UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 8 June, 2020 3 hours

PAPER 1

Before you begin read these instructions carefully

Candidates are required to comply with the Code of Conduct for Part IA Online Examinations. This is a closed-book examination.

You must begin ***each answer*** *on a* ***separate sheet****.*

Answers must be ***handwritten*** *(unless you have an approved adjustment).*

You should ensure that your answers are ***legible****; otherwise you place yourself at a significant disadvantage. You are advised to write on one side of the paper only.*

Candidates have THREE HOURS to complete the examination

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions a beta quality mark.

Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the two Applied courses (Vectors & Matrices and Differential Equations) or on the two Pure courses (Analysis I and Probability).*

At the end of the examination:

Separate your answers to each question. Make sure that the question number, e.g. 7D, and your Blind Grade ID, e.g. 1234A, are written clearly on the first page of each answer.

Scan ***each*** *answer into a* ***separate*** *PDF file.*

Name each PDF file by the relevant question number, for example 7D.pdf *for question number 7D.*

Complete the PDF cover sheet to show all the questions you have attempted.

Sign the PDF declaration that you have complied with the Code of Conduct.

Upload the PDF for each answer, together with your PDF declaration and coversheet, to Moodle and submit.

If these administrative tasks take you more than 45 minutes then email your Tutor explaining why.

SECTION I

1C Vectors and Matrices

Given a non-zero complex number $z = x + iy$, where x and y are real, find expressions for the real and imaginary parts of the following functions of z in terms of x and y:

(i) e^z ,

(ii) $\sin z$,

(iii) $\dfrac{1}{z} - \dfrac{1}{\bar{z}}$,

(iv) $z^3 - z^2\bar{z} - z\bar{z}^2 + \bar{z}^3$,

where $\bar{z}$ is the complex conjugate of z.

Now assume $x > 0$ and find expressions for the real and imaginary parts of all solutions to

(v) $w = \log z$.

2A Differential Equations

Solve the differential equation

$$\frac{dy}{dx} = \frac{1}{x + e^{2y}},$$

subject to the initial condition $y(1) = 0$.

3E Analysis I

(a) Let f be continuous in $[a, b]$, and let g be strictly monotonic in $[\alpha, \beta]$, with a continuous derivative there, and suppose that $a = g(\alpha)$ and $b = g(\beta)$. Prove that

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(u))g'(u)du\,.$$

[Any version of the fundamental theorem of calculus may be used providing it is quoted correctly.]

(b) Justifying carefully the steps in your argument, show that the improper Riemann integral

$$\int_0^{e^{-1}} \frac{dx}{x(\log \frac{1}{x})^\theta}$$

converges for $\theta > 1$, and evaluate it.

4F Probability

A robot factory begins with a single generation-0 robot. Each generation-n robot independently builds some number of generation-$(n+1)$ robots before breaking down. The number of generation-$(n+1)$ robots built by a generation-n robot is 0, 1, 2 or 3 with probabilities $\frac{1}{12}$, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{12}$ respectively. Find the expectation of the total number of generation-n robots produced by the factory. What is the probability that the factory continues producing robots forever?

[*Standard results about branching processes may be used without proof as long as they are carefully stated.*]

SECTION II

5C Vectors and Matrices

(a) Let A, B, and C be three distinct points in the plane $\mathbb{R}^2$ which are not collinear, and let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be their position vectors.

Show that the set L_{AB} of points in $\mathbb{R}^2$ equidistant from A and B is given by an equation of the form

$$\mathbf{n}_{AB} \cdot \mathbf{x} = p_{AB}\,,$$

where $\mathbf{n}_{AB}$ is a unit vector and p_{AB} is a scalar, to be determined. Show that L_{AB} is perpendicular to $\overrightarrow{AB}$.

Show that if $\mathbf{x}$ satisfies

$$\mathbf{n}_{AB} \cdot \mathbf{x} = p_{AB} \quad \text{and} \quad \mathbf{n}_{BC} \cdot \mathbf{x} = p_{BC}$$

then

$$\mathbf{n}_{CA} \cdot \mathbf{x} = p_{CA}\,.$$

How do you interpret this result geometrically?

(b) Let $\mathbf{a}$ and $\mathbf{u}$ be constant vectors in $\mathbb{R}^3$. Explain why the vectors $\mathbf{x}$ satisfying

$$\mathbf{x} \times \mathbf{u} = \mathbf{a} \times \mathbf{u}$$

describe a line in $\mathbb{R}^3$. Find an expression for the shortest distance between two lines $\mathbf{x} \times \mathbf{u}_k = \mathbf{a}_k \times \mathbf{u}_k$, where $k = 1, 2$.

6A Vectors and Matrices

What does it mean to say an $n \times n$ matrix is *Hermitian*?

What does it mean to say an $n \times n$ matrix is *unitary*?

Show that the eigenvalues of a Hermitian matrix are real and that eigenvectors corresponding to distinct eigenvalues are orthogonal.

Suppose that A is an $n \times n$ Hermitian matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding normalised eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Let U denote the matrix whose columns are $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Show directly that U is unitary and $UDU^\dagger = A$, where D is a diagonal matrix you should specify.

If U is unitary and D diagonal, must it be the case that $UDU^\dagger$ is Hermitian? Give a proof or counterexample.

Find a unitary matrix U and a diagonal matrix D such that

$$UDU^\dagger = \begin{pmatrix} 2 & 0 & 3i \\ 0 & 2 & 0 \\ -3i & 0 & 2 \end{pmatrix}.$$

7A Differential Equations

Show that for each $t > 0$ and $x \in \mathbb{R}$ the function

$$K(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

satisfies the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,.$$

For $t > 0$ and $x \in \mathbb{R}$ define the function $u = u(x,t)$ by the integral

$$u(x,t) = \int_{-\infty}^{\infty} K(x-y,t) f(y)\, dy\,.$$

Show that u satisfies the heat equation and $\lim_{t\to 0^+} u(x,t) = f(x)$. [*Hint: You may find it helpful to consider the substitution* $Y = (x-y)/\sqrt{4t}$.]

Burgers' equation is

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = \frac{\partial^2 w}{\partial x^2}\,.$$

By considering the transformation

$$w(x,t) = -2\frac{1}{u}\frac{\partial u}{\partial x}\,,$$

solve Burgers' equation with the initial condition $\lim_{t\to 0^+} w(x,t) = g(x)$.

8A Differential Equations

Solve the system of differential equations for $x(t), y(t), z(t)$,

$$\begin{aligned}
\dot{x} &= 3z - x,\\
\dot{y} &= 3x + 2y - 3z + \cos t - 2\sin t,\\
\dot{z} &= 3x - z,
\end{aligned}$$

subject to the initial conditions $x(0) = y(0) = 0$, $z(0) = 1$.

9D Analysis I

(a) State Rolle's theorem. Show that if $f : \mathbb{R} \to \mathbb{R}$ is $N+1$ times differentiable and $x \in \mathbb{R}$ then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(N)}(0)}{N!}x^N + \frac{f^{(N+1)}(\theta x)}{(N+1)!}x^{N+1},$$

for some $0 < \theta < 1$. Hence, or otherwise, show that if $f'(x) = 0$ for all $x \in \mathbb{R}$ then f is constant.

(b) Let $s : \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that

$$s'(x) = c(x), \quad c'(x) = -s(x), \quad s(0) = 0 \quad \text{and} \quad c(0) = 1.$$

Prove that

(i) $s(x)c(a-x) + c(x)s(a-x)$ is independent of x,

(ii) $s(x+y) = s(x)c(y) + c(x)s(y)$,

(iii) $s(x)^2 + c(x)^2 = 1$.

Show that $c(1) > 0$ and $c(2) < 0$. Deduce there exists $1 < k < 2$ such that $s(2k) = c(k) = 0$ and $s(x+4k) = s(x)$.

10F Analysis I

(a) Let (x_n) be a bounded sequence of real numbers. Show that (x_n) has a convergent subsequence.

(b) Let (z_n) be a bounded sequence of complex numbers. For each $n \geqslant 1$, write $z_n = x_n + iy_n$. Show that (z_n) has a subsequence (z_{n_j}) such that (x_{n_j}) converges. Hence, or otherwise, show that (z_n) has a convergent subsequence.

(c) Write $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the set of positive integers. Let M be a positive real number, and for each $i \in \mathbb{N}$, let $X^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, \ldots)$ be a sequence of real numbers with $|x_j^{(i)}| \leqslant M$ for all i, $j \in \mathbb{N}$. By induction on i or otherwise, show that there exist sequences $N^{(i)} = (n_1^{(i)}, n_2^{(i)}, n_3^{(i)}, \ldots)$ of positive integers with the following properties:

- for all $i \in \mathbb{N}$, the sequence $N^{(i)}$ is strictly increasing;
- for all $i \in \mathbb{N}$, $N^{(i+1)}$ is a subsequence of $N^{(i)}$; and
- for all $k \in \mathbb{N}$ and all $i \in \mathbb{N}$ with $1 \leqslant i \leqslant k$, the sequence

  $$(x^{(i)}_{n^{(k)}_1}, x^{(i)}_{n^{(k)}_2}, x^{(i)}_{n^{(k)}_3}, \ldots)$$

  converges.

Hence, or otherwise, show that there exists a strictly increasing sequence (m_j) of positive integers such that for all $i \in \mathbb{N}$ the sequence $(x^{(i)}_{m_1}, x^{(i)}_{m_2}, x^{(i)}_{m_3}, \ldots)$ converges.

11F Probability

Let $A_1, A_2, \ldots, A_n$ be events in some probability space. State and prove the inclusion-exclusion formula for the probability $\mathbb{P}(\bigcup_{i=1}^n A_i)$. Show also that

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) \geqslant \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j).$$

Suppose now that $n \geqslant 2$ and that whenever $i \neq j$ we have $\mathbb{P}(A_i \cap A_j) \leqslant 1/n$. Show that there is a constant c independent of n such that $\sum_{i=1}^n \mathbb{P}(A_i) \leqslant c\sqrt{n}$.

12F Probability

(a) Let Z be a $N(0,1)$ random variable. Write down the probability density function (pdf) of Z, and verify that it is indeed a pdf. Find the moment generating function (mgf) $m_Z(\theta) = \mathbb{E}(e^{\theta Z})$ of Z and hence, or otherwise, verify that Z has mean 0 and variance 1.

(b) Let $(X_n)_{n\geqslant 1}$ be a sequence of IID $N(0,1)$ random variables. Let $S_n = \sum_{i=1}^n X_i$ and let $U_n = S_n/\sqrt{n}$. Find the distribution of U_n.

(c) Let $Y_n = X_n^2$. Find the mean μ and variance σ^2 of Y_1. Let $T_n = \sum_{i=1}^n Y_i$ and let $V_n = (T_n - n\mu)/\sigma\sqrt{n}$.

If $(W_n)_{n\geqslant 1}$ is a sequence of random variables and W is a random variable, what does it mean to say that $W_n \to W$ in distribution? State carefully the continuity theorem and use it to show that $V_n \to Z$ in distribution.

[You may **not** assume the central limit theorem.]

END OF PAPER