UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday, 10 June, 2020 3 hours

PAPER 2

Before you begin read these instructions carefully

Candidates are required to comply with the Code of Conduct for Part IB Online Examinations. This is a closed-book examination.

You must begin ***each answer*** *on a* ***separate sheet****.*

Answers must be ***handwritten*** *(unless you have an approved adjustment).*

You should ensure that your answers are ***legible****; otherwise you place yourself at a significant disadvantage. You are advised to write on one side of the paper only.*

Candidates have THREE HOURS to complete the examination

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may attempt ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

At the end of the examination:

Separate your answers to each question. Make sure that the question number, e.g. 7D, and your Blind Grade ID, e.g. 1234A, are written clearly on the first page of each answer.

Scan ***each*** *answer into a* ***separate*** *PDF file.*

Name each PDF file by the relevant question number, for example 7D.pdf *for question number 7D.*

Complete the PDF cover sheet to show all the questions you have attempted.

Sign the PDF declaration that you have complied with the Code of Conduct.

Upload the PDF for each answer, together with your PDF declaration and coversheet, to Moodle and submit.

If these administrative tasks take you more than 45 minutes then email your Tutor explaining why.

SECTION I

1G Groups Rings and Modules

Assume a group G acts transitively on a set Ω and that the size of Ω is a prime number. Let H be a normal subgroup of G that acts non-trivially on Ω.

Show that any two H-orbits of Ω have the same size. Deduce that the action of H on Ω is transitive.

Let $\alpha \in \Omega$ and let G_α denote the stabiliser of α in G. Show that if $H \cap G_\alpha$ is trivial, then there is a bijection $\theta\colon H \to \Omega$ under which the action of G_α on H by conjugation corresponds to the action of G_α on Ω.

2E Analysis and Topology

Let τ be the collection of subsets of $\mathbb{C}$ of the form $\mathbb{C}\backslash f^{-1}(0)$, where f is an arbitrary complex polynomial. Show that τ is a topology on $\mathbb{C}$.

Given topological spaces X and Y, define the *product topology* on $X \times Y$. Equip $\mathbb{C}^2$ with the topology given by the product of $(\mathbb{C}, \tau)$ with itself. Let g be an arbitrary two-variable complex polynomial. Is the subset $\mathbb{C}^2\backslash g^{-1}(0)$ always open in this topology? Justify your answer.

3D Variational Principles

Find the stationary points of the function $\phi = xyz$ subject to the constraint $x + a^2y^2 + z^2 = b^2$, with $a, b > 0$. What are the maximum and minimum values attained by ϕ, subject to this constraint, if we further restrict to $x \geqslant 0$?

4B Methods

Find the Fourier transform of the function

$$f(x) = \begin{cases} A\,, & |x| \leqslant 1 \\ 0\,, & |x| > 1\,. \end{cases}$$

Determine the convolution of the function $f(x)$ with itself.

State the convolution theorem for Fourier transforms. Using it, or otherwise, determine the Fourier transform of the function

$$g(x) = \begin{cases} B(2 - |x|)\,, & |x| \leqslant 2 \\ 0\,, & |x| > 2\,. \end{cases}$$

5D Electromagnetism

Two concentric spherical shells with radii R and $2R$ carry fixed, uniformly distributed charges Q_1 and Q_2 respectively. Find the electric field and electric potential at all points in space. Calculate the total energy of the electric field.

6C Fluid Dynamics

Incompressible fluid of constant viscosity μ is confined to the region $0 < y < h$ between two parallel rigid plates. Consider two parallel viscous flows: flow A is driven by the motion of one plate in the x-direction with the other plate at rest; flow B is driven by a constant pressure gradient in the x-direction with both plates at rest. The velocity mid-way between the plates is the same for both flows.

The viscous friction in these flows is known to produce heat locally at a rate

$$Q = \mu \left(\frac{\partial u}{\partial y} \right)^2$$

per unit volume, where u is the x-component of the velocity. Determine the ratio of the total rate of heat production in flow A to that in flow B.

7H Markov Chains

Let $(X_n)_{n \geqslant 0}$ be a Markov chain with state space $\{1, 2\}$ and transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

where $\alpha, \beta \in (0, 1]$. Compute $\mathbb{P}(X_n = 1 | X_0 = 1)$. Find the value of $\mathbb{P}(X_n = 1 | X_0 = 2)$.

SECTION II

8F Linear Algebra

Let V be a finite-dimensional vector space over a field. Show that an endomorphism α of V is idempotent, i.e. $\alpha^2 = \alpha$, if and only if α is a projection onto its image.

Determine whether the following statements are true or false, giving a proof or counter-example as appropriate:

(i) If $\alpha^3 = \alpha^2$, then α is idempotent.

(ii) The condition $\alpha(1-\alpha)^2 = 0$ is equivalent to α being idempotent.

(iii) If α and β are idempotent and such that $\alpha + \beta$ is also idempotent, then $\alpha\beta = 0$.

(iv) If α and β are idempotent and $\alpha\beta = 0$, then $\alpha + \beta$ is also idempotent.

9G Groups Rings and Modules

State Gauss' lemma. State and prove Eisenstein's criterion.

Define the notion of an *algebraic integer*. Show that if α is an algebraic integer, then $\{f \in \mathbb{Z}[X] : f(\alpha) = 0\}$ is a principal ideal generated by a monic, irreducible polynomial.

Let $f = X^4 + 2X^3 - 3X^2 - 4X - 11$. Show that $\mathbb{Q}[X]/(f)$ is a field. Show that $\mathbb{Z}[X]/(f)$ is an integral domain, but not a field. Justify your answers.

10E Analysis and Topology

Let $C[0,1]$ be the space of continuous real-valued functions on $[0,1]$, and let d_1, d_∞ be the metrics on it given by

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx \qquad \text{and} \qquad d_\infty(f,g) = \max_{x\in[0,1]}|f(x) - g(x)|.$$

Show that $\mathrm{id} : (C[0,1], d_\infty) \to (C[0,1], d_1)$ is a continuous map. Do d_1 and d_∞ induce the same topology on $C[0,1]$? Justify your answer.

Let d denote for any $m \in \mathbb{N}$ the uniform metric on $\mathbb{R}^m$: $d((x_i),(y_i)) = \max_i |x_i - y_i|$. Let $\mathcal{P}_n \subset C[0,1]$ be the subspace of real polynomials of degree at most n. Define a *Lipschitz map* between two metric spaces, and show that evaluation at a point gives a Lipschitz map $(C[0,1], d_\infty) \to (\mathbb{R}, d)$. Hence or otherwise find a bijection from $(\mathcal{P}_n, d_\infty)$ to $(\mathbb{R}^{n+1}, d)$ which is Lipschitz and has a Lipschitz inverse.

Let $\tilde{\mathcal{P}}_n \subset \mathcal{P}_n$ be the subset of polynomials with values in the range $[-1,1]$.

(i) Show that $(\tilde{\mathcal{P}}_n, d_\infty)$ is compact.

(ii) Show that d_1 and d_∞ induce the same topology on $\tilde{\mathcal{P}}_n$.

Any theorems that you use should be clearly stated.

[*You may use the fact that for distinct constants a_i, the following matrix is invertible:*

$$\begin{pmatrix} 1 & a_0 & a_0^2 & \dots & a_0^n \\ 1 & a_1 & a_1^2 & \dots & a_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^n \end{pmatrix}.]$$

11F Geometry

Let $H = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the hyperbolic half-plane with the metric $g_H = (dx^2 + dy^2)/y^2$. Define the *length* of a continuously differentiable curve in H with respect to g_H.

What are the *hyperbolic lines* in H? Show that for any two distinct points z, w in H, the infimum $\rho(z, w)$ of the lengths (with respect to g_H) of curves from z to w is attained by the segment $[z, w]$ of the hyperbolic line with an appropriate parameterisation.

The 'hyperbolic Pythagoras theorem' asserts that if a hyperbolic triangle ABC has angle $\pi/2$ at C then

$$\cosh c = \cosh a \cosh b\,,$$

where a, b, c are the lengths of the sides BC, AC, AB, respectively.

Let l and m be two hyperbolic lines in H such that

$$\inf\{\rho(z, w) : z \in l,\ w \in m\} = d > 0.$$

Prove that the distance d is attained by the points of intersection with a hyperbolic line h that meets each of l, m orthogonally. Give an example of two hyperbolic lines l and m such that the infimum of $\rho(z, w)$ is *not* attained by any $z \in l$, $w \in m$.

[*You may assume that every Möbius transformation that maps H onto itself is an isometry of g_H.*]

12B Complex Analysis or Complex Methods

For the function

$$f(z) = \frac{1}{z(z-2)}\,,$$

find the Laurent expansions

(i) about $z = 0$ in the annulus $0 < |z| < 2$,

(ii) about $z = 0$ in the annulus $2 < |z| < \infty$,

(iii) about $z = 1$ in the annulus $0 < |z - 1| < 1$.

What is the nature of the singularity of f, if any, at $z = 0$, $z = \infty$ and $z = 1$?

Using an integral of f, or otherwise, evaluate

$$\int_0^{2\pi} \frac{2 - \cos\theta}{5 - 4\cos\theta}\, d\theta\,.$$

13A Methods

(i) The solution to the equation

$$\frac{d}{dx}\left(x\frac{dF}{dx}\right) + \alpha^2 xF = 0$$

that is regular at the origin is $F(x) = CJ_0(\alpha x)$, where α is a real, positive parameter, J_0 is a Bessel function, and C is an arbitrary constant. The Bessel function has infinitely many zeros: $J_0(\gamma_k) = 0$ with $\gamma_k > 0$, for $k = 1, 2, \ldots$. Show that

$$\int_0^1 J_0(\alpha x)\, J_0(\beta x)\, x\, dx = \frac{\beta J_0(\alpha)J_0'(\beta) - \alpha J_0(\beta)J_0'(\alpha)}{\alpha^2 - \beta^2}\,, \quad \alpha \neq \beta\,,$$

(where α and β are real and positive) and deduce that

$$\int_0^1 J_0(\gamma_k x)\, J_0(\gamma_\ell x)\, x\, dx = 0\,, \quad k \neq \ell\,; \qquad \int_0^1 (J_0(\gamma_k x))^2\, x\, dx = \frac{1}{2}(J_0'(\gamma_k))^2\,.$$

[*Hint: For the second identity, consider* $\alpha = \gamma_k$ *and* $\beta = \gamma_k + \epsilon$ *with* ϵ *small.*]

(ii) The displacement $z(r,t)$ of the membrane of a circular drum of unit radius obeys

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right) = \frac{\partial^2 z}{\partial t^2}\,, \qquad z(1,t) = 0\,,$$

where r is the radial coordinate on the membrane surface, t is time (in certain units), and the displacement is assumed to have no angular dependence. At $t = 0$ the drum is struck, so that

$$z(r,0) = 0\,, \qquad \frac{\partial z}{\partial t}(r,0) = \begin{cases} U\,, & r < b \\ 0\,, & r > b \end{cases}$$

where U and $b < 1$ are constants. Show that the subsequent motion is given by

$$z(r,t) = \sum_{k=1}^{\infty} C_k\, J_0(\gamma_k r)\, \sin(\gamma_k t) \qquad \text{where} \qquad C_k = -2bU\frac{J_0'(\gamma_k b)}{\gamma_k^2 (J_0'(\gamma_k))^2}\,.$$

14A Quantum Mechanics

(a) The potential $V(x)$ for a particle of mass m in one dimension is such that $V \to 0$ rapidly as $x \to \pm\infty$. Let $\psi(x)$ be a wavefunction for the particle satisfying the time-independent Schrödinger equation with energy E.

Suppose ψ has the asymptotic behaviour

$$\psi(x) \sim Ae^{ikx} + Be^{-ikx} \quad (x \to -\infty)\,, \qquad \psi(x) \sim Ce^{ikx} \quad (x \to +\infty)\,,$$

where A, B, C are complex coefficients. Explain, in outline, how the probability current $j(x)$ is used in the interpretation of such a solution as a scattering process and how the transmission and reflection probabilities P_{tr} and P_{ref} are found.

Now suppose instead that $\psi(x)$ is a bound state solution. Write down the asymptotic behaviour in this case, relating an appropriate parameter to the energy E.

(b) Consider the potential

$$V(x) \;=\; -\,\frac{\hbar^2}{m}\,\frac{a^2}{\cosh^2 ax}$$

where a is a real, positive constant. Show that

$$\psi(x) \;=\; Ne^{ikx}(a\tanh ax \;-\; ik)\,,$$

where N is a complex coefficient, is a solution of the time-independent Schrödinger equation for any real k and find the energy E. Show that ψ represents a scattering process for which $P_{\mathrm{ref}} = 0$, and find P_{tr} explicitly.

Now let $k = i\lambda$ in the formula for ψ above. Show that this defines a bound state if a certain real positive value of λ is chosen and find the energy of this solution.

15D Electromagnetism

(a) A surface current $\mathbf{K} = K\mathbf{e}_x$, with K a constant and $\mathbf{e}_x$ the unit vector in the x-direction, lies in the plane $z = 0$. Use Ampère's law to determine the magnetic field above and below the plane. Confirm that the magnetic field is discontinuous across the surface, with the discontinuity given by

$$\lim_{z\to 0^+} \mathbf{e}_z \times \mathbf{B} - \lim_{z\to 0^-} \mathbf{e}_z \times \mathbf{B} = \mu_0 \mathbf{K}\,,$$

where $\mathbf{e}_z$ is the unit vector in the z-direction.

(b) A surface current $\mathbf{K}$ flows radially in the $z = 0$ plane, resulting in a pile-up of charge Q at the origin, with $dQ/dt = I$, where I is a constant.

Write down the electric field $\mathbf{E}$ due to the charge at the origin, and hence the displacement current $\epsilon_0 \partial\mathbf{E}/\partial t$.

Confirm that, away from the plane and for $\theta < \pi/2$, the magnetic field due to the displacement current is given by

$$\mathbf{B}(r,\theta) = \frac{\mu_0 I}{4\pi r}\tan\left(\frac{\theta}{2}\right)\mathbf{e}_\phi\,,$$

where (r, θ, ϕ) are the usual spherical polar coordinates. [*Hint: Use Stokes' theorem applied to a spherical cap that subtends an angle* θ*.*]

16C Fluid Dynamics

A vertical cylindrical container of radius R is partly filled with fluid of constant density to depth h. The free surface is perturbed so that the fluid occupies the region

$$0 < r < R, \quad -h < z < \zeta(r, \theta, t),$$

where (r, θ, z) are cylindrical coordinates and ζ is the perturbed height of the free surface. For small perturbations, a linearised description of surface waves in the cylinder yields the following system of equations for ζ and the velocity potential ϕ:

$$\begin{aligned}
\nabla^2 \phi &= 0\,, \qquad 0 < r < R\,, \quad -h < z < 0\,, && (1)\\
\frac{\partial \phi}{\partial t} + g\zeta &= 0 \quad \text{on} \quad z = 0\,, && (2)\\
\frac{\partial \zeta}{\partial t} - \frac{\partial \phi}{\partial z} &= 0 \quad \text{on} \quad z = 0\,, && (3)\\
\frac{\partial \phi}{\partial z} &= 0 \quad \text{on} \quad z = -h\,, && (4)\\
\frac{\partial \phi}{\partial r} &= 0 \quad \text{on} \quad r = R\,. && (5)
\end{aligned}$$

(a) Describe briefly the physical meaning of each equation.

(b) Consider axisymmetric normal modes of the form

$$\phi = \mathrm{Re}\left(\hat{\phi}(r, z)e^{-i\sigma t}\right), \qquad \zeta = \mathrm{Re}\left(\hat{\zeta}(r)e^{-i\sigma t}\right).$$

Show that the system of equations (1)–(5) admits a solution for $\hat{\phi}$ of the form

$$\hat{\phi}(r, z) = AJ_0(k_n r)Z(z)\,,$$

where A is an arbitrary amplitude, $J_0(x)$ satisfies the equation

$$\frac{d^2 J_0}{dx^2} + \frac{1}{x}\frac{dJ_0}{dx} + J_0 = 0\,,$$

the wavenumber k_n, $n = 1, 2, \ldots$ is such that $x_n = k_n R$ is one of the zeros of the function dJ_0/dx, and the function $Z(z)$ should be determined explicitly.

(c) Show that the frequency σ_n of the n-th mode is given by

$$\sigma_n^2 = \frac{g}{h}\Psi(k_n h)\,,$$

where the function $\Psi(x)$ is to be determined.

[*Hint: In cylindrical coordinates* (r, θ, z),

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\,.]$$

17C Numerical Analysis

Consider a multistep method for numerical solution of the differential equation $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$:

$$\mathbf{y}_{n+2} - \mathbf{y}_{n+1} = h\left[\,(1+\alpha)\mathbf{f}(t_{n+2}, \mathbf{y}_{n+2}) + \beta\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) - (\alpha+\beta)\mathbf{f}(t_n, \mathbf{y}_n)\,\right], \qquad (*)$$

where $n = 0, 1, \ldots,$ and α and β are constants.

(a) Define the *order* of a method for numerically solving an ODE.

(b) Show that in general an explicit method of the form $(*)$ has order 1. Determine the values of α and β for which this multistep method is of order 3.

(c) Show that the multistep method $(*)$ is convergent.

18H Statistics

Consider the general linear model $Y = X\beta^0 + \varepsilon$ where X is a known $n \times p$ design matrix with $p \geqslant 2$, $\beta^0 \in \mathbb{R}^p$ is an unknown vector of parameters, and $\varepsilon \in \mathbb{R}^n$ is a vector of stochastic errors with $\mathbb{E}(\varepsilon_i) = 0$, $\mathrm{var}(\varepsilon_i) = \sigma^2 > 0$ and $\mathrm{cov}(\varepsilon_i, \varepsilon_j) = 0$ for all $i, j = 1, \ldots, n$ with $i \neq j$. Suppose X has full column rank.

(a) Write down the least squares estimate $\hat{\beta}$ of β^0 and show that it minimises the least squares objective $S(\beta) = \|Y - X\beta\|^2$ over $\beta \in \mathbb{R}^p$.

(b) Write down the variance–covariance matrix $\mathrm{cov}(\hat{\beta})$.

(c) Let $\tilde{\beta} \in \mathbb{R}^p$ minimise $S(\beta)$ over $\beta \in \mathbb{R}^p$ subject to $\beta_p = 0$. Let Z be the $n \times (p-1)$ submatrix of X that excludes the final column. Write down $\mathrm{cov}(\tilde{\beta})$.

(d) Let P and P_0 be $n \times n$ orthogonal projections onto the column spaces of X and Z respectively. Show that for all $u \in \mathbb{R}^n$, $u^T P u \geqslant u^T P_0 u$.

(e) Show that for all $x \in \mathbb{R}^p$,

$$\mathrm{var}(x^T\tilde{\beta}) \leqslant \mathrm{var}(x^T\hat{\beta}).$$

[*Hint: Argue that $x = X^T u$ for some $u \in \mathbb{R}^n$.*]

19H Optimisation

State and prove the Lagrangian sufficiency theorem.

Solve, using the Lagrangian method, the optimisation problem

$$\begin{aligned} \text{maximise} \quad & x + y + 2a\sqrt{1+z} \\ \text{subject to} \quad & x + \frac{1}{2}y^2 + z = b\,, \\ & x, z \geqslant 0\,, \end{aligned}$$

where the constants a and b satisfy $a \geqslant 1$ and $b \geqslant 1/2$.

[*You need not prove that your solution is unique.*]

END OF PAPER