MATHEMATICAL TRIPOS   Part IA

Saturday, 5 June, 2021   10:00am to 1:00pm

PAPER 2

Before you begin read these instructions carefully

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

*Candidates may obtain credit from attempts on **all four** questions from Section I and **at most five** questions from Section II. Of the Section II questions, no more than three may be on the same course.*

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

*Complete a gold cover sheet **for each question** that you have attempted, and place it at the front of your answer to that question.*

*Complete a green master cover sheet listing **all the questions** that you have attempted.*

Every cover sheet must also show your Blind Grade Number and desk number.

*Tie up your answers and cover sheets into **a single bundle**, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

> **You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

SECTION I

1A   Differential Equations
Solve the difference equation

$$y_{n+2} - 4y_{n+1} + 4y_n = n$$

subject to the initial conditions $y_0 = 1$ and $y_1 = 0$.

2A   Differential Equations
Let y_1 and y_2 be two linearly independent solutions to the differential equation

$$\frac{d^2 y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0.$$

Show that the Wronskian $W = y_1 y_2' - y_2 y_1'$ satisfies

$$\frac{dW}{dx} + pW = 0.$$

Deduce that if $y_2(x_0) = 0$ then

$$y_2(x) = y_1(x) \int_{x_0}^x \frac{W(t)}{y_1(t)^2}\, dt.$$

Given that $y_1(x) = x^3$ satisfies the equation

$$x^2\frac{d^2 y}{dx^2} - x\frac{dy}{dx} - 3y = 0$$

find the solution which satisfies $y(1) = 0$ and $y'(1) = 1$.

3D   Probability
A coin has probability p of landing heads. Let q_n be the probability that the number of heads after n tosses is even. Give an expression for q_{n+1} in terms of q_n. Hence, or otherwise, find q_n.

4F   Probability

Let X be a continuous random variable taking values in $[0, \sqrt{3}]$. Let the probability density function of X be

$$f_X(x) = \frac{c}{1 + x^2}, \quad \text{for } x \in [0, \sqrt{3}],$$

where c is a constant.

Find the value of c and calculate the mean, variance and median of X.

[*Recall that the median of X is the number m such that $\mathbb{P}(X \leqslant m) = \frac{1}{2}$.*]

SECTION II

5A   Differential Equations

For a linear, second order differential equation define the terms *ordinary point, singular point* and *regular singular point.*

For $a, b \in \mathbb{R}$ and $b \notin \mathbb{Z}$ consider the following differential equation

$$x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (b - x)\frac{\mathrm{d}y}{\mathrm{d}x} - ay = 0\,. \tag{$*$}$$

Find coefficients $c_m(a, b)$ such that the function $y_1 = M(x, a, b)$, where

$$M(x, a, b) = \sum_{m=0}^{\infty} c_m(a, b)x^m,$$

satisfies $(*)$. By making the substitution $y = x^{1-b}u(x)$, or otherwise, find a second linearly independent solution of the form $y_2 = x^{1-b}M(x, \alpha, \beta)$ for suitable α, β.

Suppose now that $b = 1$. By considering a limit of the form

$$\lim_{b \to 1} \frac{y_2 - y_1}{b - 1}\,,$$

or otherwise, obtain two linearly independent solutions to $(*)$ in terms of M and derivatives thereof.

6A   Differential Equations

By means of the change of variables $\eta = x - t$ and $\xi = x + t$, show that the wave equation for $u = u(x,t)$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{$*$}$$

is equivalent to the equation

$$\frac{\partial^2 U}{\partial \eta \, \partial \xi} = 0$$

where $U(\eta, \xi) = u(x,t)$. Hence show that the solution to $(*)$ on $x \in \mathbf{R}$ and $t > 0$, subject to the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x)$$

is

$$u(x,t) = \frac{1}{2}\left[f(x-t) + f(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,\mathrm{d}y\,.$$

Deduce that if $f(x) = 0$ and $g(x) = 0$ on the interval $|x - x_0| > r$ then $u(x,t) = 0$ on $|x - x_0| > r + t$.

Suppose now that $y = y(x,t)$ is a solution to the wave equation $(*)$ on the finite interval $0 < x < L$ and obeys the boundary conditions

$$y(0,t) = y(L,t) = 0$$

for all t. The energy is defined by

$$E(t) = \frac{1}{2}\int_0^L \left[\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2\right]\mathrm{d}x\,.$$

By considering $\mathrm{d}E/\mathrm{d}t$, or otherwise, show that the energy remains constant in time.

7A   Differential Equations

The function $\theta = \theta(t)$ takes values in the interval $(-\pi, \pi]$ and satisfies the differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + (\lambda - 2\mu)\sin\theta + \frac{2\mu\sin\theta}{\sqrt{5 + 4\cos\theta}} = 0\,, \tag{$*$}$$

where λ and μ are positive constants.

Let $\omega = \dot{\theta}$. Express $(*)$ in terms of a pair of first order differential equations in (θ, ω). Show that if $3\lambda < 4\mu$ then there are three fixed points in the region $0 \leqslant \theta \leqslant \pi$.

Classify all the fixed points of the system in the case $3\lambda < 4\mu$. Sketch the phase portrait in the case $\lambda = 1$ and $\mu = 3/2$.

Comment briefly on the case when $3\lambda > 4\mu$.

8A   Differential Equations

For an $n \times n$ matrix A, define the matrix exponential by

$$\exp(A) = \sum_{m=0}^{\infty} \frac{A^m}{m!} \,,$$

where $A^0 \equiv I$, with I being the $n \times n$ identity matrix. [You may assume that $\exp((s+t)A) = \exp(sA)\exp(tA)$ for real numbers s, t and you do not need to consider issues of convergence.] Show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \exp(tA) = A \exp(tA) \,.$$

Deduce that the unique solution to the initial value problem

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{y}_0, \quad \text{where } \mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix},$$

is $\mathbf{y}(t) = \exp(tA)\mathbf{y}_0$.

Let $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{f} = \mathbf{f}(t)$ be vectors of length n and A a real $n \times n$ matrix. By considering a suitable integrating factor, show that the unique solution to

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} - A\mathbf{x} = \mathbf{f}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{$*$}$$

is given by

$$\mathbf{x}(t) = \exp(tA)\mathbf{x}_0 + \int_0^t \exp[(t-s)A]\mathbf{f}(s)\,\mathrm{d}s \,.$$

Hence, or otherwise, solve the system of differential equations $(*)$ when

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{pmatrix}, \quad \mathbf{f}(t) = \begin{pmatrix} \sin t \\ 3\sin t \\ 0 \end{pmatrix}, \quad \mathbf{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

[*Hint: Compute A^2 and show that $A^3 = 0$.*]

9E   Probability

(a) (i) Define the *conditional probability* $\mathbb{P}(A|B)$ of the event A given the event B. Let $\{B_j : 1 \leqslant j \leqslant n\}$ be a partition of the sample space such that $\mathbb{P}(B_j) > 0$ for all j. Show that, if $\mathbb{P}(A) > 0$,

$$\mathbb{P}(B_j|A) = \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\sum_{k=1}^{n} \mathbb{P}(A|B_k)\mathbb{P}(B_k)}.$$

(ii) There are n urns, the rth of which contains $r - 1$ red balls and $n - r$ blue balls. Alice picks an urn (uniformly) at random and removes two balls without replacement. Find the probability that the first ball is blue, and the conditional probability that the second ball is blue, given that the first is blue. [You may assume, if you wish, that $\sum_{i=1}^{n-1} i(i-1) = \frac{1}{3}n(n-1)(n-2)$.]

(b) (i) What is meant by saying that two events A and B are *independent*? Two fair (6-sided) dice are rolled. Let A_t be the event that the sum of the numbers shown is t, and let B_i be the event that the first die shows i. For what values of t and i are the two events A_t and B_i independent?

(ii) The casino at Monte Corona features the following game: three coins each show heads with probability 3/5 and tails otherwise. The first counts 10 points for a head and 2 for a tail; the second counts 4 points for both a head and a tail; and the third counts 3 points for a head and 20 for a tail. You and your opponent each choose a coin. You cannot both choose the same coin. Each of you tosses your coin once and the person with the larger score wins the jackpot. Would you prefer to be the first or the second to choose a coin?

10E Probability

(a) Alanya repeatedly rolls a fair six-sided die. What is the probability that the first number she rolls is a 1, given that she rolls a 1 before she rolls a 6?

(b) Let $(X_n)_{n \geqslant 0}$ be a simple symmetric random walk on the integers starting at $x \in \mathbb{Z}$, that is,

$$
X_n = \begin{cases} x & \text{if } n = 0 \\ x + \sum_{i=1}^n Y_i & \text{if } n \geqslant 1 \end{cases},
$$

where $(Y_n)_{n \geqslant 1}$ is a sequence of IID random variables with $\mathbb{P}(Y_n = 1) = \mathbb{P}(Y_n = -1) = \frac{1}{2}$. Let $T = \min\{n \geqslant 0 : X_n = 0\}$ be the time that the walk first hits 0.

(i) Let n be a positive integer. For $0 < x < n$, calculate the probability that the walk hits 0 before it hits n.

(ii) Let $x = 1$ and let A be the event that the walk hits 0 before it hits 3. Find $\mathbb{P}(X_1 = 0 | A)$. Hence find $\mathbb{E}(T | A)$.

(iii) Let $x = 1$ and let B be the event that the walk hits 0 before it hits 4. Find $\mathbb{E}(T | B)$.

11D Probability

Let Δ be the disc of radius 1 with centre at the origin O. Let P be a random point uniformly distributed in Δ. Let (R, Θ) be the polar coordinates of P. Show that R and Θ are independent and find their probability density functions f_R and f_Θ.

Let A, B and C be three random points selected independently and uniformly in Δ. Find the expected area of triangle OAB and hence find the probability that C lies in the interior of triangle OAB.

Find the probability that O, A, B and C are the vertices of a convex quadrilateral.

12F Probability

State and prove Chebyshev's inequality.

Let $(X_i)_{i \geqslant 1}$ be a sequence of independent, identically distributed random variables such that

$$\mathbb{P}(X_i = 1) = p \text{ and } \mathbb{P}(X_i = 0) = 1 - p$$

for some $p \in [0,1]$, and let $f : [0,1] \to \mathbb{R}$ be a continuous function.

(i) Prove that

$$B_n(p) := \mathbb{E}\left(f\left(\frac{X_1 + \cdots + X_n}{n} \right) \right)$$

is a polynomial function of p, for any natural number n.

(ii) Let $\delta > 0$. Prove that

$$\sum_{k \in K_\delta} \binom{n}{k} p^k (1-p)^{n-k} \leqslant \frac{1}{4n\delta^2} \,,$$

where K_δ is the set of natural numbers $0 \leqslant k \leqslant n$ such that $|k/n - p| > \delta$.

(iii) Show that

$$\sup_{p \in [0,1]} |f(p) - B_n(p)| \to 0$$

as $n \to \infty$. [You may use without proof that, for any $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| \leqslant \epsilon$ for all $x, y \in [0,1]$ with $|x - y| \leqslant \delta$.]

END OF PAPER