UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday, 11 June, 2021 10:00am to 1:00pm

PAPER 4

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

Consider functions $f : X \to Y$ and $g : Y \to X$. Which of the following statements are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) If $g \circ f$ is surjective then f is surjective.

(ii) If $g \circ f$ is injective then f is injective.

(iii) If $g \circ f$ is injective then g is injective.

If $X = \{1, \ldots, m\}$ and $Y = \{1, \ldots, n\}$ with $m < n$, and $g \circ f$ is the identity on X, then how many possibilities are there for the pair of functions f and g?

2E Numbers and Sets

The *Fibonacci numbers* F_n are defined by $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ $(n \geqslant 1)$. Let $a_n = F_{n+1}/F_n$ be the ratio of successive Fibonacci numbers.

(i) Show that $a_{n+1} = 1 + 1/a_n$. Hence prove by induction that

$$(-1)^n a_{n+2} \leqslant (-1)^n a_n$$

for all $n \geqslant 1$. Deduce that the sequence a_{2n} is monotonically decreasing.

(ii) Prove that

$$F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1}$$

for all $n \geqslant 1$. Hence show that $a_{n+1} - a_n \to 0$ as $n \to \infty$.

(iii) Explain without detailed justification why the sequence a_n has a limit.

3C Dynamics and Relativity

A trolley travels with initial speed v_0 along a frictionless, horizontal, linear track. It slows down by ejecting gas in the direction of motion. The gas is emitted at a constant mass ejection rate α and with constant speed u relative to the trolley. The trolley and its supply of gas initially have a combined mass of m_0. How much time is spent ejecting gas before the trolley stops? [Assume that the trolley carries sufficient gas.]

4C Dynamics and Relativity

A rigid body composed of N particles with positions $\mathbf{x}_i$, and masses m_i ($i = 1, 2, \ldots, N$), rotates about the z-axis with constant angular speed ω. Show that the body's kinetic energy is $T = \frac{1}{2}I\omega^2$, where you should give an expression for the moment of inertia I in terms of the particle masses and positions.

Consider a solid cuboid of uniform density, mass M, and dimensions $2a \times 2b \times 2c$. Choose coordinate axes so that the cuboid is described by the points (x, y, z) with $-a \leqslant x \leqslant a$, $-b \leqslant y \leqslant b$, and $-c \leqslant z \leqslant c$. In terms of M, a, b, and c, find the cuboid's moment of inertia I for rotations about the z-axis.

SECTION II

5E Numbers and Sets

(a) Let S be the set of all functions $f : \mathbb{N} \to \mathbb{R}$. Define $\delta : S \to S$ by

$$(\delta f)(n) = f(n+1) - f(n)\,.$$

(i) Define the binomial coefficient $\binom{n}{r}$ for $0 \leqslant r \leqslant n$. Setting $\binom{n}{r} = 0$ when $r > n$, prove from your definition that if $f_r(n) = \binom{n}{r}$ then $\delta f_r = f_{r-1}$.

(ii) Show that if $f \in S$ is integer-valued and $\delta^{k+1} f = 0$, then

$$f(n) = c_0 \binom{n}{k} + c_1 \binom{n}{k-1} + \cdots + c_{k-1} \binom{n}{1} + c_k$$

for some integers $c_0, \ldots, c_k$.

(b) State the binomial theorem. Show that

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2} \binom{n}{n/2} & \text{if } n \text{ is even} \end{cases}.$$

6E Numbers and Sets

(a) (i) By considering Euclid's algorithm, show that the highest common factor of two positive integers a and b can be written in the form $\alpha a + \beta b$ for suitable integers α and β. Find an integer solution of

$$15x + 21y + 35z = 1\,.$$

Is your solution unique?

(ii) Suppose that n and m are coprime. Show that the simultaneous congruences

$$\begin{aligned} x &\equiv a \pmod{n}\,, \\ x &\equiv b \pmod{m} \end{aligned}$$

have the same set of solutions as $x \equiv c \pmod{mn}$ for some $c \in \mathbb{N}$. Hence solve (i.e. find all solutions of) the simultaneous congruences

$$\begin{aligned} 3x &\equiv 1 \pmod{5}\,, \\ 5x &\equiv 1 \pmod{7}\,, \\ 7x &\equiv 1 \pmod{3}\,. \end{aligned}$$

(b) State the inclusion–exclusion principle.

For integers $r, n \geqslant 1$, denote by $\phi_r(n)$ the number of ordered r-tuples $(x_1, \ldots, x_r)$ of integers x_i satisfying $1 \leqslant x_i \leqslant n$ for $i = 1, \ldots, r$ and such that the greatest common divisor of $\{n, x_1, \ldots, x_r\}$ is 1. Show that

$$\phi_r(n) = n^r \prod_{p|n} (1 - \frac{1}{p^r})$$

where the product is over all prime numbers p dividing n.

7E Numbers and Sets

(a) Prove that every real number $\alpha \in (0,1]$ can be written in the form $\alpha = \sum_{n=1}^{\infty} 2^{-b_n}$ where (b_n) is a strictly increasing sequence of positive integers.

Are such expressions unique?

(b) Let $\theta \in \mathbb{R}$ be a root of $f(x) = \alpha_d x^d + \cdots + \alpha_1 x + \alpha_0$, where $\alpha_0, \ldots, \alpha_d \in \mathbb{Z}$. Suppose that f has no rational roots, except possibly θ.

(i) Show that if $s, t \in \mathbb{R}$ then

$$|f(s) - f(t)| \leqslant A(\max\{|s|, |t|, 1\})^{d-1}|s-t|\,.$$

where A is a constant depending only on f.

(ii) Deduce that if $p, q \in \mathbb{Z}$ with $q > 0$ and $0 < \left|\theta - \frac{p}{q}\right| < 1$ then

$$\left|\theta - \frac{p}{q}\right| \geqslant \frac{1}{A}\left(\frac{1}{|\theta|+1}\right)^{d-1}\frac{1}{q^d}\,.$$

(c) Prove that $\alpha = \sum_{n=1}^{\infty} 2^{-n!}$ is transcendental.

(d) Let β and γ be transcendental numbers. What of the following statements are always true and which can be false? Briefly justify your answers.

(i) $\beta\gamma$ is transcendental.

(ii) β^n is transcendental for every $n \in \mathbb{N}$.

8E Numbers and Sets

(a) Prove that a countable union of countable sets is countable.

(b) (i) Show that the set $\mathbb{N}^{\mathbb{N}}$ of all functions $\mathbb{N} \to \mathbb{N}$ is uncountable.

(ii) Determine the countability or otherwise of each of the two sets

$$A = \{f \in \mathbb{N}^{\mathbb{N}} : f(n) \leqslant f(n+1) \text{ for all } n \in \mathbb{N}\},$$
$$B = \{f \in \mathbb{N}^{\mathbb{N}} : f(n) \geqslant f(n+1) \text{ for all } n \in \mathbb{N}\}.$$

Justify your answers.

(c) A *permutation* σ of the natural numbers $\mathbb{N}$ is a mapping $\sigma \in \mathbb{N}^{\mathbb{N}}$ that is bijective. Determine the countability or otherwise of each of the two sets C and D of permutations, justifying your answers:

(i) C is the set of all permutations σ of $\mathbb{N}$ such that $\sigma(j) = j$ for all sufficiently large j.

(ii) D is the set all permutations σ of $\mathbb{N}$ such that

$$\sigma(j) = j - 1 \text{ or } j \text{ or } j + 1$$

for each j.

9C Dynamics and Relativity

A particle of mass m follows a one-dimensional trajectory $x(t)$ in the presence of a variable force $F(x,t)$. Write down an expression for the work done by this force as the particle moves from $x(t_a) = a$ to $x(t_b) = b$. Assuming that this is the only force acting on the particle, show that the work done by the force is equal to the change in the particle's kinetic energy.

What does it mean if a force is said to be *conservative*?

A particle moves in a force field given by

$$F(x) = \begin{cases} -F_0\, e^{-x/\lambda} & x \geqslant 0 \\ F_0\, e^{x/\lambda} & x < 0 \end{cases}$$

where F_0 and λ are positive constants. The particle starts at the origin $x = 0$ with initial velocity $v_0 > 0$. Show that, as the particle's position increases from $x = 0$ to larger $x > 0$, the particle's velocity v at position x is given by

$$v(x) = \sqrt{v_0^2 + v_e^2\left(e^{-|x|/\lambda} - 1\right)}$$

where you should determine v_e. What determines whether the particle will escape to infinity or oscillate about the origin? Sketch $v(x)$ versus x for each of these cases, carefully identifying any significant velocities or positions.

In the case of oscillatory motion, find the period of oscillation in terms of v_0, v_e, and λ. [*Hint: You may use the fact that*

$$\int_w^1 \frac{du}{u\sqrt{u-w}} = \frac{2\cos^{-1}\sqrt{w}}{\sqrt{w}}$$

for $0 < w < 1$.]

10C Dynamics and Relativity

(a) A mass m is acted upon by a central force

$$\mathbf{F} = -\frac{km}{r^3}\mathbf{r}$$

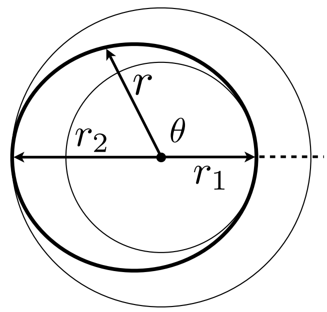
where k is a positive constant and $\mathbf{r}$ is the displacement of the mass from the origin. Show that the angular momentum and energy of the mass are conserved.

(b) Working in plane polar coordinates (r, θ), or otherwise, show that the distance $r = |\mathbf{r}|$ between the mass and the origin obeys the following differential equation

$$\ddot{r} = -\frac{k}{r^2} + \frac{h^2}{r^3}$$

where h is the angular momentum per unit mass.

(c) A satellite is initially in a *circular* orbit of radius r_1 and experiences the force described above. At $\theta = 0$ and time t_1, the satellite emits a short rocket burst putting it on an *elliptical* orbit with its closest distance to the centre r_1 and farthest distance r_2. When $\theta = \pi$ and the time is t_2, the satellite reaches the farthest distance and a second short rocket burst puts the rocket on a *circular* orbit of radius r_2. (See figure.) [Assume that the duration of the rocket bursts is negligible.]



(i) Show that the satellite's angular momentum per unit mass while in the elliptical orbit is

$$h = \sqrt{\frac{Ckr_1r_2}{r_1 + r_2}}$$

where C is a number you should determine.

(ii) What is the change in speed as a result of the rocket burst at time t_1? And what is the change in speed at t_2?

(iii) Given that the elliptical orbit can be described by

$$r = \frac{h^2}{k(1 + e\cos\theta)}$$

where e is the eccentricity of the orbit, find $t_2 - t_1$ in terms of r_1, r_2, and k. [*Hint: The area of an ellipse is equal to* πab, *where* a *and* b *are its semi-major and semi-minor axes; these are related to the eccentricity by* $e = \sqrt{1 - \frac{b^2}{a^2}}$.]

11C Dynamics and Relativity

Consider an inertial frame of reference S and a frame of reference S' which is rotating with constant angular velocity $\boldsymbol{\omega}$ relative to S. Assume that the two frames have a common origin O.

Let $\mathbf{A}$ be any vector. Explain why the derivative of $\mathbf{A}$ in frame S is related to its derivative in S' by the following equation

$$\left(\frac{d\mathbf{A}}{dt}\right)_S = \left(\frac{d\mathbf{A}}{dt}\right)_{S'} + \boldsymbol{\omega} \times \mathbf{A}\,.$$

[*Hint: It may be useful to use Cartesian basis vectors in both frames.*]

Let $\mathbf{r}(t)$ be the position vector of a particle, measured from O. Derive the expression relating the particle's acceleration as observed in S, $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_S$, to the acceleration observed in S', $\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S'}$, written in terms of $\mathbf{r}$, $\boldsymbol{\omega}$ and $\left(\frac{d\mathbf{r}}{dt}\right)_{S'}$.

A small bead of mass m is threaded on a smooth, rigid, circular wire of radius R. At any given instant, the wire hangs in a vertical plane with respect to a downward gravitational acceleration $\mathbf{g}$. The wire is rotating with constant angular velocity $\boldsymbol{\omega}$ about its vertical diameter. Let $\theta(t)$ be the angle between the downward vertical and the radial line going from the centre of the hoop to the bead.

(i) Show that $\theta(t)$ satisfies the following equation of motion

$$\ddot{\theta} = \left(\omega^2 \cos\theta - \frac{g}{R}\right)\sin\theta\,.$$

(ii) Find any equilibrium angles and determine their stability.

(iii) Find the force of the wire on the bead as a function of θ and $\dot{\theta}$.

12C Dynamics and Relativity

Write down the expression for the momentum of a particle of rest mass m, moving with velocity $\mathbf{v}$ where $v = |\mathbf{v}|$ is near the speed of light c. Write down the corresponding 4-momentum.

Such a particle experiences a force $\mathbf{F}$. Why is the following expression for the particle's acceleration,

$$\mathbf{a} = \frac{\mathbf{F}}{m},$$

not generally correct? Show that the force can be written as follows

$$\mathbf{F} = m\gamma \left(\frac{\gamma^2}{c^2}(\mathbf{v}\cdot\mathbf{a})\mathbf{v} + \mathbf{a} \right) .$$

Invert this expression to find the particle's acceleration as the sum of two vectors, one parallel to $\mathbf{F}$ and one parallel to $\mathbf{v}$.

A particle with rest mass m and charge q is in the presence of a constant electric field $\mathbf{E}$ which exerts a force $\mathbf{F} = q\mathbf{E}$ on the particle. If the particle is at rest at $t = 0$, its motion will be in the direction of $\mathbf{E}$ for $t > 0$. Determine the particle's speed for $t > 0$. How does the velocity behave as $t \to \infty$?

[*Hint: You may find that trigonometric substitution is helpful in evaluating an integral.*]

END OF PAPER