UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday, 15 June, 2021 10:00am to 1:00pm

PAPER 1

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

Let V be a vector space over $\mathbb{R}$, $\dim V = n$, and let $\langle , \rangle$ be a non-degenerate anti-symmetric bilinear form on V.

Let $v \in V$, $v \neq 0$. Show that $v^{\perp}$ is of dimension $n-1$ and $v \in v^{\perp}$. Show that if $W \subseteq v^{\perp}$ is a subspace with $W \oplus \mathbb{R}v = v^{\perp}$, then the restriction of $\langle , \rangle$ to W is non-degenerate.

Conclude that the dimension of V is even.

2F Geometry

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and let $\Sigma = f^{-1}(0)$ (assumed not empty). Show that if the differential $Df_p \neq 0$ for all $p \in \Sigma$, then Σ is a smooth surface in $\mathbb{R}^3$.

Is $\{(x,y,z) \in \mathbb{R}^3 : \; x^2 + y^2 = \cosh(z^2)\}$ a smooth surface? Is every surface $\Sigma \subset \mathbb{R}^3$ of the form $f^{-1}(0)$ for some smooth $f : \mathbb{R}^3 \to \mathbb{R}$? Justify your answers.

3B Complex Analysis or Complex Methods

Let $x > 0$, $x \neq 2$, and let C_x denote the positively oriented circle of radius x centred at the origin. Define

$$g(x) = \oint_{C_x} \frac{z^2 + e^z}{z^2(z-2)}\, dz.$$

Evaluate $g(x)$ for $x \in (0, \infty) \setminus \{2\}$.

4D Variational Principles

Let D be a bounded region of $\mathbb{R}^2$, with boundary ∂D. Let $u(x,y)$ be a smooth function defined on D, subject to the boundary condition that $u = 0$ on ∂D and the normalization condition that

$$\int_D u^2 \, dx\, dy = 1\,.$$

Let $I[u]$ be the functional

$$I[u] = \int_D |\boldsymbol{\nabla} u|^2 \, dx\, dy\,.$$

Show that $I[u]$ has a stationary value, subject to the stated boundary and normalization conditions, when u satisfies a partial differential equation of the form

$$\nabla^2 u + \lambda u = 0$$

in D, where λ is a constant.

Determine how λ is related to the stationary value of the functional $I[u]$. [*Hint: Consider* $\boldsymbol{\nabla} \cdot (u \boldsymbol{\nabla} u)$.]

5B Numerical Analysis

Prove, from first principles, that there is an algorithm that can determine whether any real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite or not, with the computational cost (number of arithmetic operations) bounded by $\mathcal{O}(n^3)$.

[*Hint: Consider the LDL decomposition.*]

6H Statistics

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(p) random variables, where $n \geqslant 3$ and $p \in (0,1)$ is unknown.

(a) What does it mean for a statistic T to be *sufficient* for p? Find such a sufficient statistic T.

(b) State and prove the Rao–Blackwell theorem.

(c) By considering the estimator X_1X_2 of p^2, find an unbiased estimator of p^2 that is a function of the statistic T found in part (a), and has variance strictly smaller than that of X_1X_2.

7H Optimisation

(a) Let $f_i : \mathbb{R}^d \to \mathbb{R}$ be a convex function for each $i = 1, \ldots, m$. Show that

$$x \mapsto \max_{i=1,\ldots,m} f_i(x) \qquad \text{and} \qquad x \mapsto \sum_{i=1}^{m} f_i(x)$$

are both convex functions.

(b) Fix $c \in \mathbb{R}^d$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is convex, then $g : \mathbb{R}^d \to \mathbb{R}$ given by $g(x) = f(c^T x)$ is convex.

(c) Fix vectors $a_1, \ldots, a_n \in \mathbb{R}^d$. Let $Q : \mathbb{R}^d \to \mathbb{R}$ be given by

$$Q(\beta) = \sum_{i=1}^{n} \log(1 + e^{a_i^T \beta}) + \sum_{j=1}^{d} |\beta_j|.$$

Show that Q is convex. [You may use any result from the course provided you state it.]

SECTION II

8E Linear Algebra

Let $d \geqslant 1$, and let $J_d = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & \dots & \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_d(\mathbb{C})$.

(a) (i) Compute J_d^n, for all $n \geqslant 0$.

(ii) Hence, or otherwise, compute $(\lambda I + J_d)^n$, for all $n \geqslant 0$.

(b) Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $\varphi \in \mathrm{End}(V)$. Suppose $\varphi^n = 0$ for some $n > 1$.

(i) Determine the possible eigenvalues of φ.

(ii) What are the possible Jordan blocks of φ?

(iii) Show that if $\varphi^2 = 0$, there exists a decomposition

$$V = U \oplus W_1 \oplus W_2,$$

where $\varphi(U) = \varphi(W_1) = 0$, $\varphi(W_2) = W_1$, and $\dim W_2 = \dim W_1$.

9G Groups, Rings and Modules

Show that a ring R is Noetherian if and only if every ideal of R is finitely generated. Show that if $\phi\colon R \to S$ is a surjective ring homomorphism and R is Noetherian, then S is Noetherian.

State and prove Hilbert's Basis Theorem.

Let $\alpha \in \mathbb{C}$. Is $\mathbb{Z}[\alpha]$ Noetherian? Justify your answer.

Give, with proof, an example of a Unique Factorization Domain that is not Noetherian.

Let R be the ring of continuous functions $\mathbb{R} \to \mathbb{R}$. Is R Noetherian? Justify your answer.

10F Analysis and Topology

Let $f : X \to Y$ be a map between metric spaces. Prove that the following two statements are equivalent:

(i) $f^{-1}(A) \subset X$ is open whenever $A \subset Y$ is open.

(ii) $f(x_n) \to f(a)$ for any sequence $x_n \to a$.

For $f : X \to Y$ as above, determine which of the following statements are always true and which may be false, giving a proof or a counterexample as appropriate.

(a) If X is compact and f is continuous, then f is uniformly continuous.

(b) If X is compact and f is continuous, then Y is compact.

(c) If X is connected, f is continuous and $f(X)$ is dense in Y, then Y is connected.

(d) If the set $\{(x,y) \in X \times Y : \; y = f(x)\}$ is closed in $X \times Y$ and Y is compact, then f is continuous.

11F Geometry

Let $S \subset \mathbb{R}^3$ be an oriented surface. Define the *Gauss map* N and show that the differential DN_p of the Gauss map at any point $p \in S$ is a self-adjoint linear map. Define the *Gauss curvature* κ and compute κ in a given parametrisation.

A point $p \in S$ is called umbilic if DN_p has a repeated eigenvalue. Let $S \subset \mathbb{R}^3$ be a surface such that every point is umbilic and there is a parametrisation $\phi : \mathbb{R}^2 \to S$ such that $S = \phi(\mathbb{R}^2)$. Prove that S is part of a plane or part of a sphere. [*Hint: consider the symmetry of the mixed partial derivatives* $n_{uv} = n_{vu}$, *where* $n(u,v) = N(\phi(u,v))$ *for* $(u,v) \in \mathbb{R}^2$.]

12G Complex Analysis or Complex Methods

(a) State a theorem establishing Laurent series of analytic functions on suitable domains. Give a formula for the n^{th} Laurent coefficient.

Define the notion of *isolated singularity.* State the classification of an isolated singularity in terms of Laurent coefficients.

Compute the Laurent series of

$$f(z) = \frac{1}{z(z-1)}$$

on the annuli $A_1 = \{z : 0 < |z| < 1\}$ and $A_2 = \{z : 1 < |z|\}$. Using this example, comment on the statement that Laurent coefficients are unique. Classify the singularity of f at 0.

(b) Let U be an open subset of the complex plane, let $a \in U$ and let $U' = U \setminus \{a\}$. Assume that f is an analytic function on U' with $|f(z)| \to \infty$ as $z \to a$. By considering the Laurent series of $g(z) = \frac{1}{f(z)}$ at a, classify the singularity of f at a in terms of the Laurent coefficients. [You may assume that a continuous function on U that is analytic on U' is analytic on U.]

Now let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function with $|f(z)| \to \infty$ as $z \to \infty$. By considering Laurent series at 0 of $f(z)$ and of $h(z) = f\left(\frac{1}{z}\right)$, show that f is a polynomial.

(c) Classify, giving reasons, the singularity at the origin of each of the following functions and in each case compute the residue:

$$g(z) = \frac{\exp(z) - 1}{z \log(z+1)} \qquad \text{and} \qquad h(z) = \sin(z)\sin(1/z)\ .$$

13C Methods

(a) By introducing the variables $\xi = x + ct$ and $\eta = x - ct$ (where c is a constant), derive d'Alembert's solution of the initial value problem for the wave equation:

$$u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$$

where $-\infty < x < \infty$, $t \geqslant 0$ and ϕ and ψ are given functions (and subscripts denote partial derivatives).

(b) Consider the forced wave equation with homogeneous initial conditions:

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad u(x,0) = 0, \quad u_t(x,0) = 0$$

where $-\infty < x < \infty$, $t \geqslant 0$ and f is a given function. You may assume that the solution is given by

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy\, ds.$$

For the forced wave equation $u_{tt} - c^2 u_{xx} = f(x,t)$, now in the half space $x \geqslant 0$ (and with $t \geqslant 0$ as before), find (in terms of f) the solution for $u(x,t)$ that satisfies the (inhomogeneous) initial conditions

$$u(x,0) = \sin x, \quad u_t(x,0) = 0, \qquad \text{for } x \geqslant 0$$

and the boundary condition $u(0,t) = 0$ for $t \geqslant 0$.

14C Quantum Mechanics

Consider a quantum mechanical particle of mass m in a one-dimensional stepped potential well $U(x)$ given by:

$$U(x) = \begin{cases} \infty & \text{for } x < 0 \text{ and } x > a \\ 0 & \text{for } 0 \leqslant x \leqslant a/2 \\ U_0 & \text{for } a/2 < x \leqslant a \end{cases}$$

where $a > 0$ and $U_0 \geqslant 0$ are constants.

(i) Show that all energy levels E of the particle are non-negative. Show that any level E with $0 < E < U_0$ satisfies

$$\frac{1}{k} \tan \frac{ka}{2} = -\frac{1}{l} \tanh \frac{la}{2}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}} > 0 \quad \text{and} \quad l = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} > 0.$$

(ii) Suppose that initially $U_0 = 0$ and the particle is in the ground state of the potential well. U_0 is then changed to a value $U_0 > 0$ (while the particle's wavefunction stays the same) and the energy of the particle is measured. For $0 < E < U_0$, give an expression in terms of E for prob(E), the probability that the energy measurement will find the particle having energy E. The expression may be left in terms of integrals that you need not evaluate.

15D Electromagnetism

(a) Show that the magnetic flux passing through a simple, closed curve C can be written as

$$\Phi = \oint_C \mathbf{A} \cdot \mathbf{dx}\,,$$

where $\mathbf{A}$ is the magnetic vector potential. Explain why this integral is independent of the choice of gauge.

(b) Show that the magnetic vector potential due to a static electric current density $\mathbf{J}$, in the Coulomb gauge, satisfies Poisson's equation

$$-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}\,.$$

Hence obtain an expression for the magnetic vector potential due to a static, thin wire, in the form of a simple, closed curve C, that carries an electric current I. [You may assume that the electric current density of the wire can be written as

$$\mathbf{J}(\mathbf{x}) = I \int_C \delta^{(3)}(\mathbf{x} - \mathbf{x}')\, \mathbf{dx}'\,,$$

where $\delta^{(3)}$ is the three-dimensional Dirac delta function.]

(c) Consider two thin wires, in the form of simple, closed curves C_1 and C_2, that carry electric currents I_1 and I_2, respectively. Let Φ_{ij} (where $i, j \in \{1, 2\}$) be the magnetic flux passing through the curve C_i due to the current I_j flowing around C_j. The inductances are defined by $L_{ij} = \Phi_{ij}/I_j$. By combining the results of parts (a) and (b), or otherwise, derive Neumann's formula for the mutual inductance,

$$L_{12} = L_{21} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\mathbf{dx}_1 \cdot \mathbf{dx}_2}{|\mathbf{x}_1 - \mathbf{x}_2|}\,.$$

Suppose that C_1 is a circular loop of radius a, centred at $(0, 0, 0)$ and lying in the plane $z = 0$, and that C_2 is a different circular loop of radius b, centred at $(0, 0, c)$ and lying in the plane $z = c$. Show that the mutual inductance of the two loops is

$$\frac{\mu_0}{4} \sqrt{a^2 + b^2 + c^2}\, f(q)\,,$$

where

$$q = \frac{2ab}{a^2 + b^2 + c^2}$$

and the function $f(q)$ is defined, for $0 < q < 1$, by the integral

$$f(q) = \int_0^{2\pi} \frac{q \cos\theta\, d\theta}{\sqrt{1 - q\cos\theta}}\,.$$

16A Fluid Dynamics

A two-dimensional flow is given by a velocity potential

$$\phi(x, y, t) = \epsilon y \sin(x - t),$$

where ϵ is a constant.

(a) Find the corresponding velocity field $\mathbf{u}(x, y, t)$. Determine $\boldsymbol{\nabla} \cdot \mathbf{u}$.

(b) The time-average $\langle \psi \rangle(x, y)$ of a quantity $\psi(x, y, t)$ is defined as

$$\langle \psi \rangle(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \psi(x, y, t) dt.$$

Show that the time-average of this velocity field is zero everywhere. Write down an expression for the acceleration of fluid particles, and find the time-average of this expression at a fixed point (x, y).

(c) Now assume that $|\epsilon| \ll 1$. The material particle at $(0, 0)$ at $t = 0$ is marked with dye. Write down equations for its subsequent motion. Verify that its position (x, y) for $t > 0$ is given (correct to terms of order ϵ^2) by

$$\begin{aligned} x &= \epsilon^2 \left(\frac{1}{4} \sin 2t + \frac{t}{2} - \sin t \right), \\ y &= \epsilon(\cos t - 1). \end{aligned}$$

Deduce the time-average velocity of the dyed particle correct to this order.

17B Numerical Analysis

For the ordinary differential equation

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}), \quad \boldsymbol{y}(0) = \tilde{\boldsymbol{y}}_0, \quad t \geqslant 0, \qquad (*)$$

where $\boldsymbol{y}(t) \in \mathbb{R}^N$ and the function $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is analytic, consider an explicit one-step method described as the mapping

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{\varphi}(t_n, \boldsymbol{y}_n, h). \qquad (\dagger)$$

Here $\boldsymbol{\varphi} : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}^N$, $n = 0, 1, \ldots$ and $t_n = nh$ with time step $h > 0$, producing numerical approximations $\boldsymbol{y}_n$ to the exact solution $\boldsymbol{y}(t_n)$ of equation $(*)$, with $\boldsymbol{y}_0$ being the initial value of the numerical solution.

(i) Define *the local error* of a one-step method.

(ii) Let $\|\cdot\|$ be a norm on $\mathbb{R}^N$ and suppose that

$$\|\boldsymbol{\varphi}(t, \boldsymbol{u}, h) - \boldsymbol{\varphi}(t, \boldsymbol{v}, h)\| \leqslant L\|\boldsymbol{u} - \boldsymbol{v}\|,$$

for all $h > 0$, $t \in \mathbb{R}$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^N$, where L is some positive constant. Let $t^* > 0$ be given and $\boldsymbol{e}_0 = \boldsymbol{y}_0 - \boldsymbol{y}(0)$ denote the initial error (potentially non-zero). Show that if the local error of the one-step method $(\dagger)$ is $\mathcal{O}(h^{p+1})$, then

$$\max_{n=0,\ldots,\lfloor t^*/h \rfloor} \|\boldsymbol{y}_n - \boldsymbol{y}(nh)\| \leqslant e^{t^* L}\|\boldsymbol{e}_0\| + \mathcal{O}(h^p), \quad h \to 0. \qquad (\dagger\dagger)$$

(iii) Let $N = 1$ and consider equation $(*)$ where f is time-independent satisfying $|f(u) - f(v)| \leqslant K|u - v|$ for all $u, v \in \mathbb{R}$, where K is a positive constant. Consider the one-step method given by

$$y_{n+1} = y_n + \frac{1}{4}h(k_1 + 3k_2), \qquad k_1 = f(y_n), \quad k_2 = f(y_n + \frac{2}{3}hk_1).$$

Use part (ii) to show that for this method we have that equation $(\dagger\dagger)$ holds (with a potentially different constant L) for $p = 2$.

18H Statistics

(a) Show that if $W_1, \ldots, W_n$ are independent random variables with common Exp(1) distribution, then $\sum_{i=1}^n W_i \sim \Gamma(n,1)$. [*Hint: If* $W \sim \Gamma(\alpha,\lambda)$ *then* $\mathbb{E}e^{tW} = \{\lambda/(\lambda-t)\}^\alpha$ *if* $t < \lambda$ *and* ∞ *otherwise.*]

(b) Show that if $X \sim U(0,1)$ then $-\log X \sim \text{Exp}(1)$.

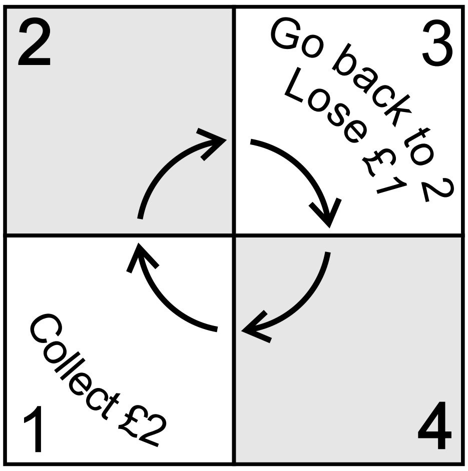
(c) State the Neyman–Pearson lemma.

(d) Let $X_1, \ldots, X_n$ be independent random variables with common density proportional to $x^\theta \mathbf{1}_{(0,1)}(x)$ for $\theta \geqslant 0$. Find a most powerful test of size α of $H_0 : \theta = 0$ against $H_1 : \theta = 1$, giving the critical region in terms of a quantile of an appropriate gamma distribution. Find a uniformly most powerful test of size α of $H_0 : \theta = 0$ against $H_1 : \theta > 0$.

19H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain with transition matrix P. What is a *stopping time* of $(X_n)_{n\geqslant 0}$? What is the *strong Markov property*?

The exciting game of 'Unopoly' is played by a single player on a board of 4 squares. The player starts with £m (where $m \in \mathbb{N}$). During each turn, the player tosses a fair coin and moves one or two places in a clockwise direction ($1 \to 2 \to 3 \to 4 \to 1$) according to whether the coin lands heads or tails respectively. The player collects £2 each time they pass (or land on) square 1. If the player lands on square 3 however, they immediately lose £1 and go back to square 2. The game continues indefinitely unless the player is on square 2 with £0, in which case the player loses the game and the game ends.



(a) By setting up an appropriate Markov chain, show that if the player is at square 2 with £m, where $m \geqslant 1$, the probability that they are ever at square 2 with £$(m-1)$ is 2/3.

(b) Find the probability of losing the game when the player starts on square 1 with £m, where $m \geqslant 1$.

[*Hint: Take the state space of your Markov chain to be* $\{1,2,4\} \times \{0,1,\ldots\}$.]

END OF PAPER