmetic operations) of computing HAH^{-1} can be $\mathcal{O}(n^2)$ operations, as opposed to $\mathcal{O}(n^3)$ for standard matrix products.

(b) Show that for any $A \in \mathbb{R}^{n\times n}$ there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$QAQ^T = T = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,n} \\ t_{2,1} & t_{2,2} & t_{2,3} & \cdots & t_{2,n} \\ 0 & t_{3,2} & t_{3,3} & \cdots & t_{3,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{n,n-1} & t_{n,n} \end{bmatrix}.$$

In particular, T has zero entries below the first subdiagonal. Show that one can compute such a T and Q (they may not be unique) using $\mathcal{O}(n^3)$ arithmetic operations.

[*Hint: Multiply A from the left and right with Householder reflections.*]

18H Markov Chains

Let P be a transition matrix on state space I. What does it mean for a distribution π to be an *invariant distribution*? What does it mean for π and P to be in *detailed balance*? Show that if π and P are in detailed balance, then π is an invariant distribution.

(a) Assuming that an invariant distribution exists, state the relationship between this and

(i) the expected return time to a state i;

(ii) the expected time spent in a state i between visits to a state k.

(b) Let $(X_n)_{n\geqslant 0}$ be a Markov chain with transition matrix $P = (p_{ij})_{i,j\in I}$ where $I = \{0, 1, 2, \ldots\}$. The transition probabilities are given for $i \geqslant 1$ by

$$p_{ij} = \begin{cases} q^{-(i+2)} & \text{if } j = i+1, \\ q^{-i} & \text{if } j = i-1, \\ 1 - q^{-(i+2)} - q^{-i} & \text{if } j = i, \end{cases}$$

where $q \geqslant 2$. For $p \in (0, 1]$ let $p_{01} = p = 1 - p_{00}$. Compute the following, justifying your answers:

(i) The expected time spent in states $\{2, 4, 6, \ldots\}$ between visits to state 1;

(ii) The expected time taken to return to state 1, starting from 1;

(iii) The expected time taken to hit state 0 starting from 1.

END OF PAPER