UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday, 10 September, 2020 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Separate your answers to each question.

Complete a gold cover sheet ***for each question*** *that you have attempted, and place it at the front of your answer to that question.*

Complete a green master cover sheet listing ***all the questions*** *that you have attempted.*

Every cover sheet must also show your Blind Grade ID and desk number.

Tie up your answers and cover sheets into ***a single bundle****, with the master cover sheet on the top, and then the cover sheet and answer for each question, in the numerical order of the questions.*

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet
Script paper
Rough paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Let $N \geqslant 3$ be an odd integer and b an integer with $(b, N) = 1$. What does it mean to say that N is an *Euler pseudoprime to base* b?

Show that if N is not an Euler pseudoprime to some base b_0, then it is not an Euler pseudoprime to at least half the bases $\{1 \leqslant b < N : (b, N) = 1\}$.

Show that if N is odd and composite, then there exists an integer b such that N is not an Euler pseudoprime to base b.

2H Topics in Analysis

State Runge's theorem about the uniform approximation of holomorphic functions by polynomials.

Explicitly construct, with a brief justification, a sequence of polynomials which converges uniformly to $1/z$ on the semicircle $\{z : |z| = 1,\ \mathrm{Re}(z) \leqslant 0\}$.

Does there exist a sequence of polynomials converging uniformly to $1/z$ on $\{z : |z| = 1,\ z \neq 1\}$? Give a justification.

3I Coding and Cryptography

Let N and p be very large positive integers with p a prime and $p > N$. The Chair of the Committee is able to inscribe pairs of very large integers on discs. The Chair wishes to inscribe a collection of discs in such a way that any Committee member who acquires r of the discs and knows the prime p can deduce the integer N, but owning $r - 1$ discs will give no information whatsoever. What strategy should the Chair follow?

[You may use without proof standard properties of the determinant of the $r \times r$ Vandermonde matrix.]

4F Automata and Formal Languages

Define a *context-free grammar* G, a *sentence* of G and the *language* $\mathcal{L}(G)$ generated by G.

For the alphabet $\Sigma = \{a, b\}$, which of the following languages over Σ are context-free?

(i) $\{a^{2m}b^{2m} \mid m \geqslant 0\}$,

(ii) $\{a^{m^2}b^{m^2} \mid m \geqslant 0\}$.

[*You may assume standard results without proof if clearly stated.*]

5J Statistical Modelling

Suppose we have data $(Y_1, x_1^T), \dots, (Y_n, x_n^T)$, where the Y_i are independent conditional on the design matrix X whose rows are the $x_i^T, i = 1, \dots, n$. Suppose that given x_i, the true probability density function of Y_i is f_{x_i}, so that the data is generated from an element of a model $\mathcal{F} := \{(f_{x_i}(\,\cdot\,;\theta))_{i=1}^n\,, \theta \in \Theta\}$ for some $\Theta \subseteq \mathbb{R}^q$ and $q \in \mathbb{N}$.

(a) Define the *log-likelihood function* for $\mathcal{F}$, the *maximum likelihood estimator* of θ and *Akaike's Information Criterion* (AIC) for $\mathcal{F}$.

From now on let $\mathcal{F}$ be the normal linear model, i.e. $Y := (Y_1, \dots, Y_n)^T = X\beta + \varepsilon$, where $X \in \mathbb{R}^{n \times p}$ has full column rank and $\varepsilon \sim N_n(0, \sigma^2 I)$.

(b) Let $\hat{\sigma}^2$ denote the maximum likelihood estimator of σ^2. Show that the AIC of $\mathcal{F}$ is

$$n(1 + \log(2\pi\hat{\sigma}^2)) + 2(p+1).$$

(c) Let χ^2_{n-p} be a chi-squared distribution on $n-p$ degrees of freedom. Using any results from the course, show that the distribution of the AIC of $\mathcal{F}$ is

$$n\log(\chi^2_{n-p}) + n(\log(2\pi\sigma^2/n) + 1) + 2(p+1).$$

[*Hint:* $\hat{\sigma}^2 := n^{-1}\|Y - X\hat{\beta}\|^2 = n^{-1}\|(I-P)\varepsilon\|^2$, *where* $\hat{\beta}$ *is the maximum likelihood estimator of* β *and* P *is the projection matrix onto the column space of* X.]

6B Mathematical Biology

Consider a model for the common cold in which the population is partitioned into susceptible (S), infective (I), and recovered (R) categories, which satisfy

$$\frac{dS}{dt} = \alpha R - \beta SI\,,$$
$$\frac{dI}{dt} = \beta SI - \gamma I\,,$$
$$\frac{dR}{dt} = \gamma I - \alpha R\,,$$

where α, β and γ are positive constants.

(i) Show that the sum $N \equiv S + I + R$ does not change in time.

(ii) Determine the condition, in terms of β, γ and N, for an endemic steady state to exist, that is, a time-independent state with a non-zero number of infectives.

(iii) By considering a reduced set of equations for S and I only, show that the endemic steady state identified in (ii) above, if it exists, is stable.

7E Further Complex Methods

The Weierstrass elliptic function is defined by

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{m,n} \left[\frac{1}{(z-\omega_{m,n})^2} - \frac{1}{{\omega_{m,n}}^2} \right] ,$$

where $\omega_{m,n} = m\omega_1 + n\omega_2$, with non-zero periods (ω_1, ω_2) such that ω_1/ω_2 is not real, and where (m,n) are integers not both zero.

(i) Show that, in a neighbourhood of $z = 0$,

$$\mathcal{P}(z) = \frac{1}{z^2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6) ,$$

where

$$g_2 = 60 \sum_{m,n} (\omega_{m,n})^{-4}, \qquad g_3 = 140 \sum_{m,n} (\omega_{m,n})^{-6}.$$

(ii) Deduce that $\mathcal{P}$ satisfies

$$\left(\frac{d\mathcal{P}}{dz} \right)^2 = 4\mathcal{P}^3 - g_2 \mathcal{P} - g_3.$$

8B Classical Dynamics

A particle of mass m experiences a repulsive central force of magnitude k/r^2, where $r = |\mathbf{r}|$ is its distance from the origin. Write down the Hamiltonian of the system.

The Laplace–Runge–Lenz vector for this system is defined by

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} + mk\,\hat{\mathbf{r}} ,$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum and $\hat{\mathbf{r}} = \mathbf{r}/r$ is the radial unit vector. Show that

$$\{\mathbf{L}, H\} = \{\mathbf{A}, H\} = \mathbf{0} ,$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. What are the integrals of motion of the system? Show that the polar equation of the orbit can be written as

$$r = \frac{\lambda}{e \cos\theta - 1} ,$$

where λ and e are non-negative constants.

9D Cosmology

At temperature T, with $\beta = 1/(k_B T)$, the distribution of ultra-relativistic particles with momentum $\mathbf{p}$ is given by

$$n(\mathbf{p}) = \frac{1}{e^{\beta pc} \mp 1},$$

where the minus sign is for bosons and the plus sign for fermions, and with $p = |\mathbf{p}|$.

Show that the total number of fermions, n_{f}, is related to the total number of bosons, n_{b}, by $n_{\mathrm{f}} = \frac{3}{4} n_{\mathrm{b}}$.

Show that the total energy density of fermions, ρ_{f}, is related to the total energy density of bosons, ρ_{b}, by $\rho_{\mathrm{f}} = \frac{7}{8}\rho_{\mathrm{b}}$.

10C Quantum Information and Computation

For $\phi \in [0, 2\pi)$ and $|\psi\rangle \in \mathbb{C}^4$ consider the operator

$$R_\psi^\phi = \mathbb{I} - \left(1 - e^{i\phi}\right) |\psi\rangle \langle\psi| .$$

Let U be a unitary operator on $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ with action on $|00\rangle$ given as follows

$$U\,|00\rangle = \sqrt{p}\,|g\rangle + \sqrt{1-p}\,|b\rangle =: |\psi_{\mathrm{in}}\rangle\,, \qquad (\dagger)$$

where p is a constant in $[0, 1]$ and $|g\rangle\,, |b\rangle \in \mathbb{C}^4$ are orthonormal states.

(i) Give an explicit expression of the state $R_g^\phi U\,|00\rangle$.

(ii) Find a $|\psi\rangle \in \mathbb{C}^4$ for which $R_\psi^\pi = U R_{00}^\pi U^\dagger$.

(iii) Choosing $p = 1/4$ in equation ($\dagger$), calculate the state $U R_{00}^\pi U^\dagger R_g^\phi U\,|00\rangle$. For what choice of $\phi \in [0, 2\pi)$ is this state proportional to $|g\rangle$?

(iv) Describe how the above considerations can be used to find a marked element g in a list of four items $\{g, b_1, b_2, b_3\}$. Assume that you have the state $|00\rangle$ and can act on it with a unitary operator that prepares the uniform superposition of four orthonormal basis states $|g\rangle\,, |b_1\rangle\,, |b_2\rangle\,, |b_3\rangle$ of $\mathbb{C}^4$. [You may use the operators U (defined in ($\dagger$)), $U^\dagger$ and R_ψ^ϕ for any choice of $\phi \in [0, 2\pi)$ and any $|\psi\rangle \in \mathbb{C}^4$.]

SECTION II

11H Number Theory

Let p be an odd prime.

(i) Define the *Legendre symbol* $\left(\frac{x}{p}\right)$, and show that when $(x,p)=1$, then $\left(\frac{x^{-1}}{p}\right)=\left(\frac{x}{p}\right)$.

(ii) State and prove Gauss's lemma, and use it to evaluate $\left(\frac{-1}{p}\right)$. [You may assume Euler's criterion.]

(iii) Prove that

$$\sum_{x=1}^{p}\left(\frac{x}{p}\right)=0,$$

and deduce that

$$\sum_{x=1}^{p}\left(\frac{x(x+1)}{p}\right)=-1.$$

Hence or otherwise determine the number of pairs of consecutive integers $z, z+1$ such that $1 \leqslant z, z+1 \leqslant p-1$ and both z and $z+1$ are quadratic residues mod p.

12F Automata and Formal Languages

Give the definition of a *deterministic finite state automaton* and of a *regular language.*

State and prove the pumping lemma for regular languages.

Let $S=\{2^n \mid n=0,1,2,\dots\}$ be the subset of $\mathbb{N}$ consisting of the powers of 2. If we write the elements of S in base 2 (with no preceding zeros), is S a regular language over $\{0,1\}$?

Now suppose we write the elements of S in base 10 (again with no preceding zeros). Show that S is not a regular language over $\{0,1,2,3,4,5,6,7,8,9\}$. [*Hint: Give a proof by contradiction; use the above lemma to obtain a sequence* $a_1, a_2, \dots$ *of powers of 2, then consider* $a_{i+1}-10^d a_i$ *for* $i=1,2,3,\dots$ *and a suitable fixed* d.]

13B Mathematical Biology

The larva of a parasitic worm disperses in one dimension while laying eggs at rate $\lambda > 0$. The larvae die at rate μ and have diffusivity D, so that their density, $n(x,t)$, obeys

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} - \mu n\,, \qquad (D>0,\ \mu>0).$$

The eggs do not diffuse, so that their density, $e(x,t)$, obeys

$$\frac{\partial e}{\partial t} = \lambda n\ .$$

At $t = 0$ there are no eggs and N larvae concentrated at $x = 0$, so that $n(x,0) = N\delta(x)$.

(i) Determine $n(x,t)$ for $t > 0$. Show that $n(x,t) \to 0$ as $t \to \infty$.

(ii) Determine the limit of $e(x,t)$ as $t \to \infty$.

(iii) Provide a physical explanation for the remnant density of the eggs identified in part (ii).

[You may quote without proof the results

$$\begin{aligned}\int_{-\infty}^{\infty} \exp(-x^2)dx &= \sqrt{\pi}\\ \int_{-\infty}^{\infty} \frac{\exp(ikx)}{k^2+\alpha^2}dk &= \pi\exp(-\alpha|x|)/\alpha\ , \quad \alpha>0\ .\ ]\end{aligned}$$

14D Cosmology

In an expanding spacetime, the density contrast $\delta(\mathbf{x},t)$ satisfies the linearised equation

$$\ddot{\delta} + 2H\dot{\delta} - c_s^2\left(\frac{1}{a^2}\nabla^2 + k_J^2\right)\delta = 0\,, \qquad (*)$$

where a is the scale factor, H is the Hubble parameter, c_s is a constant, and k_J is the Jeans wavenumber, defined by

$$c_s^2 k_J^2 = \frac{4\pi G}{c^2}\bar{\rho}(t)\,,$$

with $\bar{\rho}(t)$ the background, homogeneous energy density.

(i) Solve for $\delta(\mathbf{x},t)$ in a static universe, with $a = 1$ and $H = 0$ and $\bar{\rho}$ constant. Identify two regimes: one in which sound waves propagate, and one in which there is an instability.

(ii) In a matter-dominated universe with $\bar{\rho} \sim 1/a^3$, use the Friedmann equation $H^2 = 8\pi G\bar{\rho}/3c^2$ to find the growing and decaying long-wavelength modes of δ as a function of a.

(iii) Assuming $c_s^2 \approx c_s^2 k_J^2 \approx 0$ in equation $(*)$, find the growth of matter perturbations in a radiation-dominated universe and find the growth of matter perturbations in a curvature-dominated universe.

15C Quantum Information and Computation

Consider the quantum oracle U_f for a function $f : B_n \to B_n$ which acts on the state $|x\rangle\,|y\rangle$ of $2n$ qubits as follows:

$$U_f\,|x\rangle\,|y\rangle = |x\rangle\,|y \oplus f(x)\rangle\,. \tag{1}$$

The function f is promised to have the following property: there exists a $z \in B_n$ such that for any $x, y \in B_n$,

$$[f(x) = f(y)] \quad \text{if and only if} \quad x \oplus y \in \{0^n, z\}, \tag{2}$$

where $0^n \equiv (0, 0, \ldots, 0) \in B_n$.

(a) What is the nature of the function f for the case in which $z = 0^n$, and for the case in which $z \neq 0^n$?

(b) Suppose initially each of the $2n$ qubits are in the state $|0\rangle$. They are then subject to the following operations:

1. Each of the first n qubits forming an input register are acted on by Hadamard gates;
2. The $2n$ qubits are then acted on by the quantum oracle U_f;
3. Next, the qubits in the input register are individually acted on by Hadamard gates.

(i) List the states of the $2n$ qubits after each of the above operations; the expression for the final state should involve the n-bit "dot product" which is defined as follows:

$$a \cdot b = (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n) \bmod 2,$$

where $a, b \in B_n$ with $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$.

(ii) Justify that if $z = 0^n$ then for any $y \in B_n$ and any $\varphi(x, y) \in \{-1, +1\}$, the following identity holds:

$$\left\| \sum_{x \in B_n} \varphi(x, y)\,|f(x)\rangle \right\|^2 = \left\| \sum_{x \in B_n} \varphi(x, y)\,|x\rangle \right\|^2. \tag{3}$$

(iii) For the case $z = 0^n$, what is the probability that a measurement of the input register, relative to the computational basis of $\mathbb{C}^n$ results in a string $y \in B_n$?

(iv) For the case $z \neq 0^n$, show that the probability that the above-mentioned measurement of the input register results in a string $y \in B_n$, is equal to the following:

zero for all strings $y \in B_n$ satisfying $y \cdot z = 1$, and

$2^{-(n-1)}$ for any fixed string $y \in B_n$ satisfying $y \cdot z = 0$.

[State any identity you may employ. You may use $(x \oplus z) \cdot y = (x \cdot y) \oplus (z \cdot y)$, $\forall\, x, y, z \in B_n$.]

16H Logic and Set Theory

Let $(V, \in)$ be a model of ZF. Give the definition of a *class* and a *function class* in V. Use the concept of function class to give a short, informal statement of the Axiom of Replacement.

Let $z_0 = \omega$ and, for each $n \in \omega$, let $z_{n+1} = \mathcal{P} z_n$. Show that $y = \{z_n | n \in \omega\}$ is a set.

We say that a set x is small if there is an injection from x to z_n for some $n \in \omega$. Let $\mathbf{HS}$ be the class of sets x such that every member of $\mathrm{TC}(\{x\})$ is small, where $\mathrm{TC}(\{x\})$ is the transitive closure of $\{x\}$. Show that $n \in \mathbf{HS}$ for all $n \in \omega$ and deduce that $\omega \in \mathbf{HS}$. Show further that $z_n \in \mathbf{HS}$ for all $n \in \omega$. Deduce that $y \in \mathbf{HS}$.

Is $(\mathbf{HS}, \in)$ a model of ZF? Justify your answer.

[*Recall that* $0 = \emptyset$ *and that* $n + 1 = n \cup \{n\}$ *for all* $n \in \omega$*.*]

17G Graph Theory

(i) State and prove Turán's theorem.

(ii) Let G be a graph of order $2n \geqslant 4$ with $n^2 + 1$ edges. Show that G must contain a triangle, and that if $n = 2$ then G contains two triangles.

(iii) Show that if every edge of G lies in a triangle then G contains at least $(n^2 + 1)/3$ triangles.

(iv) Suppose that G has some edge uv contained in no triangles. Show that $\Gamma(u) \cap \Gamma(v) = \emptyset$, and that if $|\Gamma(u)| + |\Gamma(v)| = 2n$ then $\Gamma(u)$ and $\Gamma(v)$ are not both independent sets.

By induction on n, or otherwise, show that every graph of order $2n \geqslant 4$ with $n^2 + 1$ edges contains at least n triangles. [*Hint: If uv is an edge that is contained in no triangles, consider $G - u - v$.*]

18G Galois Theory

(a) Let L/K be a Galois extension of fields, with $\mathrm{Aut}(L/K) = A_{10}$, the alternating group on 10 elements. Find $[L : K]$.

Let $f(x) = x^2 + bx + c \in K[x]$ be an irreducible polynomial, $\mathit{char}\, K \neq 2$. Show that $f(x)$ remains irreducible in $L[x]$.

(b) Let $L = \mathbb{Q}[\xi_{11}]$, where ξ_{11} is a primitive 11^{th} root of unity.

Determine all subfields $M \subseteq L$. Which are Galois over $\mathbb{Q}$?

For each proper subfield M, show that an element in M which is not in $\mathbb{Q}$ must be primitive, and give an example of such an element explicitly in terms of ξ_{11} for each M. [You do not need to justify that your examples are not in $\mathbb{Q}$.]

Find a primitive element for the extension $L/\mathbb{Q}$.

19F Representation Theory

State Mackey's restriction formula and Frobenius reciprocity for characters. Deduce Mackey's irreducibility criterion for an induced representation.

For $n \geqslant 2$ show that if S_{n-1} is the subgroup of S_n consisting of the elements that fix n, and W is a complex representation of S_{n-1}, then $\mathrm{Ind}_{S_{n-1}}^{S_n} W$ is not irreducible.

20F Algebraic Topology

Let K be a simplicial complex with four vertices $v_1, \ldots, v_4$ with simplices $\langle v_1, v_2, v_3\rangle$, $\langle v_1, v_4\rangle$ and $\langle v_2, v_4\rangle$ and their faces.

(a) Draw a picture of $|K|$, labelling the vertices.

(b) Using the definition of homology, calculate $H_n(K)$ for all n.

(c) Let L be the subcomplex of K consisting of the vertices v_1, v_2, v_4 and the 1-simplices $\langle v_1, v_2\rangle$, $\langle v_1, v_4\rangle$, $\langle v_2, v_4\rangle$. Let $i : L \to K$ be the inclusion. Construct a simplicial map $j : K \to L$ such that the topological realisation $|j|$ of j is a homotopy inverse to $|i|$. Construct an explicit chain homotopy $h : C_\bullet(K) \to C_\bullet(K)$ between $i_\bullet \circ j_\bullet$ and $\mathrm{id}_{C_\bullet(K)}$, and verify that h is a chain homotopy.

21I Linear Analysis

Let H be a separable complex Hilbert space.

(a) For an operator $T : H \to H$, define the *spectrum* and *point spectrum*. Define what it means for T to be: (i) a *compact operator*; (ii) a *self-adjoint operator* and (iii) a *finite rank operator*.

(b) Suppose $T : H \to H$ is compact. Prove that given any $\delta > 0$, there exists a finite-dimensional subspace $E \subset H$ such that $\|T(e_n) - P_E T(e_n)\| < \delta$ for each n, where $\{e_1, e_2, e_3, \ldots\}$ is an orthonormal basis for H and P_E denotes the orthogonal projection onto E. Deduce that a compact operator is the operator norm limit of finite rank operators.

(c) Suppose that $S : H \to H$ has finite rank and $\lambda \in \mathbb{C} \setminus \{0\}$ is not an eigenvalue of S. Prove that $S - \lambda I$ is surjective. [You may wish to consider the action of $S(S - \lambda I)$ on $\ker(S)^{\perp}$.]

(d) Suppose $T : H \to H$ is compact and $\lambda \in \mathbb{C} \setminus \{0\}$ is not an eigenvalue of T. Prove that the image of $T - \lambda I$ is dense in H.

Prove also that $T - \lambda I$ is bounded below, i.e. prove also that there exists a constant $c > 0$ such that $\|(T - \lambda I)x\| \geqslant c\|x\|$ for all $x \in H$. Deduce that $T - \lambda I$ is surjective.

22I Analysis of Functions

Let X be a Banach space.

(a) Define the *dual space* X', giving an expression for $\|\Lambda\|_{X'}$ for $\Lambda \in X'$. If $Y = L^p(\mathbb{R}^n)$ for some $1 \leqslant p < \infty$, identify Y' giving an expression for a general element of Y'. [You need not prove your assertion.]

(b) For a sequence $(\Lambda_i)_{i=1}^{\infty}$ with $\Lambda_i \in X'$, what is meant by: (i) $\Lambda_i \to \Lambda$, (ii) $\Lambda_i \rightharpoonup \Lambda$ (iii) $\Lambda_i \overset{*}{\rightharpoonup} \Lambda$? Show that (i) $\implies$ (ii) $\implies$ (iii). Find a sequence $(f_i)_{i=1}^{\infty}$ with $f_i \in L^{\infty}(\mathbb{R}) = (L^1(\mathbb{R}))'$ such that, for some $f, g \in L^{\infty}(\mathbb{R}^n)$:

$$f_i \overset{*}{\rightharpoonup} f, \qquad f_i^2 \overset{*}{\rightharpoonup} g, \qquad g \neq f^2.$$

(c) For $f \in C_c^0(\mathbb{R}^n)$, let $\Lambda : C_c^0(\mathbb{R}^n) \to \mathbb{C}$ be the map $\Lambda f = f(0)$. Show that Λ may be extended to a continuous linear map $\widetilde{\Lambda} : L^{\infty}(\mathbb{R}^n) \to \mathbb{C}$, and deduce that $(L^{\infty}(\mathbb{R}^n))' \neq L^1(\mathbb{R}^n)$. For which $1 \leqslant p \leqslant \infty$ is $L^p(\mathbb{R}^n)$ reflexive? [You may use without proof the Hahn–Banach theorem].

23F Riemann Surfaces

Let $\Lambda = \langle \lambda, \mu \rangle \subseteq \mathbb{C}$ be a lattice. Give the definition of the associated Weierstrass $\wp$-function as an infinite sum, and prove that it converges. [You may use without proof the fact that

$$\sum_{w \in \Lambda \setminus \{0\}} \frac{1}{|w|^t}$$

converges if and only if $t > 2$.]

Consider the half-lattice points

$$z_1 = \lambda/2\,, \quad z_2 = \mu/2\,, \quad z_3 = (\lambda + \mu)/2\,,$$

and let $e_i = \wp(z_i)$. Using basic properties of $\wp$, explain why the values e_1, e_2, e_3 are distinct.

Give an example of a lattice Λ and a conformal equivalence $\theta : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ such that θ acts transitively on the images of the half-lattice points z_1, z_2, z_3.

24F Algebraic Geometry

(i) Suppose $f(x, y) = 0$ is an affine equation whose projective completion is a smooth projective curve. Give a basis for the vector space of holomorphic differential forms on this curve. [You are not required to prove your assertion.]

Let $C \subset \mathbb{P}^2$ be the plane curve given by the vanishing of the polynomial

$$X_0^4 - X_1^4 - X_2^4 = 0$$

over the complex numbers.

(ii) Prove that C is nonsingular.

(iii) Let ℓ be a line in $\mathbb{P}^2$ and define D to be the divisor $\ell \cap C$. Prove that D is a canonical divisor on C.

(iv) Calculate the minimum degree d such that there exists a non-constant map

$$C \to \mathbb{P}^1$$

of degree d.

[*You may use any results from the lectures provided that they are stated clearly.*]

25I Differential Geometry

(a) Show that for a compact regular surface $S \subset \mathbb{R}^3$, there exists a point $p \in S$ such that $K(p) > 0$, where K denotes the Gaussian curvature. Show that if S is contained in a closed ball of radius R in $\mathbb{R}^3$, then there is a point p such that $K(p) \geqslant R^{-2}$.

(b) For a regular surface $S \subset \mathbb{R}^3$, give the definition of a *geodesic polar coordinate system* at a point $p \in S$. Show that in such a coordinate system, $\lim_{r\to 0} G(r,\theta) = 0$, $\lim_{r\to 0}(\sqrt{G})_r(r,\theta) = 1$, $E(r,\theta) = 1$ and $F(r,\theta) = 0$. [You may use without proof standard properties of the exponential map provided you state them clearly.]

(c) Let $S \subset \mathbb{R}^3$ be a regular surface. Show that if $K \leqslant 0$, then any geodesic polar coordinate ball $B(p,\epsilon_0) \subset S$ of radius ϵ_0 around p has area satisfying

$$\text{Area } B(p,\epsilon_0) \geqslant \pi\epsilon_0^2.$$

[You may use without proof the identity $(\sqrt{G})_{rr}(r,\theta) = -\sqrt{G}K$.]

(d) Let $S \subset \mathbb{R}^3$ be a regular surface, and now suppose $-\infty < K \leqslant C$ for some constant $0 < C < \infty$. Given any constant $0 < \gamma < 1$, show that there exists $\epsilon_0 > 0$, depending only on C and γ, so that if $B(p,\epsilon) \subset S$ is any geodesic polar coordinate ball of radius $\epsilon \leqslant \epsilon_0$, then

$$\text{Area } B(p,\epsilon) \geqslant \gamma\pi\epsilon^2.$$

[Hint: For any fixed θ_0, consider the function $f(r) := \sqrt{G}(r,\theta_0) - \alpha\sin(\sqrt{C}r)$, for all $0 < \alpha < \frac{1}{\sqrt{C}}$. Derive the relation $f'' \geqslant -Cf$ and show $f(r) > 0$ for an appropriate range of r. The following variant of Wirtinger's inequality may be useful and can be assumed without proof: if g is a C^1 function on $[0,L]$ vanishing at 0, then $\int_0^L |g(x)|^2dx \leqslant \frac{L}{2\pi}\int_0^L |g'(x)|^2dx$.]

26K Probability and Measure

Let $(X,\mathcal{A},m,T)$ be a probability measure preserving system.

(a) State what it means for $(X,\mathcal{A},m,T)$ to be *ergodic*.

(b) State Kolmogorov's 0-1 law for a sequence of independent random variables. What does it imply for the canonical model associated with an i.i.d. random process?

(c) Consider the special case when $X = [0,1]$, $\mathcal{A}$ is the σ-algebra of Borel subsets, and T is the map defined as

$$Tx = \begin{cases} 2x, & \text{if } x \in [0,\frac{1}{2}], \\ 2-2x, & \text{if } x \in [\frac{1}{2},1]. \end{cases}$$

(i) Check that the Lebesgue measure m on $[0,1]$ is indeed an invariant probability measure for T.

(ii) Let $X_0 := 1_{(0,\frac{1}{2})}$ and $X_n := X_0 \circ T^n$ for $n \geqslant 1$. Show that $(X_n)_{n\geqslant 0}$ forms a sequence of i.i.d. random variables on $(X,\mathcal{A},m)$, and that the σ-algebra $\sigma(X_0,X_1,\ldots)$ is all of $\mathcal{A}$. *[Hint: check first that for any integer $n \geqslant 0$, $T^{-n}(0,\frac{1}{2})$ is a disjoint union of 2^n intervals of length $1/2^{n+1}$.]*

(iii) Is $(X,\mathcal{A},m,T)$ ergodic? Justify your answer.

27K Applied Probability

Define a *renewal–reward process*, and state the renewal–reward theorem.

A machine M is repaired at time $t = 0$. After any repair, it functions without intervention for a time that is exponentially distributed with parameter λ, at which point it breaks down (assume the usual independence). Following any repair at time T, say, it is inspected at times T, $T + m$, $T + 2m$, $\ldots$, and instantly repaired if found to be broken (the inspection schedule is then restarted). Find the long run proportion of time that M is working. [You may express your answer in terms of an integral.]

28J Principles of Statistics

Let $\Theta = \mathbb{R}^p$, let $\mu > 0$ be a probability density function on Θ and suppose we are given a further auxiliary conditional probability density function $q(\cdot|t) > 0, t \in \Theta$, on Θ from which we can generate random draws. Consider a sequence of random variables $\{\vartheta_m : m \in \mathbb{N}\}$ generated as follows:

- For $m \in \mathbb{N}$ and given ϑ_m, generate a new draw $s_m \sim q(\cdot|\vartheta_m)$.
- Define

$$\vartheta_{m+1} = \begin{cases} s_m, & \text{with probability } \rho(\vartheta_m, s_m), \\ \vartheta_m, & \text{with probability } 1 - \rho(\vartheta_m, s_m) \end{cases}$$

where $\rho(t,s) = \min\left\{\frac{\mu(s)}{\mu(t)}\frac{q(t|s)}{q(s|t)}, 1\right\}$.

(i) Show that the Markov chain (ϑ_m) has invariant measure μ, that is, show that for all (measurable) subsets $B \subset \Theta$ and all $m \in \mathbb{N}$ we have

$$\int_\Theta \Pr(\vartheta_{m+1} \in B | \vartheta_m = t)\mu(t)dt = \int_B \mu(\theta)d\theta.$$

(ii) Now suppose that μ is the posterior probability density function arising in a statistical model $\{f(\cdot, \theta) : \theta \in \Theta\}$ with observations x and a $N(0, I_p)$ prior distribution on θ. Derive a family $\{q(\cdot \mid t) : t \in \Theta\}$ such that in the above algorithm the acceptance probability $\rho(t,s)$ is a function of the likelihood ratio $f(x,s)/f(x,t)$, and for which the probability density function $q(\cdot \mid t)$ has covariance matrix $2\delta I_p$ for all $t \in \Theta$.

29K Stochastic Financial Models

(a) Let $(B_t)_{t\geqslant 0}$ be a real-valued random process.

(i) What does it mean to say that $(B_t)_{t\geqslant 0}$ is a *Brownian motion*?

(ii) State the reflection principle for Brownian motion.

(b) Suppose that $(B_t)_{t\geqslant 0}$ is a Brownian motion and set $M_t = \sup_{s\leqslant t} B_s$ and $Z_t = M_t - B_t$.

(i) Find the joint distribution function of B_t and M_t.

(ii) Show that (M_t, Z_t) has a joint density function on $[0,\infty)^2$ given by

$$\mathbb{P}(M_t \in dy \text{ and } Z_t \in dz) = \frac{2}{\sqrt{2\pi t}}\frac{(y+z)}{t}e^{-(y+z)^2/(2t)}dydz.$$

(iii) You are given that two of the three processes $(|B_t|)_{t\geqslant 0}$, $(M_t)_{t\geqslant 0}$ and $(Z_t)_{t\geqslant 0}$ have the same distribution. Identify which two, justifying your answer.

30D Asymptotic Methods

(a) Find the leading order term of the asymptotic expansion, as $x \to \infty$, of the integral

$$I(x) = \int_0^{3\pi} e^{(t+x\cos t)}\, dt\,.$$

(b) Find the first two leading nonzero terms of the asymptotic expansion, as $x \to \infty$, of the integral

$$J(x) = \int_0^{\pi} (1-\cos t)e^{-x\ln(1+t)}\, dt\,.$$

31E Dynamical Systems

(a) A dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a fixed point at the origin. Define the terms *asymptotic stability*, *Lyapunov function* and *domain of stability* of the fixed point $\mathbf{x} = \mathbf{0}$. State and prove Lyapunov's first theorem and state (without proof) La Salle's invariance principle.

(b) Consider the system

$$\begin{aligned} \dot{x} &= -2x + x^3 + \sin(2y)\,, \\ \dot{y} &= -x - y^3. \end{aligned}$$

(i) Show that trajectories cannot leave the square $S = \{(x,y) : |x| < 1, |y| < 1\}$. Show also that there are no fixed points in S other than the origin. Is this enough to deduce that S is in the domain of stability of the origin?

(ii) Construct a Lyapunov function of the form $V = x^2/2 + g(y)$. Deduce that the origin is asymptotically stable.

(iii) Find the largest rectangle of the form $|x| < x_0, |y| < y_0$ on which V is a strict Lyapunov function. Is this enough to deduce that this region is in the domain of stability of the origin?

(iv) Purely from using the Lyapunov function V, what is the most that can be deduced about the domain of stability of the origin?

32C Integrable Systems

(a) Given a smooth vector field

$$V = V_1(x,u)\frac{\partial}{\partial x} + \phi(x,u)\frac{\partial}{\partial u}$$

on $\mathbb{R}^2$ define the *prolongation* of V of arbitrary order N.

Calculate the prolongation of order two for the group $SO(2)$ of transformations of $\mathbb{R}^2$ given for $s \in \mathbb{R}$ by

$$g^s \begin{pmatrix} u \\ x \end{pmatrix} = \begin{pmatrix} u\cos s - x\sin s \\ u\sin s + x\cos s \end{pmatrix},$$

and hence, or otherwise, calculate the prolongation of order two of the vector field $V = -x\partial_u + u\partial_x$. Show that both of the equations $u_{xx} = 0$ and $u_{xx} = (1+u_x^2)^{\frac{3}{2}}$ are invariant under this action of $SO(2)$, and interpret this geometrically.

(b) Show that the sine-Gordon equation

$$\frac{\partial^2 u}{\partial X \partial T} = \sin u$$

admits the group $\{g^s\}_{s\in\mathbb{R}}$, where

$$g^s : \begin{pmatrix} X \\ T \\ u \end{pmatrix} \mapsto \begin{pmatrix} e^s X \\ e^{-s} T \\ u \end{pmatrix}$$

as a group of Lie point symmetries. Show that there is a group invariant solution of the form $u(X,T) = F(z)$ where z is an invariant formed from the independent variables, and hence obtain a second order equation for $w = w(z)$ where $\exp[iF] = w$.

33A Principles of Quantum Mechanics

Explain what is meant by the terms *boson* and *fermion*.

Three distinguishable spin-1 particles are governed by the Hamiltonian

$$H = \frac{2\lambda}{\hbar^2}\left(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_3 \cdot \mathbf{S}_1\right),$$

where $\mathbf{S}_i$ is the spin operator of particle i and λ is a positive constant. How many spin states are possible altogether? By considering the total spin operator, determine the eigenvalues and corresponding degeneracies of the Hamiltonian.

Now consider the case that all three particles are indistinguishable and all have the same spatial wavefunction. What are the degeneracies of the Hamiltonian in this case?

34C Applications of Quantum Mechanics

(a) For the quantum scattering of a beam of particles in three dimensions off a spherically symmetric potential $V(r)$ that vanishes at large r, discuss the boundary conditions satisfied by the wavefunction ψ and define the scattering amplitude $f(\theta)$. Assuming the asymptotic form

$$\psi = \sum_{l=0}^{\infty} \frac{2l+1}{2ik} \left[(-1)^{l+1} \frac{e^{-ikr}}{r} + (1+2if_l) \frac{e^{ikr}}{r} \right] P_l(\cos\theta)\,,$$

state the constraints on f_l imposed by the unitarity of the S-matrix and define the *phase shifts* δ_l.

(b) For $V_0 > 0$, consider the specific potential

$$V(r) = \begin{cases} \infty\,, & r \leqslant a\,, \\ -V_0\,, & a < r \leqslant 2a\,, \\ 0\,, & r > 2a\,. \end{cases}$$

(i) Show that the s-wave phase shift δ_0 obeys

$$\tan(\delta_0) = \frac{k\cos(2ka) - \kappa\cot(\kappa a)\sin(2ka)}{k\sin(2ka) + \kappa\cot(\kappa a)\cos(2ka)}\,,$$

where $\kappa^2 = k^2 + 2mV_0/\hbar^2$.

(ii) Compute the scattering length a_s and find for which values of κ it diverges. Discuss briefly the physical interpretation of the divergences. [*Hint: you may find this trigonometric identity useful*

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\,. \quad]$$

35A Statistical Physics

Starting with the density of electromagnetic radiation modes in **k**-space, determine the energy E of black-body radiation in a box of volume V at temperature T.

Using the first law of thermodynamics show that

$$\left.\frac{\partial E}{\partial V}\right|_T = T \left.\frac{\partial P}{\partial T}\right|_V - P.$$

By using this relation determine the pressure P of the black-body radiation.

[You are given the following:

(i) The mean number of photons in a radiation mode of frequency ω is $\dfrac{1}{e^{\hbar\omega/T} - 1}$,

(ii) $1 + \dfrac{1}{2^4} + \dfrac{1}{3^4} + \cdots = \dfrac{\pi^4}{90}$,

(iii) You may assume P vanishes with T more rapidly than linearly, as $T \to 0$.]

36D Electrodynamics

The Maxwell stress tensor σ of the electromagnetic fields is a two-index Cartesian tensor with components

$$\sigma_{ij} = -\epsilon_0 \left(E_i E_j - \frac{1}{2}|\mathbf{E}|^2 \delta_{ij} \right) - \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}|\mathbf{B}|^2 \delta_{ij} \right) ,$$

where $i, j = 1, 2, 3$, and E_i and B_i denote the Cartesian components of the electric and magnetic fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ respectively.

(i) Consider an electromagnetic field sourced by charge and current densities denoted by $\rho(\mathbf{x}, t)$ and $\mathbf{J}(\mathbf{x}, t)$ respectively. Using Maxwell's equations and the Lorentz force law, show that the components of σ obey the equation

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial g_i}{\partial t} = -\left(\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}\right)_i ,$$

where g_i, for $i = 1, 2, 3$, are the components of a vector field $\mathbf{g}(\mathbf{x}, t)$ which you should give explicitly in terms of $\mathbf{E}$ and $\mathbf{B}$. Explain the physical interpretation of this equation and of the quantities σ and $\mathbf{g}$.

(ii) A localised source near the origin, $\mathbf{x} = 0$, emits electromagnetic radiation. Far from the source, the resulting electric and magnetic fields can be approximated as

$$\mathbf{B}(\mathbf{x}, t) \simeq \mathbf{B}_0(\mathbf{x}) \sin(\omega t - \mathbf{k} \cdot \mathbf{x}) , \qquad \mathbf{E}(\mathbf{x}, t) \simeq \mathbf{E}_0(\mathbf{x}) \sin(\omega t - \mathbf{k} \cdot \mathbf{x}) ,$$

where $\mathbf{B}_0(\mathbf{x}) = \dfrac{\mu_0 \omega^2}{4\pi r c} \hat{\mathbf{x}} \times \mathbf{p}_0$ and $\mathbf{E}_0(\mathbf{x}) = -c\hat{\mathbf{x}} \times \mathbf{B}_0(\mathbf{x})$ with $r = |\mathbf{x}|$ and $\hat{\mathbf{x}} = \mathbf{x}/r$. Here, $\mathbf{k} = (\omega/c)\hat{\mathbf{x}}$ and $\mathbf{p}_0$ is a constant vector.

Calculate the pressure exerted by these fields on a spherical shell of very large radius R centred on the origin. [You may assume that $\mathbf{E}$ and $\mathbf{B}$ vanish for $r > R$ and that the shell material is absorbant, i.e. no reflected wave is generated.]

37D General Relativity

(a) Let $(\mathcal{M}, \boldsymbol{g})$ be a four-dimensional spacetime and let $\boldsymbol{T}$ denote the rank $\binom{1}{1}$ tensor defined by

$$\boldsymbol{T} : \mathcal{T}_p^*(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}\,, \qquad (\boldsymbol{\eta}, \boldsymbol{V}) \mapsto \boldsymbol{\eta}(\boldsymbol{V})\,, \quad \forall\, \boldsymbol{\eta} \in \mathcal{T}_p^*(\mathcal{M}),\ \boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})\,.$$

Determine the components of the tensor $\boldsymbol{T}$ and use the general law for the transformation of tensor components under a change of coordinates to show that the components of $\boldsymbol{T}$ are the same in any coordinate system.

(b) In Cartesian coordinates (t, x, y, z) the Minkowski metric is given by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2\,.$$

Spheroidal coordinates (r, θ, ϕ) are defined through

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\theta\,\cos\phi\,, \\ y &= \sqrt{r^2 + a^2}\sin\theta\,\sin\phi\,, \\ z &= r\cos\theta\,, \end{aligned}$$

where $a \geqslant 0$ is a real constant.

(i) Show that the Minkowski metric in coordinates (t, r, θ, ϕ) is given by

$$ds^2 = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2\,. \qquad (\dagger)$$

(ii) Transform the metric $(\dagger)$ to null coordinates given by $u = t - r$, $R = r$ and show that $\partial/\partial R$ is not a null vector field for $a > 0$.

(iii) Determine a new azimuthal angle $\varphi = \phi - F(R)$ such that in the new coordinate system (u, R, θ, φ), the vector field $\partial/\partial R$ is null for any $a \geqslant 0$. Write down the Minkowski metric in this new coordinate system.

38B Fluid Dynamics II

(a) Briefly outline the derivation of the boundary layer equation

$$uu_x + vu_y = UdU/dx + \nu u_{yy}$$

explaining the significance of the symbols used and what sets the x-direction.

(b) Viscous fluid occupies the sector $0 < \theta < \alpha$ in cylindrical coordinates which is bounded by rigid walls and there is a line sink at the origin of strength αQ with $Q/\nu \gg 1$. Assume that vorticity is confined to boundary layers along the rigid walls $\theta = 0$ $(x > 0, y = 0)$ and $\theta = \alpha$.

(i) Find the flow outside the boundary layers and clarify why boundary layers exist at all.

(ii) Show that the boundary layer thickness along the wall $y = 0$ is proportional to

$$\delta := \left(\frac{\nu}{Q}\right)^{1/2} x\,.$$

(iii) Show that the boundary layer equation admits a similarity solution for the streamfunction $\psi(x, y)$ of the form

$$\psi = (\nu Q)^{1/2} f(\eta)\,,$$

where $\eta = y/\delta$. You should find the equation and boundary conditions satisfied by f.

(iv) Verify that

$$\frac{df}{d\eta} = \frac{5 - \cosh(\sqrt{2}\eta + c)}{1 + \cosh(\sqrt{2}\eta + c)}$$

yields a solution provided the constant c has one of two possible values. Which is the likely physical choice?

39B Waves

The dispersion relation for capillary waves on the surface of deep water is

$$\omega^2 = S^2|k|^3 \,,$$

where $S = (T/\rho)^{1/2}$, ρ is the density and T is the coefficient of surface tension. The free surface $z = \eta(x,t)$ is undisturbed for $t < 0$, when it is suddenly impacted by an object, giving the initial conditions at time $t = 0$:

$$\eta = 0 \quad \text{and} \quad \frac{\partial \eta}{\partial t} = \begin{cases} -W\,, & |x| < \epsilon\,, \\ 0\,, & |x| > \epsilon\,, \end{cases}$$

where W is a constant.

(i) Use Fourier analysis to find an integral expression for $\eta(x,t)$ when $t > 0$.

(ii) Use the method of stationary phase to find the asymptotic behaviour of $\eta(Vt,t)$ for fixed $V > 0$ as $t \to \infty$, for the case $V \ll \epsilon^{-1/2}S$. Show that the result can be written in the form

$$\eta(x,t) \sim \frac{W\epsilon\, S\, t^2}{x^{5/2}}\, F\left(\frac{x^3}{S^2 t^2}\right),$$

and determine the function F.

(iii) Give a brief physical interpretation of the link between the condition $\epsilon V^2/S^2 \ll 1$ and the simple dependence on the product $W\epsilon$.

[You are given that $\displaystyle\int_{-\infty}^{\infty} e^{\pm i a u^2}\, du = (\pi/a)^{1/2}\, e^{\pm i\pi/4}$ for $a > 0$.]

40E Numerical Analysis

(a) Give the definition of a *normal* matrix. Prove that if A is normal, then the (Euclidean) matrix ℓ_2-norm of A is equal to its spectral radius, i.e., $\|A\|_2 = \rho(A)$.

(b) The advection equation

$$u_t = u_x\,, \qquad 0 \leqslant x \leqslant 1, \qquad 0 \leqslant t < \infty,$$

is discretized by the Crank–Nicolson scheme

$$u_m^{n+1} - u_m^n = \tfrac{1}{4}\mu(u_{m+1}^{n+1} - u_{m-1}^{n+1}) + \tfrac{1}{4}\mu(u_{m+1}^n - u_{m-1}^n), \qquad m = 1, 2, \ldots, M\,, \quad n \in \mathbb{Z}_+\,.$$

Here, $\mu = \frac{k}{h}$ is the Courant number, with $k = \Delta t$, $h = \Delta x = \frac{1}{M+1}$, and u_m^n is an approximation to $u(mh, nk)$.

Using the eigenvalue analysis and carefully justifying each step, determine conditions on $\mu > 0$ for which the method is stable. [*Hint: All $M \times M$ Toeplitz anti-symmetric tridiagonal (TAT) matrices have the same set of orthogonal eigenvectors, and a TAT matrix with the elements $a_{j,j} = a$ and $a_{j,j+1} = -a_{j,j-1} = b$ has the eigenvalues $\lambda_k = a + 2\mathrm{i}b\cos\frac{\pi k}{M+1}$ where $\mathrm{i} = \sqrt{-1}$.*]

(c) Consider the same advection equation for the Cauchy problem $(x \in \mathbb{R},\ 0 \leqslant t \leqslant T)$. Now it is discretized by the two-step leapfrog scheme

$$u_m^{n+1} = \mu\,(u_{m+1}^n - u_{m-1}^n) + u_m^{n-1}.$$

Applying the Fourier technique, find the range of $\mu > 0$ for which the method is stable.

END OF PAPER