UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2021

List of Courses

Paper 2, Section I

2F Analysis and Topology

Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and let $C([0,1])$ denote the set of continuous real-valued functions on $[0,1]$. Given $f \in C([0,1])$, define the function Tf by the expression

$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy.$$

(a) Prove that T is a continuous map $C([0,1]) \to C([0,1])$ with the uniform metric on $C([0,1])$.

(b) Let d_1 be the metric on $C([0,1])$ given by

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

Is T continuous with respect to d_1?

Paper 4, Section I

2F Analysis and Topology

Let X be a topological space with an equivalence relation, $\tilde{X}$ the set of equivalence classes, $\pi : X \to \tilde{X}$, the quotient map taking a point in X to its equivalence class.

(a) Define the *quotient topology* on $\tilde{X}$ and check it is a topology.

(b) Prove that if Y is a topological space, a map $f : \tilde{X} \to Y$ is continuous if and only if $f \circ \pi$ is continuous.

(c) If X is Hausdorff, is it true that $\tilde{X}$ is also Hausdorff? Justify your answer.

Paper 1, Section II

10F Analysis and Topology

Let $f : X \to Y$ be a map between metric spaces. Prove that the following two statements are equivalent:

(i) $f^{-1}(A) \subset X$ is open whenever $A \subset Y$ is open.

(ii) $f(x_n) \to f(a)$ for any sequence $x_n \to a$.

For $f : X \to Y$ as above, determine which of the following statements are always true and which may be false, giving a proof or a counterexample as appropriate.

(a) If X is compact and f is continuous, then f is uniformly continuous.

(b) If X is compact and f is continuous, then Y is compact.

(c) If X is connected, f is continuous and $f(X)$ is dense in Y, then Y is connected.

(d) If the set $\{(x,y) \in X \times Y : \ y = f(x)\}$ is closed in $X \times Y$ and Y is compact, then f is continuous.

Paper 2, Section II

10F Analysis and Topology

Let $k_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions satisfying the following properties:

1. $k_n(x) \geqslant 0$ for all n and $x \in \mathbb{R}$ and there is $R > 0$ such that k_n vanishes outside $[-R, R]$ for all n;
2. each k_n is continuous and
$$\int_{-\infty}^{\infty} k_n(t)\, dt = 1;$$
3. given $\varepsilon > 0$ and $\delta > 0$, there exists a positive integer N such that if $n \geqslant N$, then
$$\int_{-\infty}^{-\delta} k_n(t)\, dt + \int_{\delta}^{\infty} k_n(t)\, dt < \varepsilon.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function and set

$$f_n(x) := \int_{-\infty}^{\infty} k_n(t) f(x-t)\, dt.$$

Show that f_n converges uniformly to f on any compact subset of $\mathbb{R}$.

Let $g : [0,1] \to \mathbb{R}$ be a continuous function with $g(0) = g(1) = 0$. Show that there is a sequence of polynomials p_n such that p_n converges uniformly to g on $[0,1]$. [*Hint: consider the functions*

$$k_n(t) = \begin{cases} (1-t^2)^n / c_n & t \in [-1,1] \\ 0 & \text{otherwise,} \end{cases}$$

where c_n is a suitably chosen constant.]

Paper 3, Section II

11F Analysis and Topology

Define the terms *connected* and *path-connected* for a topological space. Prove that the interval $[0,1]$ is connected and that if a topological space is path-connected, then it is connected.

Let X be an open subset of Euclidean space $\mathbb{R}^n$. Show that X is connected if and only if X is path-connected.

Let X be a topological space with the property that every point has a neighbourhood homeomorphic to an open set in $\mathbb{R}^n$. Assume X is connected; must X be also path-connected? Briefly justify your answer.

Consider the following subsets of $\mathbb{R}^2$:

$$A = \{(x,0) : \ x \in (0,1]\}, \ \ B = \{(0,y) : \ y \in [1/2,1]\}, \text{ and}$$

$$C_n = \{(1/n, y) : \ y \in [0,1]\} \text{ for } n \geqslant 1.$$

Let

$$X = A \cup B \cup \bigcup_{n \geqslant 1} C_n$$

with the subspace topology. Is X path-connected? Is X connected? Justify your answers.

Paper 4, Section II

10F Analysis and Topology

(a) Let $g : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function such that for each $t \in [0,1]$, the partial derivatives $D_i g(t,x)$ $(i = 1, \ldots, n)$ of $x \mapsto g(t,x)$ exist and are continuous on $[0,1] \times \mathbb{R}^n$. Define $G : \mathbb{R}^n \to \mathbb{R}$ by

$$G(x) = \int_0^1 g(t,x)\, dt.$$

Show that G has continuous partial derivatives $D_i G$ given by

$$D_i G(x) = \int_0^1 D_i g(t,x)\, dt$$

for $i = 1, \ldots, n$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an infinitely differentiable function, that is, partial derivatives $D_{i_1} D_{i_2} \cdots D_{i_k} f$ exist and are continuous for all $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in \{1,2\}$. Show that for any $(x_1, x_2) \in \mathbb{R}^2$,

$$f(x_1,x_2) = f(x_1,0) + x_2\, D_2 f(x_1,0) + x_2^2\, h(x_1,x_2),$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is an infinitely differentiable function.

[*Hint: You may use the fact that if* $u : \mathbb{R} \to \mathbb{R}$ *is infinitely differentiable, then*

$$u(1) = u(0) + u'(0) + \int_0^1 (1-t) u''(t)\, dt.]$$

Paper 4, Section I

3G Complex Analysis

Let f be a holomorphic function on a neighbourhood of $a \in \mathbb{C}$. Assume that f has a zero of order k at a with $k \geqslant 1$. Show that there exist $\varepsilon > 0$ and $\delta > 0$ such that for any b with $0 < |b| < \varepsilon$ there are exactly k distinct values of $z \in D(a, \delta)$ with $f(z) = b$.

Paper 3, Section II

13G Complex Analysis

Let γ be a curve (not necessarily closed) in $\mathbb{C}$ and let $[\gamma]$ denote the image of γ. Let $\phi \colon [\gamma] \to \mathbb{C}$ be a continuous function and define

$$f(z) = \int_\gamma \frac{\phi(\lambda)}{\lambda - z}\, d\lambda$$

for $z \in \mathbb{C} \setminus [\gamma]$. Show that f has a power series expansion about every $a \notin [\gamma]$.

Using Cauchy's Integral Formula, show that a holomorphic function has complex derivatives of all orders. [Properties of power series may be assumed without proof.] Let f be a holomorphic function on an open set U that contains the closed disc $\overline{D}(a, r)$. Obtain an integral formula for the derivative of f on the open disc $D(a, r)$ in terms of the values of f on the boundary of the disc.

Show that if holomorphic functions f_n on an open set U converge locally uniformly to a holomorphic function f on U, then f_n' converges locally uniformly to f'.

Let D_1 and D_2 be two overlapping closed discs. Let f be a holomorphic function on some open neighbourhood of $D = D_1 \cap D_2$. Show that there exist open neighbourhoods U_j of D_j and holomorphic functions f_j on U_j, $j = 1, 2$, such that $f(z) = f_1(z) + f_2(z)$ on $U_1 \cap U_2$.

Paper 1, Section I

3B Complex Analysis or Complex Methods

Let $x > 0$, $x \neq 2$, and let C_x denote the positively oriented circle of radius x centred at the origin. Define

$$g(x) = \oint_{C_x} \frac{z^2 + e^z}{z^2(z-2)}\, dz.$$

Evaluate $g(x)$ for $x \in (0, \infty) \setminus \{2\}$.

Paper 1, Section II

12G Complex Analysis or Complex Methods

(a) State a theorem establishing Laurent series of analytic functions on suitable domains. Give a formula for the n^{th} Laurent coefficient.

Define the notion of *isolated singularity*. State the classification of an isolated singularity in terms of Laurent coefficients.

Compute the Laurent series of

$$f(z) = \frac{1}{z(z-1)}$$

on the annuli $A_1 = \{z : 0 < |z| < 1\}$ and $A_2 = \{z : 1 < |z|\}$. Using this example, comment on the statement that Laurent coefficients are unique. Classify the singularity of f at 0.

(b) Let U be an open subset of the complex plane, let $a \in U$ and let $U' = U \setminus \{a\}$. Assume that f is an analytic function on U' with $|f(z)| \to \infty$ as $z \to a$. By considering the Laurent series of $g(z) = \frac{1}{f(z)}$ at a, classify the singularity of f at a in terms of the Laurent coefficients. [You may assume that a continuous function on U that is analytic on U' is analytic on U.]

Now let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function with $|f(z)| \to \infty$ as $z \to \infty$. By considering Laurent series at 0 of $f(z)$ and of $h(z) = f\left(\frac{1}{z}\right)$, show that f is a polynomial.

(c) Classify, giving reasons, the singularity at the origin of each of the following functions and in each case compute the residue:

$$g(z) = \frac{\exp(z) - 1}{z \log(z+1)} \qquad \text{and} \qquad h(z) = \sin(z)\sin(1/z)\ .$$

Paper 2, Section II

12B Complex Analysis or Complex Methods

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let $a > 0$, $b > 0$ be constants. Show that if

$$|f(z)| \leqslant a|z|^{n/2} + b$$

for all $z \in \mathbb{C}$, where n is a positive odd integer, then f must be a polynomial with degree not exceeding $\lfloor n/2 \rfloor$ (closest integer part rounding down).

Does there exist a function f, analytic in $\mathbb{C} \setminus \{0\}$, such that $|f(z)| \geqslant 1/\sqrt{|z|}$ for all nonzero z? Justify your answer.

(b) State Liouville's Theorem and use it to show the following.

(i) If u is a positive harmonic function on $\mathbb{R}^2$, then u is a constant function.

(ii) Let $L = \{z \,|\, z = ax + b, x \in \mathbb{R}\}$ be a line in $\mathbb{C}$ where $a, b \in \mathbb{C}$, $a \neq 0$. If $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f(\mathbb{C}) \cap L = \emptyset$, then f is a constant function.

Paper 3, Section I

3B Complex Methods

Find the value of A for which the function

$$\phi(x,y) = x\cosh y \sin x + Ay \sinh y \cos x$$

satisfies Laplace's equation. For this value of A, find a complex analytic function of which ϕ is the real part.

Paper 4, Section II

12B Complex Methods

Let $f(t)$ be defined for $t \geqslant 0$. Define the *Laplace transform* $\widehat{f}(s)$ of f. Find an expression for the Laplace transform of $\frac{df}{dt}$ in terms of $\widehat{f}$.

Three radioactive nuclei decay sequentially, so that the numbers $N_i(t)$ of the three types obey the equations

$$\begin{aligned} \frac{dN_1}{dt} &= -\lambda_1 N_1 \,, \\ \frac{dN_2}{dt} &= \lambda_1 N_1 - \lambda_2 N_2 \,, \\ \frac{dN_3}{dt} &= \lambda_2 N_2 - \lambda_3 N_3 \,, \end{aligned}$$

where $\lambda_3 > \lambda_2 > \lambda_1 > 0$ are constants. Initially, at $t = 0$, $N_1 = N$, $N_2 = 0$ and $N_3 = n$. Using Laplace transforms, find $N_3(t)$.

By taking an appropriate limit, find $N_3(t)$ when $\lambda_2 = \lambda_1 = \lambda > 0$ and $\lambda_3 > \lambda$.

Paper 2, Section I

4D Electromagnetism

State Gauss's Law in the context of electrostatics.

A simple coaxial cable consists of an inner conductor in the form of a perfectly conducting, solid cylinder of radius a, surrounded by an outer conductor in the form of a perfectly conducting, cylindrical shell of inner radius $b > a$ and outer radius $c > b$. The cylinders are coaxial and the gap between them is filled with a perfectly insulating material. The cable may be assumed to be straight and arbitrarily long.

In a steady state, the inner conductor carries an electric charge $+Q$ per unit length, and the outer conductor carries an electric charge $-Q$ per unit length. The charges are distributed in a cylindrically symmetric way and no current flows through the cable.

Determine the electrostatic potential and the electric field as functions of the cylindrical radius r, for $0 < r < \infty$. Calculate the capacitance C of the cable per unit length and the electrostatic energy U per unit length, and verify that these are related by

$$U = \frac{Q^2}{2C}\,.$$

Paper 4, Section I

5D Electromagnetism

Write down Maxwell's equations in a vacuum. Show that they admit wave solutions with

$$\mathbf{B}(\mathbf{x},t) = \mathrm{Re}\left[\mathbf{B}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right],$$

where $\mathbf{B}_0$, $\mathbf{k}$ and ω must obey certain conditions that you should determine. Find the corresponding electric field $\mathbf{E}(\mathbf{x},t)$.

A light wave, travelling in the x-direction and linearly polarised so that the magnetic field points in the z-direction, is incident upon a conductor that occupies the half-space $x > 0$. The electric and magnetic fields obey the boundary conditions $\mathbf{E}\times\mathbf{n} = \mathbf{0}$ and $\mathbf{B}\cdot\mathbf{n} = 0$ on the surface of the conductor, where $\mathbf{n}$ is the unit normal vector. Determine the contributions to the magnetic field from the incident and reflected waves in the region $x \leqslant 0$. Compute the magnetic field tangential to the surface of the conductor.

Paper 1, Section II
15D Electromagnetism

(a) Show that the magnetic flux passing through a simple, closed curve C can be written as

$$\Phi = \oint_C \mathbf{A} \cdot \mathbf{dx} \,,$$

where $\mathbf{A}$ is the magnetic vector potential. Explain why this integral is independent of the choice of gauge.

(b) Show that the magnetic vector potential due to a static electric current density $\mathbf{J}$, in the Coulomb gauge, satisfies Poisson's equation

$$-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \,.$$

Hence obtain an expression for the magnetic vector potential due to a static, thin wire, in the form of a simple, closed curve C, that carries an electric current I. [You may assume that the electric current density of the wire can be written as

$$\mathbf{J}(\mathbf{x}) = I \int_C \delta^{(3)}(\mathbf{x} - \mathbf{x}') \, \mathbf{dx}' \,,$$

where $\delta^{(3)}$ is the three-dimensional Dirac delta function.]

(c) Consider two thin wires, in the form of simple, closed curves C_1 and C_2, that carry electric currents I_1 and I_2, respectively. Let Φ_{ij} (where $i, j \in \{1, 2\}$) be the magnetic flux passing through the curve C_i due to the current I_j flowing around C_j. The inductances are defined by $L_{ij} = \Phi_{ij}/I_j$. By combining the results of parts (a) and (b), or otherwise, derive Neumann's formula for the mutual inductance,

$$L_{12} = L_{21} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\mathbf{dx}_1 \cdot \mathbf{dx}_2}{|\mathbf{x}_1 - \mathbf{x}_2|} \,.$$

Suppose that C_1 is a circular loop of radius a, centred at $(0, 0, 0)$ and lying in the plane $z = 0$, and that C_2 is a different circular loop of radius b, centred at $(0, 0, c)$ and lying in the plane $z = c$. Show that the mutual inductance of the two loops is

$$\frac{\mu_0}{4} \sqrt{a^2 + b^2 + c^2} \, f(q) \,,$$

where

$$q = \frac{2ab}{a^2 + b^2 + c^2}$$

and the function $f(q)$ is defined, for $0 < q < 1$, by the integral

$$f(q) = \int_0^{2\pi} \frac{q \cos\theta \, d\theta}{\sqrt{1 - q\cos\theta}} \,.$$

Paper 2, Section II

16D Electromagnetism

(a) Show that, for $|\mathbf{x}| \gg |\mathbf{y}|$,

$$\frac{1}{|\mathbf{x}-\mathbf{y}|} = \frac{1}{|\mathbf{x}|}\left[1 + \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|^2} + \frac{3(\mathbf{x}\cdot\mathbf{y})^2 - |\mathbf{x}|^2|\mathbf{y}|^2}{2|\mathbf{x}|^4} + O\left(\frac{|\mathbf{y}|^3}{|\mathbf{x}|^3}\right)\right].$$

(b) A particle with electric charge $q > 0$ has position vector $(a, 0, 0)$, where $a > 0$. An earthed conductor (held at zero potential) occupies the plane $x = 0$. Explain why the boundary conditions can be satisfied by introducing a fictitious 'image' particle of appropriate charge and position. Hence determine the electrostatic potential and the electric field in the region $x > 0$. Find the leading-order approximation to the potential for $|\mathbf{x}| \gg a$ and compare with that of an electric dipole. Directly calculate the total flux of the electric field through the plane $x = 0$ and comment on the result. Find the induced charge distribution on the surface of the conductor, and the total induced surface charge. Sketch the electric field lines in the plane $z = 0$.

(c) Now consider instead a particle with charge q at position $(a, b, 0)$, where $a > 0$ and $b > 0$, with earthed conductors occupying the planes $x = 0$ and $y = 0$. Find the leading-order approximation to the potential in the region $x, y > 0$ for $|\mathbf{x}| \gg a, b$ and state what type of multipole potential this is.

Paper 3, Section II

15D Electromagnetism

(a) The energy density stored in the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ is given by

$$w = \frac{\epsilon_0}{2}\,\mathbf{E}\cdot\mathbf{E} + \frac{1}{2\mu_0}\,\mathbf{B}\cdot\mathbf{B}\,.$$

Show that, in regions where no electric current flows,

$$\frac{\partial w}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0$$

for some vector field $\mathbf{S}$ that you should determine.

(b) The coordinates $x'^{\mu} = (ct', \mathbf{x}')$ in an inertial frame $\mathcal{S}'$ are related to the coordinates $x^{\mu} = (ct, \mathbf{x})$ in an inertial frame $\mathcal{S}$ by a Lorentz transformation $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$, where

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$. Here v is the relative velocity of $\mathcal{S}'$ with respect to $\mathcal{S}$ in the x-direction.

In frame $\mathcal{S}'$, there is a static electric field $\mathbf{E}'(\mathbf{x}')$ with $\partial\mathbf{E}'/\partial t' = 0$, and no magnetic field. Calculate the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ in frame $\mathcal{S}$. Show that the energy density in frame $\mathcal{S}$ is given in terms of the components of $\mathbf{E}'$ by

$$w = \frac{\epsilon_0}{2}\left[E_x'^{\,2} + \left(\frac{c^2+v^2}{c^2-v^2}\right)\left(E_y'^{\,2} + E_z'^{\,2}\right)\right].$$

Use the fact that $\partial w/\partial t' = 0$ to show that

$$\frac{\partial w}{\partial t} + \boldsymbol{\nabla}\cdot(wv\,\mathbf{e}_x) = 0\,,$$

where $\mathbf{e}_x$ is the unit vector in the x-direction.

Paper 2, Section I

5A Fluid Dynamics

Consider an axisymmetric container, initially filled with water to a depth h_I. A small circular hole of radius r_0 is opened in the base of the container at $z = 0$.

(a) Determine how the radius r of the container should vary with $z < h_I$ so that the depth of the water will decrease at a constant rate.

(b) For such a container, determine how the cross-sectional area A of the free surface should decrease with time.

[*You may assume that the flow rate through the opening is sufficiently small that Bernoulli's theorem for steady flows can be applied.*]

Paper 3, Section I

7A Fluid Dynamics

A two-dimensional flow $\mathbf{u} = (u, v)$ has a velocity field given by

$$u = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad \text{and} \quad v = \frac{2xy}{(x^2 + y^2)^2}.$$

(a) Show explicitly that this flow is incompressible and irrotational away from the origin.

(b) Find the stream function for this flow.

(c) Find the velocity potential for this flow.

Paper 1, Section II

16A Fluid Dynamics

A two-dimensional flow is given by a velocity potential

$$\phi(x, y, t) = \epsilon y \sin(x - t),$$

where ϵ is a constant.

(a) Find the corresponding velocity field $\mathbf{u}(x, y, t)$. Determine $\boldsymbol{\nabla} \cdot \mathbf{u}$.

(b) The time-average $\langle \psi \rangle(x, y)$ of a quantity $\psi(x, y, t)$ is defined as

$$\langle \psi \rangle(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \psi(x, y, t) dt.$$

Show that the time-average of this velocity field is zero everywhere. Write down an expression for the acceleration of fluid particles, and find the time-average of this expression at a fixed point (x, y).

(c) Now assume that $|\epsilon| \ll 1$. The material particle at $(0, 0)$ at $t = 0$ is marked with dye. Write down equations for its subsequent motion. Verify that its position (x, y) for $t > 0$ is given (correct to terms of order ϵ^2) by

$$\begin{aligned} x &= \epsilon^2 \left(\frac{1}{4} \sin 2t + \frac{t}{2} - \sin t \right), \\ y &= \epsilon(\cos t - 1). \end{aligned}$$

Deduce the time-average velocity of the dyed particle correct to this order.

Paper 3, Section II

16A Fluid Dynamics

A two-dimensional layer of viscous fluid lies between two rigid boundaries at $y = \pm L_0$. The boundary at $y = L_0$ oscillates in its own plane with velocity $(U_0 \cos \omega t, 0)$, while the boundary at $y = -L_0$ oscillates in its own plane with velocity $(-U_0 \cos \omega t, 0)$. Assume that there is no pressure gradient and that the fluid flows parallel to the boundary with velocity $(u(y,t), 0)$, where $u(y,t)$ can be written as $u(y,t) = \mathrm{Re}[U_0 f(y) \exp(i\omega t)]$.

(a) By exploiting the symmetry of the system or otherwise, show that

$$f(y) \quad = \quad \frac{\sinh[(1+i)\Delta \hat{y}]}{\sinh[(1+i)\Delta]}, \text{ where } \hat{y} = \frac{y}{L_0} \text{ and } \Delta = \left(\frac{\omega L_0^2}{2\nu}\right)^{1/2}.$$

(b) Hence or otherwise, show that

$$\frac{u(y,t)}{U_0} \quad = \quad \frac{\cos \omega t \, [\cosh \Delta_+ \cos \Delta_- - \cosh \Delta_- \cos \Delta_+]}{(\cosh 2\Delta - \cos 2\Delta)} + \frac{\sin \omega t \, [\sinh \Delta_+ \sin \Delta_- - \sinh \Delta_- \sin \Delta_+]}{(\cosh 2\Delta - \cos 2\Delta)},$$

where $\Delta_\pm = \Delta(1 \pm \hat{y})$.

(c) Show that, for $\Delta \ll 1$,

$$u(y,t) \simeq \frac{U_0 y}{L_0} \cos \omega t,$$

and briefly interpret this result physically.

Paper 4, Section II
16A Fluid Dynamics

Consider the spherically symmetric motion induced by the collapse of a spherical cavity of radius $a(t)$, centred on the origin. For $r < a$, there is a vacuum, while for $r > a$, there is an inviscid incompressible fluid with constant density ρ. At time $t = 0$, $a = a_0$, and the fluid is at rest and at constant pressure p_0.

(a) Consider the radial volume transport in the fluid $Q(R,t)$, defined as

$$Q(R,t) = \int_{r=R} u dS,$$

where u is the radial velocity, and dS is an infinitesimal element of the surface of a sphere of radius $R \geqslant a$. Use the incompressibility condition to establish that Q is a function of time alone.

(b) Using the expression for pressure in potential flow or otherwise, establish that

$$\frac{1}{4\pi a}\frac{dQ}{dt} - \frac{(\dot{a})^2}{2} = -\frac{p_0}{\rho},$$

where $\dot{a}(t)$ is the radial velocity of the cavity boundary.

(c) By expressing $Q(t)$ in terms of a and $\dot{a}$, show that

$$\dot{a} = -\sqrt{\frac{2p_0}{3\rho}\left(\frac{a_0^3}{a^3} - 1\right)}.$$

[*Hint: You may find it useful to assume $\dot{a}(t)$ is an explicit function of a from the outset.*]

(d) Hence write down an integral expression for the implosion time τ, i.e. the time for the radius of the cavity $a \to 0$. [Do not attempt to evaluate the integral.]

Paper 1, Section I
2F Geometry

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and let $\Sigma = f^{-1}(0)$ (assumed not empty). Show that if the differential $Df_p \neq 0$ for all $p \in \Sigma$, then Σ is a smooth surface in $\mathbb{R}^3$.

Is $\{(x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 = \cosh(z^2)\}$ a smooth surface? Is every surface $\Sigma \subset \mathbb{R}^3$ of the form $f^{-1}(0)$ for some smooth $f : \mathbb{R}^3 \to \mathbb{R}$? Justify your answers.

Paper 3, Section I
2E Geometry

State the local Gauss–Bonnet theorem for geodesic triangles on a surface. Deduce the Gauss–Bonnet theorem for closed surfaces. [Existence of a geodesic triangulation can be assumed.]

Let $S_r \subset \mathbb{R}^3$ denote the sphere with radius r centred at the origin. Show that the Gauss curvature of S_r is $1/r^2$. An octant is any of the eight regions in S_r bounded by arcs of great circles arising from the planes $x = 0$, $y = 0$, $z = 0$. Verify directly that the local Gauss–Bonnet theorem holds for an octant. [You may assume that the great circles on S_r are geodesics.]

Paper 1, Section II
11F Geometry

Let $S \subset \mathbb{R}^3$ be an oriented surface. Define the *Gauss map* N and show that the differential DN_p of the Gauss map at any point $p \in S$ is a self-adjoint linear map. Define the *Gauss curvature* κ and compute κ in a given parametrisation.

A point $p \in S$ is called umbilic if DN_p has a repeated eigenvalue. Let $S \subset \mathbb{R}^3$ be a surface such that every point is umbilic and there is a parametrisation $\phi : \mathbb{R}^2 \to S$ such that $S = \phi(\mathbb{R}^2)$. Prove that S is part of a plane or part of a sphere. [*Hint: consider the symmetry of the mixed partial derivatives* $n_{uv} = n_{vu}$, *where* $n(u,v) = N(\phi(u,v))$ *for* $(u,v) \in \mathbb{R}^2$.]

Paper 2, Section II
11E Geometry

Define $\mathbb{H}$, the *upper half plane model* for the hyperbolic plane, and show that $PSL_2(\mathbb{R})$ acts on $\mathbb{H}$ by isometries, and that these isometries preserve the orientation of $\mathbb{H}$.

Show that every orientation preserving isometry of $\mathbb{H}$ is in $PSL_2(\mathbb{R})$, and hence the full group of isometries of $\mathbb{H}$ is $G = PSL_2(\mathbb{R}) \ \cup \ PSL_2(\mathbb{R})\tau$, where $\tau z = -\bar{z}$.

Let ℓ be a hyperbolic line. Define the reflection σ_ℓ in ℓ. Now let ℓ, ℓ' be two hyperbolic lines which meet at a point $A \in \mathbb{H}$ at an angle θ. What are the possibilities for the group G generated by σ_ℓ and $\sigma_{\ell'}$? Carefully justify your answer.

Paper 3, Section II

12E Geometry

Let $S \subset \mathbb{R}^3$ be an embedded smooth surface and $\gamma : [0,1] \to S$ a parameterised smooth curve on S. What is the *energy* of γ? By applying the Euler–Lagrange equations for stationary curves to the energy function, determine the differential equations for geodesics on S explicitly in terms of a parameterisation of S.

If S contains a straight line ℓ, prove from first principles that each segment $[P,Q] \subset \ell$ (with some parameterisation) is a geodesic on S.

Let $H \subset \mathbb{R}^3$ be the hyperboloid defined by the equation $x^2 + y^2 - z^2 = 1$ and let $P = (x_0, y_0, z_0) \in H$. By considering appropriate isometries, or otherwise, display explicitly *three* distinct (as subsets of H) geodesics $\gamma : \mathbb{R} \to H$ through P in the case when $z_0 \neq 0$ and *four* distinct geodesics through P in the case when $z_0 = 0$. Justify your answer.

Let $\gamma : \mathbb{R} \to H$ be a geodesic, with coordinates $\gamma(t) = (x(t), y(t), z(t))$. Clairaut's relation asserts $\rho(t)\sin\psi(t)$ is constant, where $\rho(t) = \sqrt{x(t)^2 + y(t)^2}$ and $\psi(t)$ is the angle between $\dot{\gamma}(t)$ and the plane through the point $\gamma(t)$ and the z-axis. Deduce from Clairaut's relation that there exist infinitely many geodesics $\gamma(t)$ on H which stay in the half-space $\{z > 0\}$ for all $t \in \mathbb{R}$.

[*You may assume that if $\gamma(t)$ satisfies the geodesic equations on H then γ is defined for all $t \in \mathbb{R}$ and the Euclidean norm $\|\dot{\gamma}(t)\|$ is constant. If you use a version of the geodesic equations for a surface of revolution, then that should be proved.*]

Paper 4, Section II

11F Geometry

Define an *abstract smooth surface* and explain what it means for the surface to be *orientable*. Given two smooth surfaces S_1 and S_2 and a map $f : S_1 \to S_2$, explain what it means for f to be *smooth*.

For the cylinder

$$C = \{(x,y,z) \in \mathbb{R}^3 : \ x^2 + y^2 = 1\},$$

let $a : C \to C$ be the orientation reversing diffeomorphism $a(x,y,z) = (-x,-y,-z)$. Let S be the quotient of C by the equivalence relation $p \sim a(p)$ and let $\pi : C \to S$ be the canonical projection map. Show that S can be made into an abstract smooth surface so that π is smooth. Is S orientable? Justify your answer.

Paper 2, Section I
1G Groups, Rings and Modules

Let M be a module over a Principal Ideal Domain R and let N be a submodule of M. Show that M is finitely generated if and only if N and M/N are finitely generated.

Paper 3, Section I
1G Groups, Rings and Modules

Let G be a finite group, and let H be a proper subgroup of G of index n.

Show that there is a normal subgroup K of G such that $|G/K|$ divides $n!$ and $|G/K| \geqslant n$.

Show that if G is non-abelian and simple, then G is isomorphic to a subgroup of A_n.

Paper 1, Section II
9G Groups, Rings and Modules

Show that a ring R is Noetherian if and only if every ideal of R is finitely generated. Show that if $\phi\colon R \to S$ is a surjective ring homomorphism and R is Noetherian, then S is Noetherian.

State and prove Hilbert's Basis Theorem.

Let $\alpha \in \mathbb{C}$. Is $\mathbb{Z}[\alpha]$ Noetherian? Justify your answer.

Give, with proof, an example of a Unique Factorization Domain that is not Noetherian.

Let R be the ring of continuous functions $\mathbb{R} \to \mathbb{R}$. Is R Noetherian? Justify your answer.

Paper 2, Section II
9G Groups, Rings and Modules

Let M be a module over a ring R and let $S \subset M$. Define what it means that S *freely generates* M. Show that this happens if and only if for every R-module N, every function $f\colon S \to N$ extends uniquely to a homomorphism $\phi\colon M \to N$.

Let M be a free module over a (non-trivial) ring R that is generated (not necessarily freely) by a subset $T \subset M$ of size m. Show that if S is a basis of M, then S is finite with $|S| \leqslant m$. Hence, or otherwise, deduce that any two bases of M have the same number of elements. Denoting this number $\mathrm{rk}M$ and by quoting any result you need, show that if R is a Euclidean Domain and N is a submodule of M, then N is free with $\mathrm{rk}N \leqslant \mathrm{rk}M$.

State the Primary Decomposition Theorem for a finitely generated module M over a Euclidean Domain R. Deduce that any finite subgroup of the multiplicative group of a field is cyclic.

Paper 3, Section II
10G Groups, Rings and Modules

Let p be a non-zero element of a Principal Ideal Domain R. Show that the following are equivalent:

(i) p is prime;

(ii) p is irreducible;

(iii) (p) is a maximal ideal of R;

(iv) $R/(p)$ is a field;

(v) $R/(p)$ is an Integral Domain.

Let R be a Principal Ideal Domain, S an Integral Domain and $\phi\colon R \to S$ a surjective ring homomorphism. Show that either ϕ is an isomorphism or S is a field.

Show that if R is a commutative ring and $R[X]$ is a Principal Ideal Domain, then R is a field.

Let R be an Integral Domain in which every two non-zero elements have a highest common factor. Show that in R every irreducible element is prime.

Paper 4, Section II
9G Groups, Rings and Modules

Let H and P be subgroups of a finite group G. Show that the sets HxP, $x \in G$, partition G. By considering the action of H on the set of left cosets of P in G by left multiplication, or otherwise, show that

$$\frac{|HxP|}{|P|} = \frac{|H|}{|H \cap xPx^{-1}|}$$

for any $x \in G$. Deduce that if G has a Sylow p-subgroup, then so does H.

Let $p, n \in \mathbb{N}$ with p a prime. Write down the order of the group $GL_n(\mathbb{Z}/p\mathbb{Z})$. Identify in $GL_n(\mathbb{Z}/p\mathbb{Z})$ a Sylow p-subgroup and a subgroup isomorphic to the symmetric group S_n. Deduce that every finite group has a Sylow p-subgroup.

State Sylow's theorem on the number of Sylow p-subgroups of a finite group.

Let G be a group of order pq, where $p > q$ are prime numbers. Show that if G is non-abelian, then $q \,|\, p-1$.

Paper 1, Section I

1E Linear Algebra

Let V be a vector space over $\mathbb{R}$, $\dim V = n$, and let $\langle , \rangle$ be a non-degenerate anti-symmetric bilinear form on V.

Let $v \in V$, $v \neq 0$. Show that $v^\perp$ is of dimension $n-1$ and $v \in v^\perp$. Show that if $W \subseteq v^\perp$ is a subspace with $W \oplus \mathbb{R}v = v^\perp$, then the restriction of $\langle , \rangle$ to W is non-degenerate.

Conclude that the dimension of V is even.

Paper 4, Section I

1E Linear Algebra

Let $\mathrm{Mat}_n(\mathbb{C})$ be the vector space of n by n complex matrices.

Given $A \in \mathrm{Mat}_n(\mathbb{C})$, define the linear map $\varphi_A : \mathrm{Mat}_n(\mathbb{C}) \to \mathrm{Mat}_n(\mathbb{C})$,

$$X \mapsto AX - XA.$$

(i) Compute a basis of eigenvectors, and their associated eigenvalues, when A is the diagonal matrix

$$A = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{pmatrix}.$$

What is the rank of φ_A?

(ii) Now let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Write down the matrix of the linear transformation φ_A with respect to the standard basis of $\mathrm{Mat}_2(\mathbb{C})$.

What is its Jordan normal form?

Paper 1, Section II

8E Linear Algebra

Let $d \geqslant 1$, and let $J_d = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & \dots & \dots & \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_d(\mathbb{C})$.

(a) (i) Compute J_d^n, for all $n \geqslant 0$.

(ii) Hence, or otherwise, compute $(\lambda I + J_d)^n$, for all $n \geqslant 0$.

(b) Let V be a finite-dimensional vector space over $\mathbb{C}$, and let $\varphi \in \mathrm{End}(V)$. Suppose $\varphi^n = 0$ for some $n > 1$.

(i) Determine the possible eigenvalues of φ.

(ii) What are the possible Jordan blocks of φ?

(iii) Show that if $\varphi^2 = 0$, there exists a decomposition

$$V = U \oplus W_1 \oplus W_2,$$

where $\varphi(U) = \varphi(W_1) = 0$, $\varphi(W_2) = W_1$, and $\dim W_2 = \dim W_1$.

Paper 2, Section II

8E Linear Algebra

(a) Compute the characteristic polynomial and minimal polynomial of

$$A = \begin{pmatrix} -2 & -6 & -9 \\ 3 & 7 & 9 \\ -1 & -2 & -2 \end{pmatrix}.$$

Write down the Jordan normal form for A.

(b) Let V be a finite-dimensional vector space over $\mathbb{C}$, $f : V \to V$ be a linear map, and for $\alpha \in \mathbb{C}$, $n \geqslant 1$, write

$$W_{\alpha,n} := \{v \in V \mid (f - \alpha I)^n v = 0\}.$$

(i) Given $v \in W_{\alpha,n}$, $v \neq 0$, construct a non-zero eigenvector for f in terms of v.

(ii) Show that if $w_1, \dots, w_d$ are non-zero eigenvectors for f with eigenvalues $\alpha_1, \dots, \alpha_d$, and $\alpha_i \neq \alpha_j$ for all $i \neq j$, then $w_1, \dots, w_d$ are linearly independent.

(iii) Show that if $v_1 \in W_{\alpha_1,n}, \dots, v_d \in W_{\alpha_d,n}$ are all non-zero, and $\alpha_i \neq \alpha_j$ for all $i \neq j$, then $v_1, \dots, v_d$ are linearly independent.

Paper 3, Section II

9E Linear Algebra

(a)(i) State the rank-nullity theorem.

Let U and W be vector spaces. Write down the definition of their direct sum $U \oplus W$ and the inclusions $i : U \to U \oplus W$, $j : W \to U \oplus W$.

Now let U and W be subspaces of a vector space V. Define $l : U \cap W \to U \oplus W$ by $l(x) = ix - jx$.

Describe the quotient space $(U \oplus W)/\mathrm{Im}(l)$ as a subspace of V.

(ii) Let $V = \mathbb{R}^5$, and let U be the subspace of V spanned by the vectors

$$\begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 2 \\ 1 \\ -2 \end{pmatrix},$$

and W the subspace of V spanned by the vectors

$$\begin{pmatrix} 3 \\ 2 \\ -3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ -1 \\ -2 \\ 1 \end{pmatrix}.$$

Determine the dimension of $U \cap W$.

(b) Let A, B be complex n by n matrices with $\mathrm{rank}(B) = k$.

Show that $\det(A + tB)$ is a polynomial in t of degree at most k.

Show that if $k = n$ the polynomial is of degree precisely n.

Give an example where $k \geqslant 1$ but this polynomial is zero.

Paper 4, Section II

8E Linear Algebra

(a) Let V be a complex vector space of dimension n.

What is a *Hermitian form* on V?

Given a Hermitian form, define the matrix A of the form with respect to the basis $v_1, \ldots, v_n$ of V, and describe in terms of A the value of the Hermitian form on two elements of V.

Now let $w_1, \ldots, w_n$ be another basis of V. Suppose $w_i = \sum_j p_{ij} v_j$, and let $P = (p_{ij})$. Write down the matrix of the form with respect to this new basis in terms of A and P.

Let $N = V^{\perp}$. Describe the dimension of N in terms of the matrix A.

(b) Write down the matrix of the real quadratic form

$$x^2 + y^2 + 2z^2 + 2xy + 2xz - 2yz.$$

Using the Gram–Schmidt algorithm, find a basis which diagonalises the form. What are its rank and signature?

(c) Let V be a real vector space, and $\langle , \rangle$ a symmetric bilinear form on it. Let A be the matrix of this form in some basis.

Prove that the signature of $\langle , \rangle$ is the number of positive eigenvalues of A minus the number of negative eigenvalues.

Explain, using an example, why the eigenvalues themselves depend on the choice of a basis.

Paper 3, Section I
8H Markov Chains

Consider a Markov chain $(X_n)_{n\geqslant 0}$ on a state space I.

(a) Define the notion of a *communicating class*. What does it mean for a communicating class to be *closed*?

(b) Taking $I = \{1, \ldots, 6\}$, find the communicating classes associated with the transition matrix P given by

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and identify which are closed.

(c) Find the expected time for the Markov chain with transition matrix P above to reach 6 starting from 1.

Paper 4, Section I
7H Markov Chains

Show that the simple symmetric random walk on $\mathbb{Z}$ is recurrent.

Three particles perform independent simple symmetric random walks on $\mathbb{Z}$. What is the probability that they are all simultaneously at 0 infinitely often? Justify your answer.

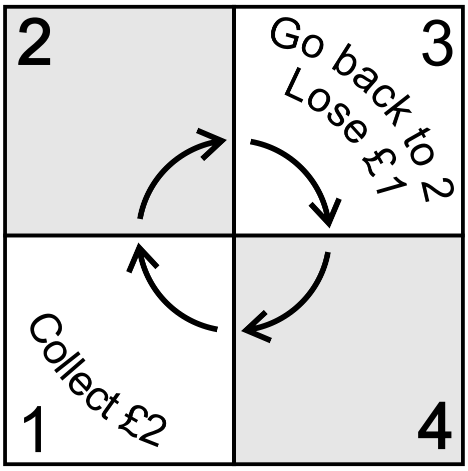
[*You may assume without proof that there exist constants* $A, B > 0$ *such that* $A\sqrt{n}(n/e)^n \leqslant n! \leqslant B\sqrt{n}(n/e)^n$ *for all positive integers* n.]

Paper 1, Section II

19H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain with transition matrix P. What is a *stopping time* of $(X_n)_{n\geqslant 0}$? What is the *strong Markov property*?

The exciting game of 'Unopoly' is played by a single player on a board of 4 squares. The player starts with £m (where $m \in \mathbb{N}$). During each turn, the player tosses a fair coin and moves one or two places in a clockwise direction ($1 \to 2 \to 3 \to 4 \to 1$) according to whether the coin lands heads or tails respectively. The player collects £2 each time they pass (or land on) square 1. If the player lands on square 3 however, they immediately lose £1 and go back to square 2. The game continues indefinitely unless the player is on square 2 with £0, in which case the player loses the game and the game ends.



(a) By setting up an appropriate Markov chain, show that if the player is at square 2 with £m, where $m \geqslant 1$, the probability that they are ever at square 2 with £$(m-1)$ is $2/3$.

(b) Find the probability of losing the game when the player starts on square 1 with £m, where $m \geqslant 1$.

[*Hint: Take the state space of your Markov chain to be* $\{1,2,4\} \times \{0,1,\ldots\}$.]

Paper 2, Section II
18H Markov Chains

Let P be a transition matrix on state space I. What does it mean for a distribution π to be an *invariant distribution*? What does it mean for π and P to be in *detailed balance*? Show that if π and P are in detailed balance, then π is an invariant distribution.

(a) Assuming that an invariant distribution exists, state the relationship between this and

(i) the expected return time to a state i;

(ii) the expected time spent in a state i between visits to a state k.

(b) Let $(X_n)_{n\geqslant 0}$ be a Markov chain with transition matrix $P=(p_{ij})_{i,j\in I}$ where $I=\{0,1,2,\ldots\}$. The transition probabilities are given for $i\geqslant 1$ by

$$p_{ij}=\begin{cases} q^{-(i+2)} & \text{if } j=i+1,\\ q^{-i} & \text{if } j=i-1,\\ 1-q^{-(i+2)}-q^{-i} & \text{if } j=i,\end{cases}$$

where $q\geqslant 2$. For $p\in(0,1]$ let $p_{01}=p=1-p_{00}$. Compute the following, justifying your answers:

(i) The expected time spent in states $\{2,4,6,\ldots\}$ between visits to state 1;

(ii) The expected time taken to return to state 1, starting from 1;

(iii) The expected time taken to hit state 0 starting from 1.

Paper 2, Section I

3C Methods

Consider the differential operator

$$\mathcal{L}y = \frac{d^2y}{dx^2} + 2\frac{dy}{dx}$$

acting on real functions $y(x)$ with $0 \leqslant x \leqslant 1$.

(i) Recast the eigenvalue equation $\mathcal{L}y = -\lambda y$ in Sturm-Liouville form $\tilde{\mathcal{L}}y = -\lambda w y$, identifying $\tilde{\mathcal{L}}$ and w.

(ii) If boundary conditions $y(0) = y(1) = 0$ are imposed, show that the eigenvalues form an infinite discrete set $\lambda_1 < \lambda_2 < \ldots$ and find the corresponding eigenfunctions $y_n(x)$ for $n = 1, 2, \ldots$. If $f(x) = x - x^2$ on $0 \leqslant x \leqslant 1$ is expanded in terms of your eigenfunctions i.e. $f(x) = \sum_{n=1}^{\infty} A_n\, y_n(x)$, give an expression for A_n. The expression can be given in terms of integrals that you need not evaluate.

Paper 3, Section I

5A Methods

Let $f(\theta)$ be a 2π-periodic function with Fourier expansion

$$f(\theta) \;=\; \frac{1}{2}a_0 \;+\; \sum_{n=1}^{\infty} \left(a_n \cos n\theta \;+\; b_n \sin n\theta \right) \,.$$

Find the Fourier coefficients a_n and b_n for

$$f(\theta) \;=\; \begin{cases} \;\;1\,, & 0 < \theta < \pi \\ -1\,, & \pi < \theta < 2\pi\,. \end{cases}$$

Hence, or otherwise, find the Fourier coefficients A_n and B_n for the 2π-periodic function F defined by

$$F(\theta) \;=\; \begin{cases} \theta\,, & 0 < \theta < \pi \\ 2\pi - \theta\,, & \pi < \theta < 2\pi\,. \end{cases}$$

Use your answers to evaluate

$$\sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \qquad \text{and} \qquad \sum_{r=0}^{\infty} \frac{1}{(2r+1)^2}\,.$$

Paper 1, Section II
13C Methods

(a) By introducing the variables $\xi = x + ct$ and $\eta = x - ct$ (where c is a constant), derive d'Alembert's solution of the initial value problem for the wave equation:

$$u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$$

where $-\infty < x < \infty$, $t \geqslant 0$ and ϕ and ψ are given functions (and subscripts denote partial derivatives).

(b) Consider the forced wave equation with homogeneous initial conditions:

$$u_{tt} - c^2 u_{xx} = f(x,t), \quad u(x,0) = 0, \quad u_t(x,0) = 0$$

where $-\infty < x < \infty$, $t \geqslant 0$ and f is a given function. You may assume that the solution is given by

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy\, ds.$$

For the forced wave equation $u_{tt} - c^2 u_{xx} = f(x,t)$, now in the half space $x \geqslant 0$ (and with $t \geqslant 0$ as before), find (in terms of f) the solution for $u(x,t)$ that satisfies the (inhomogeneous) initial conditions

$$u(x,0) = \sin x, \quad u_t(x,0) = 0, \qquad \text{for } x \geqslant 0$$

and the boundary condition $u(0,t) = 0$ for $t \geqslant 0$.

Paper 2, Section II

14A Methods

The Fourier transform $\tilde{f}(k)$ of a function $f(x)$ and its inverse are given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx, \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}dk.$$

(a) Calculate the Fourier transform of the function $f(x)$ defined by:

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < 1, \\ -1 & \text{for } -1 < x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Show that the inverse Fourier transform of $\tilde{g}(k) = e^{-\lambda|k|}$, for λ a positive real constant, is given by

$$g(x) = \frac{\lambda}{\pi(x^2 + \lambda^2)}.$$

(c) Consider the problem in the quarter plane $0 \leqslant x$, $0 \leqslant y$:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0; \\ u(x,0) &= \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise;} \end{cases} \\ u(0,y) = \lim_{x\to\infty} u(x,y) = \lim_{y\to\infty} u(x,y) &= 0. \end{aligned}$$

Use the answers from parts (a) and (b) to show that

$$u(x,y) = \frac{4xy}{\pi}\int_0^1 \frac{v\,dv}{[(x-v)^2 + y^2][(x+v)^2 + y^2]}.$$

(d) Hence solve the problem in the quarter plane $0 \leqslant x$, $0 \leqslant y$:

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= 0; \\ w(x,0) &= \begin{cases} 1 & \text{for } 0 < x < 1, \\ 0 & \text{otherwise;} \end{cases} \\ w(0,y) &= \begin{cases} 1 & \text{for } 0 < y < 1, \\ 0 & \text{otherwise;} \end{cases} \\ \lim_{x\to\infty} w(x,y) = \lim_{y\to\infty} w(x,y) &= 0. \end{aligned}$$

[*You may quote without proof any property of Fourier transforms.*]

Paper 3, Section II
14A Methods

Let $P(x)$ be a solution of Legendre's equation with eigenvalue λ,

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \lambda P = 0\,,$$

such that P and its derivatives $P^{(k)}(x) = d^kP/dx^k$, $k = 0, 1, 2, \ldots$, are regular at all points x with $-1 \leqslant x \leqslant 1$.

(a) Show by induction that

$$(1-x^2)\frac{d^2}{dx^2}\left[P^{(k)}\right] - 2(k+1)x\frac{d}{dx}\left[P^{(k)}\right] + \lambda_k P^{(k)} = 0$$

for some constant λ_k. Find λ_k explicitly and show that its value is negative when k is sufficiently large, for a fixed value of λ.

(b) Write the equation for $P^{(k)}(x)$ in part (a) in self-adjoint form. Hence deduce that if $P^{(k)}(x)$ is not identically zero, then $\lambda_k \geqslant 0$.

[*Hint: Establish a relation between integrals of the form* $\int_{-1}^{1}[P^{(k+1)}(x)]^2 f(x)\,dx$ *and* $\int_{-1}^{1}[P^{(k)}(x)]^2 g(x)\,dx$ *for certain functions* $f(x)$ *and* $g(x)$.]

(c) Use the results of parts (a) and (b) to show that if $P(x)$ is a non-zero, regular solution of Legendre's equation on $-1 \leqslant x \leqslant 1$, then $P(x)$ is a polynomial of degree n and $\lambda = n(n+1)$ for some integer $n = 0, 1, 2, \ldots$.

Paper 4, Section II

14C Methods

The function $\theta(x,t)$ obeys the diffusion equation

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 \theta}{\partial x^2}\,. \qquad (*)$$

Verify that

$$\theta(x,t) = \frac{1}{\sqrt{t}}\, e^{-x^2/4Dt}$$

is a solution of $(*)$, and by considering $\int_{-\infty}^{\infty}\theta(x,t)\,dx$, find the solution having the initial form $\theta(x,0)=\delta(x)$ at $t=0$.

Find, in terms of the error function, the solution of $(*)$ having the initial form

$$\theta(x,0) = \begin{cases} 1\,, & |x| \leqslant 1\,, \\ 0\,, & |x| > 1\,. \end{cases}$$

Sketch a graph of this solution at various times $t \geqslant 0$.

[*The error function is*

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy\,.]$$

Paper 1, Section I
5B Numerical Analysis

Prove, from first principles, that there is an algorithm that can determine whether any real symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite or not, with the computational cost (number of arithmetic operations) bounded by $\mathcal{O}(n^3)$.

[*Hint: Consider the LDL decomposition.*]

Paper 4, Section I
6B Numerical Analysis

(a) Given the data $f(0) = 0$, $f(1) = 4$, $f(2) = 2$, $f(3) = 8$, find the interpolating cubic polynomial $p_3 \in \mathbb{P}_3[x]$ in the Newton form.

(b) We add to the data one more value, $f(-2) = 10$. Find the interpolating quartic polynomial $p_4 \in \mathbb{P}_4[x]$ for the extended data in the Newton form.

Paper 1, Section II

17B Numerical Analysis

For the ordinary differential equation

$$\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}), \quad \boldsymbol{y}(0) = \tilde{\boldsymbol{y}}_0, \quad t \geqslant 0, \tag{$*$}$$

where $\boldsymbol{y}(t) \in \mathbb{R}^N$ and the function $\boldsymbol{f} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is analytic, consider an explicit one-step method described as the mapping

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{\varphi}(t_n, \boldsymbol{y}_n, h). \tag{$\dagger$}$$

Here $\boldsymbol{\varphi} : \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}^N$, $n = 0, 1, \ldots$ and $t_n = nh$ with time step $h > 0$, producing numerical approximations $\boldsymbol{y}_n$ to the exact solution $\boldsymbol{y}(t_n)$ of equation $(*)$, with $\boldsymbol{y}_0$ being the initial value of the numerical solution.

(i) Define *the local error* of a one-step method.

(ii) Let $\|\cdot\|$ be a norm on $\mathbb{R}^N$ and suppose that

$$\|\boldsymbol{\varphi}(t, \boldsymbol{u}, h) - \boldsymbol{\varphi}(t, \boldsymbol{v}, h)\| \leqslant L\|\boldsymbol{u} - \boldsymbol{v}\|,$$

for all $h > 0$, $t \in \mathbb{R}$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^N$, where L is some positive constant. Let $t^* > 0$ be given and $\boldsymbol{e}_0 = \boldsymbol{y}_0 - \boldsymbol{y}(0)$ denote the initial error (potentially non-zero). Show that if the local error of the one-step method $(\dagger)$ is $\mathcal{O}(h^{p+1})$, then

$$\max_{n=0,\ldots,\lfloor t^*/h \rfloor} \|\boldsymbol{y}_n - \boldsymbol{y}(nh)\| \leqslant e^{t^* L}\|\boldsymbol{e}_0\| + \mathcal{O}(h^p), \quad h \to 0. \tag{$\dagger\dagger$}$$

(iii) Let $N = 1$ and consider equation $(*)$ where f is time-independent satisfying $|f(u) - f(v)| \leqslant K|u - v|$ for all $u, v \in \mathbb{R}$, where K is a positive constant. Consider the one-step method given by

$$y_{n+1} = y_n + \frac{1}{4}h(k_1 + 3k_2), \qquad k_1 = f(y_n), \quad k_2 = f(y_n + \frac{2}{3}hk_1).$$

Use part (ii) to show that for this method we have that equation $(\dagger\dagger)$ holds (with a potentially different constant L) for $p = 2$.

Paper 2, Section II
17B Numerical Analysis

(a) Define *Householder reflections* and show that a real Householder reflection is symmetric and orthogonal. Moreover, show that if $H, A \in \mathbb{R}^{n\times n}$, where H is a Householder reflection and A is a full matrix, then the computational cost (number of arithmetic operations) of computing HAH^{-1} can be $\mathcal{O}(n^2)$ operations, as opposed to $\mathcal{O}(n^3)$ for standard matrix products.

(b) Show that for any $A \in \mathbb{R}^{n\times n}$ there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that

$$QAQ^T = T = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & \cdots & t_{1,n} \\ t_{2,1} & t_{2,2} & t_{2,3} & \cdots & t_{2,n} \\ 0 & t_{3,2} & t_{3,3} & \cdots & t_{3,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{n,n-1} & t_{n,n} \end{bmatrix}.$$

In particular, T has zero entries below the first subdiagonal. Show that one can compute such a T and Q (they may not be unique) using $\mathcal{O}(n^3)$ arithmetic operations.

[*Hint: Multiply A from the left and right with Householder reflections.*]

Paper 3, Section II
17B Numerical Analysis
The functions $p_0, p_1, p_2, \ldots$ are generated by the formula

$$p_n(x) = (-1)^n x^{-1/2} e^x \frac{d^n}{dx^n}\left(x^{n+1/2} e^{-x}\right), \qquad 0 \leqslant x < \infty\,.$$

(a) Show that $p_n(x)$ is a monic polynomial of degree n. Write down the explicit forms of $p_0(x)$, $p_1(x)$, $p_2(x)$.

(b) Demonstrate the orthogonality of these polynomials with respect to the scalar product

$$\langle f, g\rangle = \int_0^\infty x^{1/2} e^{-x} f(x) g(x)\, dx\,,$$

i.e. that $\langle p_n, p_m\rangle = 0$ for $m \neq n$, and show that

$$\langle p_n, p_n\rangle = n!\,\Gamma\left(n + \frac{3}{2}\right)\,,$$

where $\Gamma(y) = \int_0^\infty x^{y-1} e^{-x}\, dx$.

(c) Assuming that a three-term recurrence relation in the form

$$p_{n+1}(x) = (x - \alpha_n) p_n(x) - \beta_n p_{n-1}(x), \quad n = 1, 2, \ldots\,,$$

holds, find the explicit expressions for α_n and β_n as functions of n.

[*Hint: you may use the fact that* $\Gamma(y+1) = y\Gamma(y)$.]

Paper 1, Section I

7H Optimisation

(a) Let $f_i : \mathbb{R}^d \to \mathbb{R}$ be a convex function for each $i = 1, \ldots, m$. Show that

$$x \mapsto \max_{i=1,\ldots,m} f_i(x) \qquad \text{and} \qquad x \mapsto \sum_{i=1}^{m} f_i(x)$$

are both convex functions.

(b) Fix $c \in \mathbb{R}^d$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is convex, then $g : \mathbb{R}^d \to \mathbb{R}$ given by $g(x) = f(c^T x)$ is convex.

(c) Fix vectors $a_1, \ldots, a_n \in \mathbb{R}^d$. Let $Q : \mathbb{R}^d \to \mathbb{R}$ be given by

$$Q(\beta) = \sum_{i=1}^{n} \log(1 + e^{a_i^T \beta}) + \sum_{j=1}^{d} |\beta_j|.$$

Show that Q is convex. [You may use any result from the course provided you state it.]

Paper 2, Section I

7H Optimisation

Find the solution to the following optimisation problem using the simplex algorithm:

$$\begin{aligned} \text{maximise} \quad & 3x_1 + 6x_2 + 4x_3 \\ \text{subject to} \quad & 2x_1 + 3x_2 + x_3 \leqslant 7, \\ & 4x_1 + 2x_2 + 2x_3 \leqslant 5, \\ & x_1 + x_2 + 2x_3 \leqslant 2, \qquad x_1, x_2, x_3 \geqslant 0. \end{aligned}$$

Write down the dual problem and give its solution.

Paper 3, Section II

19H Optimisation

Explain what is meant by a *two-person zero-sum game* with $m \times n$ payoff matrix A, and define what is meant by an *optimal strategy* for each player. What are the relationships between the optimal strategies and the value of the game?

Suppose now that

$$A = \begin{pmatrix} 0 & 1 & 1 & -4 \\ -1 & 0 & 2 & 2 \\ -1 & -2 & 0 & 3 \\ 4 & -2 & -3 & 0 \end{pmatrix}.$$

Show that if strategy $p = (p_1, p_2, p_3, p_4)^T$ is optimal for player I, it must also be optimal for player II. What is the value of the game in this case? Justify your answer.

Explain why we must have $(Ap)_i \leqslant 0$ for all i. Hence or otherwise, find the optimal strategy p and prove that it is unique.

Paper 4, Section II
18H Optimisation

(a) Consider the linear program

$$P: \qquad \text{maximise over } x \geqslant 0, \quad c^T x \quad \text{subject to} \quad Ax = b,$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. What is meant by a *basic feasible solution*?

(b) Prove that if P has a finite maximum, then there exists a solution that is a basic feasible solution.

(c) Now consider the optimisation problem

$$Q: \qquad \text{maximise over } x \geqslant 0, \quad \frac{c^T x}{d^T x} \quad \text{subject to} \quad Ax = b, \quad d^T x > 0,$$

where matrix A and vectors c, b are as in the problem P, and $d \in \mathbb{R}^n$. Suppose there exists a solution x^* to Q. Further consider the linear program

$$R: \qquad \text{maximise over } y \geqslant 0,\ t \geqslant 0, \quad c^T y \quad \text{subject to} \quad Ay = bt, \quad d^T y = 1.$$

(i) Suppose $d_i > 0$ for all $i = 1, \ldots, n$. Show that the maximum of R is finite and at least as large as that of Q.

(ii) Suppose, in addition to the condition in part (i), that the entries of A are strictly positive. Show that the maximum of R is equal to that of Q.

(iii) Let $\mathcal{B}$ be the set of basic feasible solutions of the linear program P. Assuming the conditions in parts (i) and (ii) above, show that

$$\frac{c^T x^*}{d^T x^*} = \max_{x \in \mathcal{B}} \frac{c^T x}{d^T x}.$$

[*Hint: Argue that if* (y, t) *is in the set* $\mathcal{A}$ *of basic feasible solutions to* R*, then* $y/t \in \mathcal{B}$.]

Paper 3, Section I

6C Quantum Mechanics

The electron in a hydrogen-like atom moves in a spherically symmetric potential $V(r) = -K/r$ where K is a positive constant and r is the radial coordinate of spherical polar coordinates. The two lowest energy spherically symmetric normalised states of the electron are given by

$$\chi_1(r) = \frac{1}{\sqrt{\pi}\, a^{3/2}}\, e^{-r/a} \quad \text{and} \quad \chi_2(r) = \frac{1}{4\sqrt{2\pi}\, a^{3/2}} \left(2 - \frac{r}{a}\right) e^{-r/2a}$$

where $a = \hbar^2/mK$ and m is the mass of the electron. For any spherically symmetric function $f(r)$, the Laplacian is given by $\nabla^2 f = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$.

(i) Suppose that the electron is in the state $\chi(r) = \frac{1}{2}\chi_1(r) + \frac{\sqrt{3}}{2}\chi_2(r)$ and its energy is measured. Find the expectation value of the result.

(ii) Suppose now that the electron is in state $\chi(r)$ (as above) at time $t = 0$. Let $R(t)$ be the expectation value of a measurement of the electron's radial position r at time t. Show that the value of $R(t)$ oscillates sinusoidally about a constant level and determine the frequency of the oscillation.

Paper 4, Section I

4C Quantum Mechanics

Let $\Psi(x,t)$ be the wavefunction for a particle of mass m moving in one dimension in a potential $U(x)$. Show that, with suitable boundary conditions as $x \to \pm\infty$,

$$\frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx \;=\; 0\,.$$

Why is this important for the interpretation of quantum mechanics?

Verify the result above by first calculating $|\Psi(x,t)|^2$ for the free particle solution

$$\Psi(x,t) \;=\; C f(t)^{1/2} \exp\Big(-\frac{1}{2} f(t)\, x^2\Big) \quad \text{with} \quad f(t) = \Big(\alpha + \frac{i\hbar}{m} t\Big)^{-1},$$

where C and $\alpha > 0$ are real constants, and then considering the resulting integral.

Paper 1, Section II

14C Quantum Mechanics

Consider a quantum mechanical particle of mass m in a one-dimensional stepped potential well $U(x)$ given by:

$$U(x) = \begin{cases} \infty & \text{for } x < 0 \text{ and } x > a \\ 0 & \text{for } 0 \leqslant x \leqslant a/2 \\ U_0 & \text{for } a/2 < x \leqslant a \end{cases}$$

where $a > 0$ and $U_0 \geqslant 0$ are constants.

(i) Show that all energy levels E of the particle are non-negative. Show that any level E with $0 < E < U_0$ satisfies

$$\frac{1}{k} \tan \frac{ka}{2} = -\frac{1}{l} \tanh \frac{la}{2}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}} > 0 \quad \text{and} \quad l = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}} > 0.$$

(ii) Suppose that initially $U_0 = 0$ and the particle is in the ground state of the potential well. U_0 is then changed to a value $U_0 > 0$ (while the particle's wavefunction stays the same) and the energy of the particle is measured. For $0 < E < U_0$, give an expression in terms of E for prob(E), the probability that the energy measurement will find the particle having energy E. The expression may be left in terms of integrals that you need not evaluate.

Paper 2, Section II

15C Quantum Mechanics

(a) Write down the expressions for the probability density ρ and associated current density j of a quantum particle in one dimension with wavefunction $\psi(x,t)$. Show that if ψ is a stationary state then the function j is constant.

For the non-normalisable free particle wavefunction $\psi(x,t) = Ae^{ikx-iEt/\hbar}$ (where E and k are real constants and A is a complex constant) compute the functions ρ and j, and briefly give a physical interpretation of the functions ψ, ρ and j in this case.

(b) A quantum particle of mass m and energy $E > 0$ moving in one dimension is incident from the left in the potential $V(x)$ given by

$$V(x) = \begin{cases} -V_0 & 0 \leqslant x \leqslant a \\ 0 & x < 0 \text{ or } x > a \end{cases}$$

where a and V_0 are positive constants. Write down the form of the wavefunction in the regions $x < 0$, $0 \leqslant x \leqslant a$ and $x > a$.

Suppose now that $V_0 = 3E$. Show that the probability T of transmission of the particle into the region $x > a$ is given by

$$T = \frac{16}{16 + 9\sin^2\left(\frac{a\sqrt{8mE}}{\hbar}\right)}.$$

Paper 4, Section II
15C Quantum Mechanics

(a) Consider the angular momentum operators $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$ and $\hat{\mathbf{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$ where

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \quad \hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \text{ and } \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z.$$

Use the standard commutation relations for these operators to show that

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y \qquad \text{obeys} \qquad [\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm \quad \text{and} \quad [\hat{\mathbf{L}}^2, \hat{L}_\pm] = 0\,.$$

Deduce that if φ is a joint eigenstate of $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ with angular momentum quantum numbers m and ℓ respectively, then $\hat{L}_\pm\varphi$ are also joint eigenstates, provided they are non-zero, with quantum numbers $m \pm 1$ and ℓ.

(b) A harmonic oscillator of mass M in three dimensions has Hamiltonian

$$\hat{H} \;=\; \frac{1}{2M}(\,\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\,) \;+\; \frac{1}{2}\,M\omega^2(\,\hat{x}^2 + \hat{y}^2 + \hat{z}^2\,)\,.$$

Find eigenstates of $\hat{H}$ in terms of eigenstates ψ_n for an oscillator in one dimension with $n = 0, 1, 2, \ldots$ and eigenvalues $\hbar\omega(n + \frac{1}{2})$; hence determine the eigenvalues E of $\hat{H}$.

Verify that the ground state for $\hat{H}$ is a joint eigenstate of $\hat{L}_z$ and $\hat{\mathbf{L}}^2$ with $\ell = m = 0$. At the first excited energy level, find an eigenstate of $\hat{L}_z$ with $m = 0$ and construct from this two eigenstates of $\hat{L}_z$ with $m = \pm 1$.

Why should you expect to find joint eigenstates of $\hat{L}_z$, $\hat{\mathbf{L}}^2$ and $\hat{H}$?

[*The first two eigenstates for an oscillator in one dimension are* $\psi_0(x) = C_0 \exp(-M\omega x^2/2\hbar)$ *and* $\psi_1(x) = C_1 x \exp(-M\omega x^2/2\hbar)$, *where* C_0 *and* C_1 *are normalisation constants.*]

Paper 1, Section I
6H Statistics

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(p) random variables, where $n \geqslant 3$ and $p \in (0,1)$ is unknown.

(a) What does it mean for a statistic T to be *sufficient* for p? Find such a sufficient statistic T.

(b) State and prove the Rao–Blackwell theorem.

(c) By considering the estimator $X_1 X_2$ of p^2, find an unbiased estimator of p^2 that is a function of the statistic T found in part (a), and has variance strictly smaller than that of $X_1 X_2$.

Paper 2, Section I
6H Statistics

The efficacy of a new drug was tested as follows. Fifty patients were given the drug, and another fifty patients were given a placebo. A week later, the numbers of patients whose symptoms had gone entirely, improved, stayed the same and got worse were recorded, as summarised in the following table.

	Drug	Placebo
symptoms gone	14	6
improved	21	19
same	10	10
worse	5	15

Conduct a 5% significance level test of the null hypothesis that the medicine and placebo have the same effect, against the alternative that their effects differ.

[*Hint: You may find some of the following values relevant:*

Distribution	χ^2_1	χ^2_2	χ^2_3	χ^2_4	χ^2_6	χ^2_8
95th percentile	3.84	5.99	7.81	9.48	12.59	15.51

]

Paper 1, Section II
18H Statistics

(a) Show that if $W_1, \ldots, W_n$ are independent random variables with common Exp(1) distribution, then $\sum_{i=1}^n W_i \sim \Gamma(n,1)$. [*Hint: If* $W \sim \Gamma(\alpha, \lambda)$ *then* $\mathbb{E}e^{tW} = \{\lambda/(\lambda - t)\}^\alpha$ *if* $t < \lambda$ *and* ∞ *otherwise.*]

(b) Show that if $X \sim U(0,1)$ then $-\log X \sim \text{Exp}(1)$.

(c) State the Neyman–Pearson lemma.

(d) Let $X_1, \ldots, X_n$ be independent random variables with common density proportional to $x^\theta \mathbf{1}_{(0,1)}(x)$ for $\theta \geqslant 0$. Find a most powerful test of size α of $H_0 : \theta = 0$ against $H_1 : \theta = 1$, giving the critical region in terms of a quantile of an appropriate gamma distribution. Find a uniformly most powerful test of size α of $H_0 : \theta = 0$ against $H_1 : \theta > 0$.

Paper 3, Section II
18H Statistics

Consider the normal linear model $Y = X\beta + \varepsilon$ where X is a known $n \times p$ design matrix with $n-2 > p \geqslant 1$, $\beta \in \mathbb{R}^p$ is an unknown vector of parameters, and $\varepsilon \sim N_n(0, \sigma^2 I)$ is a vector of normal errors with each component having variance $\sigma^2 > 0$. Suppose X has full column rank.

(i) Write down the maximum likelihood estimators, $\hat{\beta}$ and $\hat{\sigma}^2$, for β and σ^2 respectively. [You need not derive these.]

(ii) Show that $\hat{\beta}$ is independent of $\hat{\sigma}^2$.

(iii) Find the distributions of $\hat{\beta}$ and $n\hat{\sigma}^2/\sigma^2$.

(iv) Consider the following test statistic for testing the null hypothesis $H_0 : \ \beta = 0$ against the alternative $\beta \neq 0$:

$$T := \frac{\|\hat{\beta}\|^2}{n\hat{\sigma}^2}.$$

Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_p > 0$ be the eigenvalues of $X^T X$. Show that under H_0, T has the same distribution as

$$\frac{\sum_{j=1}^p \lambda_j^{-1} W_j}{Z}$$

where $Z \sim \chi^2_{n-p}$ and $W_1, \ldots, W_p$ are independent χ^2_1 random variables, independent of Z.

[*Hint: You may use the fact that* $X = UDV^T$ *where* $U \in \mathbb{R}^{n \times p}$ *has orthonormal columns,* $V \in \mathbb{R}^{p \times p}$ *is an orthogonal matrix and* $D \in \mathbb{R}^{p \times p}$ *is a diagonal matrix with* $D_{ii} = \sqrt{\lambda_i}$.]

(v) Find $\mathbb{E}T$ when $\beta \neq 0$. [*Hint: If* $R \sim \chi^2_\nu$ *with* $\nu > 2$, *then* $\mathbb{E}(1/R) = 1/(\nu - 2)$.]

Paper 4, Section II

17H Statistics

Suppose we wish to estimate the probability $\theta \in (0,1)$ that a potentially biased coin lands heads up when tossed. After n independent tosses, we observe X heads.

(a) Write down the maximum likelihood estimator $\hat{\theta}$ of θ.

(b) Find the mean squared error $f(\theta)$ of $\hat{\theta}$ as a function of θ. Compute $\sup_{\theta \in (0,1)} f(\theta)$.

(c) Suppose a uniform prior is placed on θ. Find the Bayes estimator of θ under squared error loss $L(\theta, a) = (\theta - a)^2$.

(d) Now find the Bayes estimator $\tilde{\theta}$ under the loss $L(\theta, a) = \theta^{\alpha-1}(1-\theta)^{\beta-1}(\theta - a)^2$, where $\alpha, \beta \geqslant 1$. Show that

$$\tilde{\theta} = w\hat{\theta} + (1-w)\theta_0, \qquad (*)$$

where w and θ_0 depend on n, α and β.

(e) Determine the mean squared error $g_{w,\theta_0}(\theta)$ of $\tilde{\theta}$ as defined by (*).

(f) For what range of values of w do we have $\sup_{\theta \in (0,1)} g_{w,1/2}(\theta) \leqslant \sup_{\theta \in (0,1)} f(\theta)$?

[*Hint: The mean of a Beta*(a,b) *distribution is* $a/(a+b)$ *and its density* $p(u)$ *at* $u \in [0,1]$ *is* $c_{a,b}u^{a-1}(1-u)^{b-1}$, *where* $c_{a,b}$ *is a normalising constant.*]

Paper 1, Section I
4D Variational Principles

Let D be a bounded region of $\mathbb{R}^2$, with boundary ∂D. Let $u(x,y)$ be a smooth function defined on D, subject to the boundary condition that $u = 0$ on ∂D and the normalization condition that

$$\int_D u^2 \, dx \, dy = 1 \, .$$

Let $I[u]$ be the functional

$$I[u] = \int_D |\boldsymbol{\nabla} u|^2 \, dx \, dy \, .$$

Show that $I[u]$ has a stationary value, subject to the stated boundary and normalization conditions, when u satisfies a partial differential equation of the form

$$\nabla^2 u + \lambda u = 0$$

in D, where λ is a constant.

Determine how λ is related to the stationary value of the functional $I[u]$. [*Hint: Consider* $\boldsymbol{\nabla} \cdot (u \boldsymbol{\nabla} u)$.]

Paper 3, Section I
4D Variational Principles

Find the function $y(x)$ that gives a stationary value of the functional

$$I[y] = \int_0^1 \left(y'^2 + yy' + y' + y^2 + yx^2 \right) dx \, ,$$

subject to the boundary conditions $y(0) = -1$ and $y(1) = e - e^{-1} - \frac{3}{2}$.

Paper 2, Section II

13D Variational Principles

A particle of unit mass moves in a smooth one-dimensional potential $V(x)$. Its path $x(t)$ is such that the action integral

$$S[x] = \int_a^b L(x, \dot{x})\, dt$$

has a stationary value, where a and $b > a$ are constants, a dot denotes differentiation with respect to time t,

$$L(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - V(x)$$

is the Lagrangian function and the initial and final positions $x(a)$ and $x(b)$ are fixed.

By considering $S[x + \epsilon\xi]$ for suitably restricted functions $\xi(t)$, derive the differential equation governing the motion of the particle and obtain an integral expression for the second variation $\delta^2 S$.

If $x(t)$ is a solution of the equation of motion and $x(t) + \epsilon u(t) + O(\epsilon^2)$ is also a solution of the equation of motion in the limit $\epsilon \to 0$, show that $u(t)$ satisfies the equation

$$\ddot{u} + V''(x)\, u = 0\, .$$

If $u(t)$ satisfies this equation and is non-vanishing for $a \leqslant t \leqslant b$, show that

$$\delta^2 S = \frac{1}{2}\int_a^b \left(\dot{\xi} - \frac{\dot{u}\xi}{u}\right)^2 dt\, .$$

Consider the simple harmonic oscillator, for which

$$V(x) = \frac{1}{2}\omega^2 x^2\, ,$$

where $2\pi/\omega$ is the oscillation period. Show that the solution of the equation of motion is a local minimum of the action integral, provided that the time difference $b - a$ is less than half an oscillation period.

Paper 4, Section II
13D Variational Principles

(a) Consider the functional

$$I[y] = \int_a^b L(y, y'; x)\, dx\,,$$

where $0 < a < b$, and $y(x)$ is subject to the requirement that $y(a)$ and $y(b)$ are some fixed constants. Derive the equation satisfied by $y(x)$ when $\delta I = 0$ for all variations δy that respect the boundary conditions.

(b) Consider the function

$$L(y, y'; x) = \frac{\sqrt{1 + y'^2}}{x}\,.$$

Verify that, if $y(x)$ describes an arc of a circle, with centre on the y-axis, then $\delta I = 0$.

(c) Consider the function

$$L(y, y'; x) = \frac{\sqrt{1 + y'^2}}{y}\,.$$

Find $y(x)$ such that $\delta I = 0$ subject to the requirement that $y(a) = a$ and $y(b) = \sqrt{2ab - b^2}$, with $b < 2a$. Sketch the curve $y(x)$.