UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part IA

Thursday, 3 June, 2021 10:00am to 1:00pm

## PAPER 1

## *Before you begin read these instructions carefully*

*The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.*

*Candidates may obtain credit from attempts on* ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

*Write on* ***one side*** *of the paper only and begin each answer on a separate sheet.*

*Write legibly; otherwise you place yourself at a grave disadvantage.*

## *At the end of the examination:*

*Separate your answers to each question.*

*Complete a gold cover sheet* ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

*Complete a green master cover sheet listing* ***all the questions*** *that you have attempted.*

***Every cover sheet must also show your Blind Grade Number and desk number.***

*Tie up your answers and cover sheets into* ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

***STATIONERY REQUIREMENTS***
*Gold cover sheets*
*Green master cover sheet*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

## SECTION I

**1C Vectors and Matrices**

(a) Find all complex solutions to the equation $z^i = 1$.

(b) Write down an equation for the numbers $z$ which describe, in the complex plane, a circle with radius 5 centred at $c = 5i$. Find the points on the circle at which it intersects the line passing through $c$ and $z_0 = \frac{15}{4}$.

**2B Vectors and Matrices**

The matrix

$$A = \begin{pmatrix} 2 & -1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix}$$

represents a linear map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ with respect to the bases

$$B = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right\}, \quad C = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Find the matrix $A'$ that represents $\Phi$ with respect to the bases

$$B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \quad C' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

**3F Analysis I**

State and prove the alternating series test. Hence show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Show also that

$$\frac{7}{12} \leqslant \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \leqslant \frac{47}{60}.$$

**4F Analysis I**

State and prove the Bolzano–Weierstrass theorem.

Consider a bounded sequence $(x_n)$. Prove that if every convergent subsequence of $(x_n)$ converges to the same limit $L$ then $(x_n)$ converges to $L$.

## SECTION II

**5C Vectors and Matrices**

Using the standard formula relating products of the Levi-Civita symbol $\epsilon_{ijk}$ to products of the Kronecker $\delta_{ij}$, prove

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\,.$$

Define the scalar triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$ in terms of the dot and cross product. Show that

$$[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]^2\,.$$

Given a basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ for $\mathbb{R}^3$ which is not necessarily orthonormal, let

$$\mathbf{e}_1' = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}\,, \quad \mathbf{e}_2' = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}\,, \quad \mathbf{e}_3' = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]}\,.$$

Show that $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$ is also a basis for $\mathbb{R}^3$. [You may assume that three linearly independent vectors in $\mathbb{R}^3$ form a basis.]

The vectors $\mathbf{e}_1''$, $\mathbf{e}_2''$, $\mathbf{e}_3''$ are constructed from $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$ in the same way that $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$ are constructed from $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Show that

$$\mathbf{e}_1'' = \mathbf{e}_1\,, \quad \mathbf{e}_2'' = \mathbf{e}_2\,, \quad \mathbf{e}_3'' = \mathbf{e}_3\,.$$

An infinite lattice consists of all points with position vectors given by

$$\mathbf{R} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3 \quad \text{with } n_1, n_2, n_3 \ \in \mathbb{Z}\,.$$

Find all points with position vectors $\mathbf{K}$ such that $\mathbf{K} \cdot \mathbf{R}$ is an integer for all integers $n_1$, $n_2$, $n_3$.

**6A Vectors and Matrices**

(a) For an $n \times n$ matrix $A$ define the *characteristic polynomial* $\chi_A$ and the *characteristic equation*.

The Cayley–Hamilton theorem states that every $n \times n$ matrix satisfies its own characteristic equation. Verify this in the case $n = 2$.

(b) Define the adjugate matrix $\operatorname{adj}(A)$ of an $n \times n$ matrix $A$ in terms of the minors of $A$. You may assume that

$$A \operatorname{adj}(A) = \operatorname{adj}(A)\, A = \det(A) I\,,$$

where $I$ is the $n \times n$ identity matrix. Show that if $A$ and $B$ are non-singular $n \times n$ matrices then

$$\operatorname{adj}(AB) = \operatorname{adj}(B) \operatorname{adj}(A)\,. \qquad (*)$$

(c) Let $M$ be an arbitrary $n \times n$ matrix. Explain why

(i) there is an $\alpha > 0$ such that $M - tI$ is non-singular for $0 < t < \alpha$;

(ii) the entries of $\operatorname{adj}(M - tI)$ are polynomials in $t$.

Using parts (i) and (ii), or otherwise, show that $(*)$ holds for all matrices $A, B$.

(d) The characteristic polynomial of the arbitrary $n \times n$ matrix $A$ is

$$\chi_A(z) = (-1)^n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0\,.$$

By considering $\operatorname{adj}(A - tI)$, or otherwise, show that

$$\operatorname{adj}(A) = (-1)^{n-1} A^{n-1} - c_{n-1} A^{n-2} - \cdots - c_2 A - c_1 I\,.$$

[You may assume the Cayley–Hamilton theorem.]

**7A Vectors and Matrices**

Let $A$ be a real, symmetric $n \times n$ matrix.

We say that $A$ is *positive semi-definite* if $\mathbf{x}^T A\mathbf{x} \geqslant 0$ for all $\mathbf{x} \in \mathbb{R}^n$. Prove that $A$ is positive semi-definite if and only if all the eigenvalues of $A$ are non-negative. [You may quote results from the course, provided that they are clearly stated.]

We say that $A$ has a *principal square root* $B$ if $A = B^2$ for some symmetric, positive semi-definite $n \times n$ matrix $B$. If such a $B$ exists we write $B = \sqrt{A}$. Show that if $A$ is positive semi-definite then $\sqrt{A}$ exists.

Let $M$ be a real, non-singular $n \times n$ matrix. Show that $M^T M$ is symmetric and positive semi-definite. Deduce that $\sqrt{M^T M}$ exists and is non-singular. By considering the matrix

$$M\left(\sqrt{M^T M}\right)^{-1},$$

or otherwise, show $M = RP$ for some orthogonal $n \times n$ matrix $R$ and a symmetric, positive semi-definite $n \times n$ matrix $P$.

Describe the transformation $RP$ geometrically in the case $n = 3$.

**8B Vectors and Matrices**

(a) Consider the matrix

$$A = \begin{pmatrix} \mu & 1 & 1 \\ 2 & -\mu & 0 \\ -\mu & 2 & 1 \end{pmatrix}.$$

Find the kernel of $A$ for each real value of the constant $\mu$. Hence find how many solutions $\mathbf{x} \in \mathbb{R}^3$ there are to

$$A\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

depending on the value of $\mu$. [There is no need to find expressions for the solution(s).]

(b) Consider the reflection map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$\Phi : \mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n}$$

where $\mathbf{n}$ is a unit vector normal to the plane of reflection.

(i) Find the matrix $H$ which corresponds to the map $\Phi$ in terms of the components of $\mathbf{n}$.

(ii) Prove that a reflection in a plane with unit normal $\mathbf{n}$ followed by a reflection in a plane with unit normal vector $\mathbf{m}$ (both containing the origin) is equivalent to a rotation along the line of intersection of the planes with an angle twice that between the planes.

[*Hint: Choose your coordinate axes carefully.*]

(iii) Briefly explain why a rotation followed by a reflection or vice-versa can never be equivalent to another rotation.

**9F Analysis I**

(a) State the intermediate value theorem. Show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous bijection and $x_1 < x_2 < x_3$ then either $f(x_1) < f(x_2) < f(x_3)$ or $f(x_1) > f(x_2) > f(x_3)$. Deduce that $f$ is either strictly increasing or strictly decreasing.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions. Which of the following statements are true, and which can be false? Give a proof or counterexample as appropriate.

(i) If $f$ and $g$ are continuous then $f \circ g$ is continuous.

(ii) If $g$ is strictly increasing and $f \circ g$ is continuous then $f$ is continuous.

(iii) If $f$ is continuous and a bijection then $f^{-1}$ is continuous.

(iv) If $f$ is differentiable and a bijection then $f^{-1}$ is differentiable.

**10F Analysis I**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function.

(a) Let $m = \min_{x \in [a,b]} f(x)$ and $M = \max_{x \in [a,b]} f(x)$. If $g : [a, b] \to \mathbb{R}$ is a positive continuous function, prove that

$$m \int_a^b g(x)dx \leqslant \int_a^b f(x)g(x)dx \leqslant M \int_a^b g(x)dx$$

directly from the definition of the Riemann integral.

(b) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that

$$\int_0^{1/\sqrt{n}} nf(x)e^{-nx}dx \to f(0)$$

as $n \to \infty$, and deduce that

$$\int_0^1 nf(x)e^{-nx}dx \to f(0)$$

as $n \to \infty$.

**11F Analysis I**

Let $f : \mathbb{R} \to \mathbb{R}$ be $n$-times differentiable, for some $n > 0$.

(a) State and prove Taylor's theorem for $f$, with the Lagrange form of the remainder. [You may assume Rolle's theorem.]

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function such that $f(0) = 1$ and $f'(0) = 0$, and satisfying the differential equation $f''(x) = -f(x)$. Prove carefully that

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} .$$

**12F Analysis I**

(a) Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with $a_n \in \mathbb{C}$. Show that there exists $R \in [0, \infty]$ (called the *radius of convergence*) such that the series is absolutely convergent when $|z| < R$ but is divergent when $|z| > R$.

Suppose that the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ is $R = 2$. For a fixed positive integer $k$, find the radii of convergence of the following series. [You may assume that $\lim_{n\to\infty} |a_n|^{1/n}$ exists.]

(i) $\displaystyle\sum_{n=0}^{\infty} a_n^k z^n$ .

(ii) $\displaystyle\sum_{n=0}^{\infty} a_n z^{kn}$ .

(iii) $\displaystyle\sum_{n=0}^{\infty} a_n z^{n^2}$ .

(b) Suppose that there exist values of $z$ for which $\sum_{n=0}^{\infty} b_n e^{nz}$ converges and values for which it diverges. Show that there exists a real number $S$ such that $\sum_{n=0}^{\infty} b_n e^{nz}$ diverges whenever $\mathrm{Re}(z) > S$ and converges whenever $\mathrm{Re}(z) < S$.

Determine the set of values of $z$ for which

$$\sum_{n=0}^{\infty} \frac{2^n e^{inz}}{(n+1)^2}$$

converges.

**END OF PAPER**