UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Wednesday, 9 June, 2021 10:00am to 1:00pm

PAPER 3

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

Let G be a finite group and denote the *centre* of G by $Z(G)$. Prove that if the quotient group $G/Z(G)$ is cyclic then G is abelian. Does there exist a group H such that

(i) $|H/Z(H)| = 7$?

(ii) $|H/Z(H)| = 6$?

Justify your answers.

2D Groups

Let g and h be elements of a group G. What does it mean to say g and h are *conjugate* in G? Prove that if two elements in a group are conjugate then they have the same order.

Define the Möbius group $\mathcal{M}$. Prove that if $g, h \in \mathcal{M}$ are conjugate they have the same number of fixed points. Quoting clearly any results you use, show that any nontrivial element of $\mathcal{M}$ of finite order has precisely 2 fixed points.

3B Vector Calculus

(a) Prove that

$$\boldsymbol{\nabla} \times (\psi \mathbf{A}) = \psi\, \boldsymbol{\nabla} \times \mathbf{A} + \boldsymbol{\nabla} \psi \times \mathbf{A}\,,$$
$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \boldsymbol{\nabla} \times \mathbf{A} - \mathbf{A} \cdot \boldsymbol{\nabla} \times \mathbf{B}\,,$$

where $\mathbf{A}$ and $\mathbf{B}$ are differentiable vector fields and ψ is a differentiable scalar field.

(b) Find the solution of $\nabla^2 u = 16r^2$ on the two-dimensional domain $\mathcal{D}$ when

(i) $\mathcal{D}$ is the unit disc $0 \leqslant r \leqslant 1$, and $u = 1$ on $r = 1$;

(ii) $\mathcal{D}$ is the annulus $1 \leqslant r \leqslant 2$, and $u = 1$ on both $r = 1$ and $r = 2$.

[*Hint: the Laplacian in plane polar coordinates is:*

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,. \quad]$$

4B Vector Calculus

(a) What is meant by an *antisymmetric* tensor of second rank? Show that if a second rank tensor is antisymmetric in one Cartesian coordinate system, it is antisymmetric in every Cartesian coordinate system.

(b) Consider the vector field $\mathbf{F} = (y,\, z,\, x)$ and the second rank tensor defined by $T_{ij} = \partial F_i / \partial x_j$. Calculate the components of the antisymmetric part of T_{ij} and verify that it equals $-(1/2)\epsilon_{ijk} B_k$, where ϵ_{ijk} is the alternating tensor and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{F}$.

SECTION II

5D Groups

(a) Let x be an element of a finite group G. Define the *order* of x and the *order* of G. State and prove Lagrange's theorem. Deduce that the order of x divides the order of G.

(b) If G is a group of order n, and d is a divisor of n where $d < n$, is it always true that G must contain an element of order d? Justify your answer.

(c) Denote the cyclic group of order m by C_m.

(i) Prove that if m and n are coprime then the direct product $C_m \times C_n$ is cyclic.

(ii) Show that if a finite group G has all non-identity elements of order 2, then G is isomorphic to $C_2 \times \cdots \times C_2$. [The direct product theorem may be used without proof.]

(d) Let G be a finite group and H a subgroup of G.

(i) Let x be an element of order d in G. If r is the least positive integer such that $x^r \in H$, show that r divides d.

(ii) Suppose further that H has index n. If $x \in G$, show that $x^k \in H$ for some k such that $0 < k \leqslant n$. Is it always the case that the least positive such k is a factor of n? Justify your answer.

6D Groups

(a) Let G be a finite group acting on a set X. For $x \in X$, define the *orbit* $\mathrm{Orb}(x)$ and the *stabiliser* $\mathrm{Stab}(x)$ of x. Show that $\mathrm{Stab}(x)$ is a subgroup of G. State and prove the orbit-stabiliser theorem.

(b) Let $n \geqslant k \geqslant 1$ be integers. Let $G = S_n$, the symmetric group of degree n, and X be the set of all ordered k-tuples $(x_1, \ldots, x_k)$ with $x_i \in \{1, 2, \ldots, n\}$. Then G acts on X, where the action is defined by $\sigma(x_1, \ldots, x_k) = (\sigma(x_1), \ldots, \sigma(x_k))$ for $\sigma \in S_n$ and $(x_1, \ldots, x_k) \in X$. For $x = (1, 2, \ldots, k) \in X$, determine $\mathrm{Orb}(x)$ and $\mathrm{Stab}(x)$ and verify that the orbit-stabiliser theorem holds in this case.

(c) We say that G acts *doubly transitively* on X if, whenever (x_1, x_2) and (y_1, y_2) are elements of $X \times X$ with $x_1 \neq x_2$ and $y_1 \neq y_2$, there exists some $g \in G$ such that $gx_1 = y_1$ and $gx_2 = y_2$.

Assume that G is a finite group that acts doubly transitively on X, and let $x \in X$. Show that if H is a subgroup of G that properly contains $\mathrm{Stab}(x)$ (that is, $\mathrm{Stab}(x) \subseteq H$ but $\mathrm{Stab}(x) \neq H$) then the action of H on X is transitive. Deduce that $H = G$.

7D Groups

Let G be a finite group of order n. Show that G is isomorphic to a subgroup H of S_n, the symmetric group of degree n. Furthermore show that this isomorphism can be chosen so that any nontrivial element of H has no fixed points.

Suppose n is even. Prove that G contains an element of order 2.

What does it mean for an element of S_m to be odd? Suppose H is a subgroup of S_m for some m, and H contains an odd element. Prove that precisely half of the elements of H are odd.

Now suppose $n = 4k + 2$ for some positive integer k. Prove that G is not simple. [*Hint: Consider the sign of an element of order 2.*]

Can a nonabelian group of even order be simple?

8D Groups

(a) Let A be an abelian group (not necessarily finite). We define the *generalised dihedral group* to be the set of pairs

$$D(A) = \{(a,\ \varepsilon) : a \in A,\ \varepsilon = \pm 1\}\,,$$

with multiplication given by

$$(a,\ \varepsilon)(b,\ \eta) = (ab^{\varepsilon},\ \varepsilon\eta)\,.$$

The identity is $(e, 1)$ and the inverse of (a, ε) is $(a^{-\varepsilon}, \varepsilon)$. You may assume that this multiplication defines a group operation on $D(A)$.

(i) Identify A with the set of all pairs in which $\varepsilon = +1$. Show that A is a subgroup of $D(A)$. By considering the index of A in $D(A)$, or otherwise, show that A is a normal subgroup of $D(A)$.

(ii) Show that every element of $D(A)$ not in A has order 2. Show that $D(A)$ is abelian if and only if $a^2 = e$ for all $a \in A$. If $D(A)$ is non-abelian, what is the centre of $D(A)$? Justify your answer.

(b) Let $\mathrm{O}(2)$ denote the group of 2×2 orthogonal matrices. Show that all elements of $\mathrm{O}(2)$ have determinant 1 or -1. Show that every element of $\mathrm{SO}(2)$ is a rotation. Let $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Show that $\mathrm{O}(2)$ decomposes as a union $\mathrm{SO}(2) \cup \mathrm{SO}(2)J$.

[You may assume standard properties of determinants.]

(c) Let B be the (abelian) group $\{z \in \mathbb{C} : |z| = 1\}$, with multiplication of complex numbers as the group operation. Write down, without proof, isomorphisms $\mathrm{SO}(2) \cong B \cong \mathbb{R}/\mathbb{Z}$ where $\mathbb{R}$ denotes the additive group of real numbers and $\mathbb{Z}$ the subgroup of integers. Deduce that $\mathrm{O}(2) \cong D(B)$, the generalised dihedral group defined in part (a).

9B Vector Calculus

(a) Given a space curve $\mathbf{r}(t) = (\, x(t),\, y(t),\, z(t)\,)$, with t a parameter (not necessarily arc-length), give mathematical expressions for the unit tangent, unit normal, and unit binormal vectors.

(b) Consider the closed curve given by

$$x = 2\cos^3 t, \qquad y = \sin^3 t, \qquad z = \sqrt{3}\sin^3 t, \qquad (*)$$

where $t \in [0,\, 2\pi)$.

Show that the unit tangent vector $\mathbf{T}$ may be written as

$$\mathbf{T} = \pm\frac{1}{2}\left(-2\cos t,\, \sin t,\, \sqrt{3}\sin t\right),$$

with each sign associated with a certain range of t, which you should specify.

Calculate the unit normal and the unit binormal vectors, and hence deduce that the curve lies in a plane.

(c) A closed space curve $\mathcal{C}$ lies in a plane with unit normal $\mathbf{n} = (a,\, b,\, c)$. Use Stokes' theorem to prove that the planar area enclosed by $\mathcal{C}$ is the absolute value of the line integral

$$\frac{1}{2}\int_{\mathcal{C}} (bz - cy)dx + (cx - az)dy + (ay - bx)dz.$$

Hence show that the planar area enclosed by the curve given by $(*)$ is $(3/2)\pi$.

10B Vector Calculus

(a) By considering an appropriate double integral, show that

$$\int_0^\infty e^{-ax^2}\, dx = \sqrt{\frac{\pi}{4a}}\,,$$

where $a > 0$.

(b) Calculate $\int_0^1 x^y dy$, treating x as a constant, and hence show that

$$\int_0^\infty \frac{(e^{-u} - e^{-2u})}{u}\, du = \log 2\,.$$

(c) Consider the region $\mathcal{D}$ in the x-y plane enclosed by $x^2 + y^2 = 4$, $y = 1$, and $y = \sqrt{3}x$ with $1 < y < \sqrt{3}x$.

Sketch $\mathcal{D}$, indicating any relevant polar angles.

A surface $\mathcal{S}$ is given by $z = xy/(x^2 + y^2)$. Calculate the volume below this surface and above $\mathcal{D}$.

11B Vector Calculus

(a) By a suitable change of variables, calculate the volume enclosed by the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, where a, b, and c are constants.

(b) Suppose T_{ij} is a second rank tensor. Use the divergence theorem to show that

$$\int_{\mathcal{S}} T_{ij} n_j \, dS = \int_{\mathcal{V}} \frac{\partial T_{ij}}{\partial x_j} \, dV \,, \qquad (*)$$

where $\mathcal{S}$ is a closed surface, with unit normal n_j, and $\mathcal{V}$ is the volume it encloses.

[*Hint: Consider* $e_i T_{ij}$ *for a constant vector* e_i.]

(c) A half-ellipsoidal membrane $\mathcal{S}$ is described by the *open* surface $4x^2+4y^2+z^2 = 4$, with $z \geqslant 0$. At a given instant, air flows beneath the membrane with velocity $\mathbf{u} = (-y,\, x,\, \alpha)$, where α is a constant. The flow exerts a force on the membrane given by

$$F_i = \int_{\mathcal{S}} \beta^2 u_i u_j n_j \, dS \,,$$

where β is a constant parameter.

Show the vector $a_i = \partial(u_i u_j)/\partial x_j$ can be rewritten as $\mathbf{a} = -(x,\, y,\, 0)$.

Hence use $(*)$ to calculate the force F_i on the membrane.

12B Vector Calculus

For a given charge distribution $\rho(\mathbf{x}, t)$ and current distribution $\mathbf{J}(\mathbf{x}, t)$ in $\mathbb{R}^3$, the electric and magnetic fields, $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$, satisfy Maxwell's equations, which in suitable units, read

$$\nabla \cdot \mathbf{E} = \rho \,, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \,,$$

$$\nabla \cdot \mathbf{B} = 0 \,, \qquad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \,.$$

The Poynting vector $\mathbf{P}$ is defined as $\mathbf{P} = \mathbf{E} \times \mathbf{B}$.

(a) For a closed surface $\mathcal{S}$ around a volume $\mathcal{V}$, show that

$$\int_{\mathcal{S}} \mathbf{P} \cdot d\mathbf{S} = -\int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{J} \, dV - \frac{\partial}{\partial t} \int_{\mathcal{V}} \frac{|\mathbf{E}|^2 + |\mathbf{B}|^2}{2} \, dV \,. \qquad (*)$$

(b) Suppose $\mathbf{J} = \mathbf{0}$ and consider an electromagnetic wave

$$\mathbf{E} = E_0 \, \hat{\mathbf{y}} \cos(kx - \omega t) \quad \text{and} \quad \mathbf{B} = B_0 \, \hat{\mathbf{z}} \cos(kx - \omega t) \,,$$

where E_0, B_0, k and ω are positive constants. Show that these fields satisfy Maxwell's equations for appropriate E_0, ω, and ρ.

Confirm the wave satisfies the integral identity $(*)$ by considering its propagation through a box $\mathcal{V}$, defined by $0 \leqslant x \leqslant \pi/(2k)$, $0 \leqslant y \leqslant L$, and $0 \leqslant z \leqslant L$.

END OF PAPER