UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB 2020

List of Courses

Paper 2, Section I

2E Analysis and Topology

Let τ be the collection of subsets of $\mathbb{C}$ of the form $\mathbb{C}\backslash f^{-1}(0)$, where f is an arbitrary complex polynomial. Show that τ is a topology on $\mathbb{C}$.

Given topological spaces X and Y, define the *product topology* on $X \times Y$. Equip $\mathbb{C}^2$ with the topology given by the product of $(\mathbb{C}, \tau)$ with itself. Let g be an arbitrary two-variable complex polynomial. Is the subset $\mathbb{C}^2\backslash g^{-1}(0)$ always open in this topology? Justify your answer.

Paper 1, Section II

10E Analysis and Topology

State what it means for a function $f : \mathbb{R}^m \to \mathbb{R}^r$ to be *differentiable* at a point $x \in \mathbb{R}^m$, and define its derivative $f'(x)$.

Let $\mathcal{M}_n$ be the vector space of $n \times n$ real-valued matrices, and let $p : \mathcal{M}_n \to \mathcal{M}_n$ be given by $p(A) = A^3 - 3A - I$. Show that p is differentiable at any $A \in \mathcal{M}_n$, and calculate its derivative.

State the inverse function theorem for a function f. In the case when $f(0) = 0$ and $f'(0) = I$, prove the existence of a continuous local inverse function in a neighbourhood of 0. [The rest of the proof of the inverse function theorem is not expected.]

Show that there exists a positive ϵ such that there is a continuously differentiable function $q : D_\epsilon(I) \to \mathcal{M}_n$ such that $p \circ q = \mathrm{id}|_{D_\epsilon(I)}$. Is it possible to find a continuously differentiable inverse to p on the whole of $\mathcal{M}_n$? Justify your answer.

Paper 2, Section II

10E Analysis and Topology

Let $C[0,1]$ be the space of continuous real-valued functions on $[0,1]$, and let d_1, d_∞ be the metrics on it given by

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx \qquad \text{and} \qquad d_\infty(f,g) = \max_{x\in[0,1]}|f(x) - g(x)|.$$

Show that $\mathrm{id} : (C[0,1], d_\infty) \to (C[0,1], d_1)$ is a continuous map. Do d_1 and d_∞ induce the same topology on $C[0,1]$? Justify your answer.

Let d denote for any $m \in \mathbb{N}$ the uniform metric on $\mathbb{R}^m$: $d((x_i),(y_i)) = \max_i |x_i - y_i|$. Let $\mathcal{P}_n \subset C[0,1]$ be the subspace of real polynomials of degree at most n. Define a *Lipschitz map* between two metric spaces, and show that evaluation at a point gives a Lipschitz map $(C[0,1], d_\infty) \to (\mathbb{R}, d)$. Hence or otherwise find a bijection from $(\mathcal{P}_n, d_\infty)$ to $(\mathbb{R}^{n+1}, d)$ which is Lipschitz and has a Lipschitz inverse.

Let $\tilde{\mathcal{P}}_n \subset \mathcal{P}_n$ be the subset of polynomials with values in the range $[-1,1]$.

(i) Show that $(\tilde{\mathcal{P}}_n, d_\infty)$ is compact.

(ii) Show that d_1 and d_∞ induce the same topology on $\tilde{\mathcal{P}}_n$.

Any theorems that you use should be clearly stated.

[*You may use the fact that for distinct constants a_i, the following matrix is invertible:*

$$\begin{pmatrix} 1 & a_0 & a_0^2 & \dots & a_0^n \\ 1 & a_1 & a_1^2 & \dots & a_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a_n & a_n^2 & \dots & a_n^n \end{pmatrix}.]$$

Paper 1, Section I

3G Complex Analysis or Complex Methods

Let D be the open disc with centre $e^{2\pi i/6}$ and radius 1, and let L be the open lower half plane. Starting with a suitable Möbius map, find a conformal equivalence (or conformal bijection) of $D \cap L$ onto the open unit disc.

Paper 1, Section II

12G Complex Analysis or Complex Methods

Let $\ell(z)$ be an analytic branch of $\log z$ on a domain $D \subset \mathbb{C} \setminus \{0\}$. Write down an analytic branch of $z^{1/2}$ on D. Show that if $\psi_1(z)$ and $\psi_2(z)$ are two analytic branches of $z^{1/2}$ on D, then either $\psi_1(z) = \psi_2(z)$ for all $z \in D$ or $\psi_1(z) = -\psi_2(z)$ for all $z \in D$.

Describe the principal value or branch $\sigma_1(z)$ of $z^{1/2}$ on $D_1 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leqslant 0\}$. Describe a branch $\sigma_2(z)$ of $z^{1/2}$ on $D_2 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geqslant 0\}$.

Construct an analytic branch $\varphi(z)$ of $\sqrt{1-z^2}$ on $\mathbb{C} \setminus \{x \in \mathbb{R} : -1 \leqslant x \leqslant 1\}$ with $\varphi(2i) = \sqrt{5}$. [If you choose to use σ_1 and σ_2 in your construction, then you may assume without proof that they are analytic.]

Show that for $0 < |z| < 1$ we have $\varphi(1/z) = -i\sigma_1(1-z^2)/z$. Hence find the first three terms of the Laurent series of $\varphi(1/z)$ about 0.

Set $f(z) = \varphi(z)/(1+z^2)$ for $|z| > 1$ and $g(z) = f(1/z)/z^2$ for $0 < |z| < 1$. Compute the residue of g at 0 and use it to compute the integral

$$\int_{|z|=2} f(z)\, dz\,.$$

Paper 2, Section II

12B Complex Analysis or Complex Methods

For the function

$$f(z) = \frac{1}{z(z-2)}\,,$$

find the Laurent expansions

(i) about $z = 0$ in the annulus $0 < |z| < 2$,

(ii) about $z = 0$ in the annulus $2 < |z| < \infty$,

(iii) about $z = 1$ in the annulus $0 < |z-1| < 1$.

What is the nature of the singularity of f, if any, at $z = 0$, $z = \infty$ and $z = 1$?

Using an integral of f, or otherwise, evaluate

$$\int_0^{2\pi} \frac{2-\cos\theta}{5-4\cos\theta}\, d\theta\,.$$

Paper 2, Section I

5D Electromagnetism

Two concentric spherical shells with radii R and $2R$ carry fixed, uniformly distributed charges Q_1 and Q_2 respectively. Find the electric field and electric potential at all points in space. Calculate the total energy of the electric field.

Paper 1, Section II

16D Electromagnetism

Write down the electric potential due to a point charge Q at the origin.

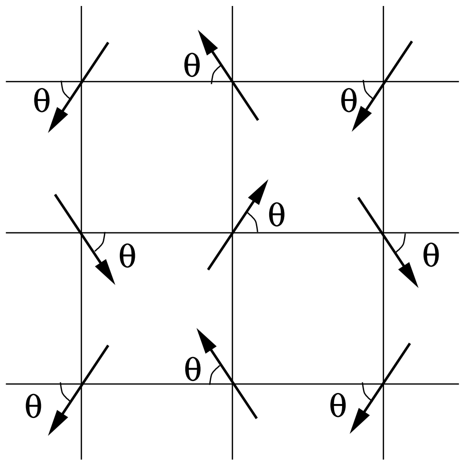
A dipole consists of a charge Q at the origin, and a charge $-Q$ at position $-\mathbf{d}$. Show that, at large distances, the electric potential due to such a dipole is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{x}}{|\mathbf{x}|^3},$$

where $\mathbf{p} = Q\mathbf{d}$ is the dipole moment. Hence show that the potential energy between two dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$, with separation $\mathbf{r}$, where $|\mathbf{r}| \gg |\mathbf{d}|$, is

$$U = \frac{1}{8\pi\epsilon_0}\left(\frac{\mathbf{p}_1\cdot\mathbf{p}_2}{r^3} - \frac{3(\mathbf{p}_1\cdot\mathbf{r})(\mathbf{p}_2\cdot\mathbf{r})}{r^5}\right).$$

Dipoles are arranged on an infinite chessboard so that they make an angle θ with the horizontal in an alternating pattern as shown in the figure. Compute the energy between a given dipole and its four nearest neighbours, and show that this is independent of θ.



Paper 2, Section II

15D Electromagnetism

(a) A surface current $\mathbf{K} = K\mathbf{e}_x$, with K a constant and $\mathbf{e}_x$ the unit vector in the x-direction, lies in the plane $z = 0$. Use Ampère's law to determine the magnetic field above and below the plane. Confirm that the magnetic field is discontinuous across the surface, with the discontinuity given by

$$\lim_{z\to 0^+} \mathbf{e}_z \times \mathbf{B} - \lim_{z\to 0^-} \mathbf{e}_z \times \mathbf{B} = \mu_0 \mathbf{K}\,,$$

where $\mathbf{e}_z$ is the unit vector in the z-direction.

(b) A surface current $\mathbf{K}$ flows radially in the $z = 0$ plane, resulting in a pile-up of charge Q at the origin, with $dQ/dt = I$, where I is a constant.

Write down the electric field $\mathbf{E}$ due to the charge at the origin, and hence the displacement current $\epsilon_0 \partial\mathbf{E}/\partial t$.

Confirm that, away from the plane and for $\theta < \pi/2$, the magnetic field due to the displacement current is given by

$$\mathbf{B}(r,\theta) = \frac{\mu_0 I}{4\pi r} \tan\left(\frac{\theta}{2}\right) \mathbf{e}_\phi\,,$$

where (r,θ,ϕ) are the usual spherical polar coordinates. [*Hint: Use Stokes' theorem applied to a spherical cap that subtends an angle* θ.]

Paper 2, Section I

6C Fluid Dynamics

Incompressible fluid of constant viscosity μ is confined to the region $0 < y < h$ between two parallel rigid plates. Consider two parallel viscous flows: flow A is driven by the motion of one plate in the x-direction with the other plate at rest; flow B is driven by a constant pressure gradient in the x-direction with both plates at rest. The velocity mid-way between the plates is the same for both flows.

The viscous friction in these flows is known to produce heat locally at a rate

$$Q = \mu \left(\frac{\partial u}{\partial y} \right)^2$$

per unit volume, where u is the x-component of the velocity. Determine the ratio of the total rate of heat production in flow A to that in flow B.

Paper 1, Section II

17C Fluid Dynamics

Steady two-dimensional potential flow of an incompressible fluid is confined to the wedge $0 < \theta < \alpha$, where (r, θ) are polar coordinates centred on the vertex of the wedge and $0 < \alpha < \pi$.

(a) Show that a velocity potential ϕ of the form

$$\phi(r, \theta) = A r^{\gamma} \cos(\lambda \theta),$$

where A, γ and λ are positive constants, satisfies the condition of incompressible flow, provided that γ and λ satisfy a certain relation to be determined.

Assuming that u_θ, the θ-component of velocity, does not change sign within the wedge, determine the values of γ and λ by using the boundary conditions.

(b) Calculate the shape of the streamlines of this flow, labelling them by the distance $r_{\min}$ of closest approach to the vertex. Sketch the streamlines.

(c) Show that the speed $|\mathbf{u}|$ and pressure p are independent of θ. Assuming that at some radius $r = r_0$ the speed and pressure are u_0 and p_0, respectively, find the pressure difference in the flow between the vertex of the wedge and r_0.

[*Hint: In polar coordinates* (r, θ),

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta} \right) \quad \textit{and} \quad \nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(r F_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$$

for a scalar f *and a vector* $\mathbf{F} = (F_r, F_\theta)$.]

Paper 2, Section II

16C Fluid Dynamics

A vertical cylindrical container of radius R is partly filled with fluid of constant density to depth h. The free surface is perturbed so that the fluid occupies the region

$$0 < r < R, \quad -h < z < \zeta(r,\theta,t),$$

where (r,θ,z) are cylindrical coordinates and ζ is the perturbed height of the free surface. For small perturbations, a linearised description of surface waves in the cylinder yields the following system of equations for ζ and the velocity potential ϕ:

$$\begin{aligned}
\nabla^2\phi &= 0\,, \qquad 0<r<R\,, \quad -h<z<0\,, && (1)\\
\frac{\partial\phi}{\partial t} + g\zeta &= 0 \quad \text{on} \quad z=0\,, && (2)\\
\frac{\partial\zeta}{\partial t} - \frac{\partial\phi}{\partial z} &= 0 \quad \text{on} \quad z=0\,, && (3)\\
\frac{\partial\phi}{\partial z} &= 0 \quad \text{on} \quad z=-h\,, && (4)\\
\frac{\partial\phi}{\partial r} &= 0 \quad \text{on} \quad r=R\,. && (5)
\end{aligned}$$

(a) Describe briefly the physical meaning of each equation.

(b) Consider axisymmetric normal modes of the form

$$\phi = \mathrm{Re}\left(\hat{\phi}(r,z)e^{-i\sigma t}\right), \qquad \zeta = \mathrm{Re}\left(\hat{\zeta}(r)e^{-i\sigma t}\right) .$$

Show that the system of equations (1)–(5) admits a solution for $\hat{\phi}$ of the form

$$\hat{\phi}(r,z) = AJ_0(k_n r)Z(z)\,,$$

where A is an arbitrary amplitude, $J_0(x)$ satisfies the equation

$$\frac{d^2J_0}{dx^2} + \frac{1}{x}\frac{dJ_0}{dx} + J_0 = 0\,,$$

the wavenumber k_n, $n=1,2,\dots$ is such that $x_n = k_nR$ is one of the zeros of the function dJ_0/dx, and the function $Z(z)$ should be determined explicitly.

(c) Show that the frequency σ_n of the n-th mode is given by

$$\sigma_n^2 = \frac{g}{h}\Psi(k_nh)\,,$$

where the function $\Psi(x)$ is to be determined.

[*Hint: In cylindrical coordinates* (r,θ,z),

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\,.]$$

Paper 1, Section I

2E Geometry

Define the *Gauss map* of a smooth embedded surface. Consider the surface of revolution S with points

$$\begin{pmatrix}(2+\cos v)\cos u\\(2+\cos v)\sin u\\\sin v\end{pmatrix}\in\mathbb{R}^3$$

for $u, v \in [0, 2\pi]$. Let f be the Gauss map of S. Describe f on the $\{y = 0\}$ cross-section of S, and use this to write down an explicit formula for f.

Let U be the upper hemisphere of the 2-sphere S^2, and K the Gauss curvature of S. Calculate $\int_{f^{-1}(U)} K\,dA$.

Paper 1, Section II

11E Geometry

Let $\mathcal{C}$ be the curve in the (x, z)-plane defined by the equation

$$(x^2-1)^2+(z^2-1)^2=5\,.$$

Sketch $\mathcal{C}$, taking care with inflection points.

Let S be the surface of revolution in $\mathbb{R}^3$ given by spinning $\mathcal{C}$ about the z-axis. Write down an equation defining S. Stating any result you use, show that S is a smooth embedded surface.

Let r be the radial coordinate on the (x, y)-plane. Show that the Gauss curvature of S vanishes when $r = 1$. Are these the only points at which the Gauss curvature of S vanishes? Briefly justify your answer.

Paper 2, Section II

11F Geometry

Let $H = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the hyperbolic half-plane with the metric $g_H = (dx^2 + dy^2)/y^2$. Define the *length* of a continuously differentiable curve in H with respect to g_H.

What are the *hyperbolic lines* in H? Show that for any two distinct points z, w in H, the infimum $\rho(z, w)$ of the lengths (with respect to g_H) of curves from z to w is attained by the segment $[z, w]$ of the hyperbolic line with an appropriate parameterisation.

The 'hyperbolic Pythagoras theorem' asserts that if a hyperbolic triangle ABC has angle $\pi/2$ at C then

$$\cosh c = \cosh a \cosh b\,,$$

where a, b, c are the lengths of the sides BC, AC, AB, respectively.

Let l and m be two hyperbolic lines in H such that

$$\inf\{\rho(z, w) : z \in l,\ w \in m\} = d > 0.$$

Prove that the distance d is attained by the points of intersection with a hyperbolic line h that meets each of l, m orthogonally. Give an example of two hyperbolic lines l and m such that the infimum of $\rho(z, w)$ is *not* attained by any $z \in l$, $w \in m$.

[*You may assume that every Möbius transformation that maps H onto itself is an isometry of g_H.*]

Paper 2, Section I

1G Groups Rings and Modules

Assume a group G acts transitively on a set Ω and that the size of Ω is a prime number. Let H be a normal subgroup of G that acts non-trivially on Ω.

Show that any two H-orbits of Ω have the same size. Deduce that the action of H on Ω is transitive.

Let $\alpha \in \Omega$ and let G_α denote the stabiliser of α in G. Show that if $H \cap G_\alpha$ is trivial, then there is a bijection $\theta\colon H \to \Omega$ under which the action of G_α on H by conjugation corresponds to the action of G_α on Ω.

Paper 1, Section II

9G Groups Rings and Modules

State the structure theorem for a finitely generated module M over a Euclidean domain R in terms of invariant factors.

Let V be a finite-dimensional vector space over a field F and let $\alpha\colon V \to V$ be a linear map. Let V_α denote the $F[X]$-module V with X acting as α. Apply the structure theorem to V_α to show the existence of a basis of V with respect to which α has the rational canonical form. Prove that the minimal polynomial and the characteristic polynomial of α can be expressed in terms of the invariant factors. [*Hint: For the characteristic polynomial apply suitable row operations.*] Deduce the Cayley–Hamilton theorem for α.

Now assume that α has matrix (a_{ij}) with respect to the basis $v_1, \ldots, v_n$ of V. Let M be the free $F[X]$-module of rank n with free basis $m_1, \ldots, m_n$ and let $\theta\colon M \to V_\alpha$ be the unique homomorphism with $\theta(m_i) = v_i$ for $1 \leqslant i \leqslant n$. Using the fact, which you need not prove, that $\ker\theta$ is generated by the elements $Xm_i - \sum_{j=1}^n a_{ji}\, m_j$, $1 \leqslant i \leqslant n$, find the invariant factors of V_α in the case that $V = \mathbb{R}^3$ and α is represented by the real matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$$

with respect to the standard basis.

Paper 2, Section II

9G Groups Rings and Modules

State Gauss' lemma. State and prove Eisenstein's criterion.

Define the notion of an *algebraic integer*. Show that if α is an algebraic integer, then $\{f \in \mathbb{Z}[X] : f(\alpha) = 0\}$ is a principal ideal generated by a monic, irreducible polynomial.

Let $f = X^4 + 2X^3 - 3X^2 - 4X - 11$. Show that $\mathbb{Q}[X]/(f)$ is a field. Show that $\mathbb{Z}[X]/(f)$ is an integral domain, but not a field. Justify your answers.

Paper 1, Section I

1F Linear Algebra

Define what it means for two $n \times n$ matrices A and B to be *similar*. Define the *Jordan normal form* of a matrix.

Determine whether the matrices

$$A = \begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{pmatrix}$$

are similar, carefully stating any theorem you use.

Paper 1, Section II

8F Linear Algebra

Let $\mathcal{M}_n$ denote the vector space of $n \times n$ matrices over a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. What is the *rank* $r(A)$ of a matrix $A \in \mathcal{M}_n$?

Show, stating accurately any preliminary results that you require, that $r(A) = n$ if and only if A is non-singular, i.e. $\det A \neq 0$.

Does $\mathcal{M}_n$ have a basis consisting of non-singular matrices? Justify your answer.

Suppose that an $n \times n$ matrix A is non-singular and every entry of A is either 0 or 1. Let c_n be the largest possible number of 1's in such an A. Show that $c_n \leqslant n^2 - n + 1$. Is this bound attained? Justify your answer.

[*Standard properties of the adjugate matrix can be assumed, if accurately stated.*]

Paper 2, Section II

8F Linear Algebra

Let V be a finite-dimensional vector space over a field. Show that an endomorphism α of V is idempotent, i.e. $\alpha^2 = \alpha$, if and only if α is a projection onto its image.

Determine whether the following statements are true or false, giving a proof or counterexample as appropriate:

(i) If $\alpha^3 = \alpha^2$, then α is idempotent.

(ii) The condition $\alpha(1-\alpha)^2 = 0$ is equivalent to α being idempotent.

(iii) If α and β are idempotent and such that $\alpha + \beta$ is also idempotent, then $\alpha\beta = 0$.

(iv) If α and β are idempotent and $\alpha\beta = 0$, then $\alpha + \beta$ is also idempotent.

Paper 2, Section I

7H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain with state space $\{1,2\}$ and transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

where $\alpha, \beta \in (0,1]$. Compute $\mathbb{P}(X_n = 1 | X_0 = 1)$. Find the value of $\mathbb{P}(X_n = 1 | X_0 = 2)$.

Paper 1, Section II

20H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain with transition matrix P. What is a *stopping time* of $(X_n)_{n\geqslant 0}$? What is the *strong Markov property*?

A porter is trying to apprehend a student who is walking along a long narrow path at night. Being unaware of the porter, the student's location Y_n at time $n \geqslant 0$ evolves as a simple symmetric random walk on $\mathbb{Z}$. The porter's initial location Z_0 is $2m$ units to the right of the student, so $Z_0 - Y_0 = 2m$ where $m \geqslant 1$. The future locations Z_{n+1} of the porter evolve as follows: The porter moves to the left (so $Z_{n+1} = Z_n - 1$) with probability $q \in (\frac{1}{2}, 1)$, and to the right with probability $1-q$ whenever $Z_n - Y_n > 2$. When $Z_n - Y_n = 2$, the porter's probability of moving left changes to $r \in (0,1)$, and the probability of moving right is $1-r$.

(a) By setting up an appropriate Markov chain, show that for $m \geqslant 2$, the expected time for the porter to be a distance $2(m-1)$ away from the student is $2/(2q-1)$.

(b) Show that the expected time for the porter to catch the student, i.e. for their locations to coincide, is

$$\frac{2}{r} + \left(m + \frac{1}{r} - 2\right)\frac{2}{2q-1}.$$

[*You may use without proof the fact that the time for the porter to catch the student is finite with probability* 1 *for any* $m \geqslant 1$.]

Paper 2, Section I

4B Methods

Find the Fourier transform of the function

$$f(x) = \begin{cases} A\,, & |x| \leqslant 1 \\ 0\,, & |x| > 1\,. \end{cases}$$

Determine the convolution of the function $f(x)$ with itself.

State the convolution theorem for Fourier transforms. Using it, or otherwise, determine the Fourier transform of the function

$$g(x) = \begin{cases} B(2 - |x|)\,, & |x| \leqslant 2 \\ 0\,, & |x| > 2\,. \end{cases}$$

Paper 1, Section II

14B Methods

Consider the equation

$$\nabla^2 \phi = \delta(x) g(y) \qquad (*)$$

on the two-dimensional strip $-\infty < x < \infty$, $0 \leqslant y \leqslant a$, where $\delta(x)$ is the delta function and $g(y)$ is a smooth function satisfying $g(0) = g(a) = 0$. $\phi(x, y)$ satisfies the boundary conditions $\phi(x, 0) = \phi(x, a) = 0$ and $\lim_{x \to \pm\infty} \phi(x, y) = 0$. By using solutions of Laplace's equation for $x < 0$ and $x > 0$, matched suitably at $x = 0$, find the solution of $(*)$ in terms of Fourier coefficients of $g(y)$.

Find the solution of $(*)$ in the limiting case $g(y) = \delta(y - c)$, where $0 < c < a$, and hence determine the Green's function $\phi(x, y)$ in the strip, satisfying

$$\nabla^2 \phi = \delta(x - b)\delta(y - c)$$

and the same boundary conditions as before.

Paper 2, Section II

13A Methods

(i) The solution to the equation

$$\frac{d}{dx}\Big(x\frac{dF}{dx}\Big) + \alpha^2 xF = 0$$

that is regular at the origin is $F(x) = CJ_0(\alpha x)$, where α is a real, positive parameter, J_0 is a Bessel function, and C is an arbitrary constant. The Bessel function has infinitely many zeros: $J_0(\gamma_k) = 0$ with $\gamma_k > 0$, for $k = 1, 2, \ldots$. Show that

$$\int_0^1 J_0(\alpha x)\, J_0(\beta x)\, x\, dx = \frac{\beta J_0(\alpha)J_0'(\beta) - \alpha J_0(\beta)J_0'(\alpha)}{\alpha^2 - \beta^2}\,, \quad \alpha \neq \beta\,,$$

(where α and β are real and positive) and deduce that

$$\int_0^1 J_0(\gamma_k x)\, J_0(\gamma_\ell x)\, x\, dx = 0\,, \quad k \neq \ell\,; \qquad \int_0^1 (J_0(\gamma_k x))^2\, x\, dx = \frac{1}{2}(J_0'(\gamma_k))^2\,.$$

[*Hint: For the second identity, consider* $\alpha = \gamma_k$ *and* $\beta = \gamma_k + \epsilon$ *with* ϵ *small.*]

(ii) The displacement $z(r,t)$ of the membrane of a circular drum of unit radius obeys

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial z}{\partial r}\Big) = \frac{\partial^2 z}{\partial t^2}\,, \qquad z(1,t) = 0\,,$$

where r is the radial coordinate on the membrane surface, t is time (in certain units), and the displacement is assumed to have no angular dependence. At $t = 0$ the drum is struck, so that

$$z(r,0) = 0\,, \qquad \frac{\partial z}{\partial t}(r,0) = \begin{cases} U\,, & r < b \\ 0\,, & r > b \end{cases}$$

where U and $b < 1$ are constants. Show that the subsequent motion is given by

$$z(r,t) = \sum_{k=1}^{\infty} C_k\, J_0(\gamma_k r)\, \sin(\gamma_k t) \qquad \text{where} \qquad C_k = -2bU\frac{J_0'(\gamma_k b)}{\gamma_k^2 (J_0'(\gamma_k))^2}\,.$$

Paper 1, Section I

5C Numerical Analysis

(a) Find an LU factorisation of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 2 & 12 \\ 0 & 5 & 7 & 32 \\ 3 & -1 & -1 & -10 \end{bmatrix},$$

where the diagonal elements of L are $L_{11} = L_{44} = 1$, $L_{22} = L_{33} = 2$.

(b) Use this factorisation to solve the linear system $A\mathbf{x} = \mathbf{b}$, where

$$\mathbf{b} = \begin{bmatrix} -3 \\ -12 \\ -30 \\ 13 \end{bmatrix}.$$

Paper 1, Section II

18C Numerical Analysis

(a) Given a set of $n+1$ distinct real points $x_0, x_1, \ldots, x_n$ and real numbers $f_0, f_1, \ldots, f_n$, show that the interpolating polynomial $p_n \in \mathbb{P}_n[x]$, $p_n(x_i) = f_i$, can be written in the form

$$p_n(x) = \sum_{k=0}^{n} a_k \prod_{j=0, j\neq k}^{n} \frac{x - x_j}{x_k - x_j}\,, \qquad x \in \mathbb{R}\,,$$

where the coefficients a_k are to be determined.

(b) Consider the approximation of the integral of a function $f \in C[a,b]$ by a finite sum,

$$\int_a^b f(x)\,dx \approx \sum_{k=0}^{s-1} w_k f(c_k)\,, \tag{1}$$

where the weights $w_0, \ldots, w_{s-1}$ and nodes $c_0, \ldots, c_{s-1} \in [a,b]$ are independent of f. Derive the expressions for the weights w_k that make the approximation (1) exact for f being any polynomial of degree $s-1$, i.e. $f \in \mathbb{P}_{s-1}[x]$.

Show that by choosing $c_0, \ldots, c_{s-1}$ to be zeros of the polynomial $q_s(x)$ of degree s, one of a sequence of orthogonal polynomials defined with respect to the scalar product

$$\langle u, v \rangle = \int_a^b u(x)v(x)dx\,, \tag{2}$$

the approximation (1) becomes exact for $f \in \mathbb{P}_{2s-1}[x]$ (i.e. for all polynomials of degree $2s-1$).

(c) On the interval $[a,b] = [-1,1]$ the scalar product (2) generates orthogonal polynomials given by

$$q_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n, \quad n = 0, 1, 2, \ldots\,.$$

Find the values of the nodes c_k for which the approximation (1) is exact for all polynomials of degree 7 (i.e. $f \in \mathbb{P}_7[x]$) but no higher.

Paper 2, Section II

17C Numerical Analysis

Consider a multistep method for numerical solution of the differential equation $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$:

$$\mathbf{y}_{n+2} - \mathbf{y}_{n+1} = h\,[\,(1+\alpha)\mathbf{f}(t_{n+2}, \mathbf{y}_{n+2}) + \beta\mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}) - (\alpha+\beta)\mathbf{f}(t_n, \mathbf{y}_n)\,]\,, \qquad (*)$$

where $n = 0, 1, \ldots$, and α and β are constants.

(a) Define the *order* of a method for numerically solving an ODE.

(b) Show that in general an explicit method of the form $(*)$ has order 1. Determine the values of α and β for which this multistep method is of order 3.

(c) Show that the multistep method $(*)$ is convergent.

Paper 1, Section I

7H Optimisation

Solve the following optimisation problem using the simplex algorithm:

$$\begin{array}{rl} \text{maximise} & x_1 + x_2 \\ \text{subject to} & |x_1 - 2x_2| \leqslant 2, \\ & 4x_1 + x_2 \leqslant 4, \qquad x_1, x_2 \geqslant 0. \end{array}$$

Suppose the constraints above are now replaced by $|x_1 - 2x_2| \leqslant 2 + \epsilon_1$ and $4x_1 + x_2 \leqslant 4 + \epsilon_2$. Give an expression for the maximum objective value that is valid for all sufficiently small non-zero ϵ_1 and ϵ_2.

Paper 2, Section II

19H Optimisation

State and prove the Lagrangian sufficiency theorem.

Solve, using the Lagrangian method, the optimisation problem

$$\begin{array}{rl} \text{maximise} & x + y + 2a\sqrt{1+z} \\ \text{subject to} & x + \frac{1}{2}y^2 + z = b\,, \\ & x, z \geqslant 0\,, \end{array}$$

where the constants a and b satisfy $a \geqslant 1$ and $b \geqslant 1/2$.

[*You need not prove that your solution is unique.*]

Paper 1, Section I

4A Quantum Mechanics

Define what it means for an operator Q to be *hermitian* and briefly explain the significance of this definition in quantum mechanics.

Define the *uncertainty* $(\Delta Q)_\psi$ of Q in a state ψ. If P is also a hermitian operator, show by considering the state $(Q + i\lambda P)\psi$, where λ is a real number, that

$$\langle Q^2\rangle_\psi \, \langle P^2\rangle_\psi \;\geqslant\; \frac{1}{4}\,|\,\langle i[Q,P]\rangle_\psi\,|^2 \,.$$

Hence deduce that

$$(\Delta Q)_\psi \, (\Delta P)_\psi \;\geqslant\; \frac{1}{2}|\,\langle i[Q,P]\rangle_\psi\,| \,.$$

Give a physical interpretation of this result.

Paper 1, Section II

15A Quantum Mechanics

Consider a quantum system with Hamiltonian H and wavefunction Ψ obeying the time-dependent Schrödinger equation. Show that if Ψ is a *stationary state* then $\langle Q\rangle_\Psi$ is independent of time, if the observable Q is independent of time.

A particle of mass m is confined to the interval $0 \leqslant x \leqslant a$ by infinite potential barriers, but moves freely otherwise. Let $\Psi(x,t)$ be the normalised wavefunction for the particle at time t, with

$$\Psi(x,0) = c_1\psi_1(x) + c_2\psi_2(x)$$

where

$$\psi_1(x) \;=\; \left(\frac{2}{a}\right)^{1/2} \sin\frac{\pi x}{a}\,, \qquad \psi_2(x) \;=\; \left(\frac{2}{a}\right)^{1/2} \sin\frac{2\pi x}{a}$$

and c_1, c_2 are complex constants. If the energy of the particle is measured at time t, what are the possible results, and what is the probability for each result to be obtained? Give brief justifications of your answers.

Calculate $\langle \hat{x}\rangle_\Psi$ at time t and show that the result oscillates with a frequency ω, to be determined. Show in addition that

$$\left|\langle \hat{x}\rangle_\Psi - \frac{a}{2}\right| \;\leqslant\; \frac{16a}{9\pi^2}\,.$$

Paper 2, Section II

14A Quantum Mechanics

(a) The potential $V(x)$ for a particle of mass m in one dimension is such that $V \to 0$ rapidly as $x \to \pm\infty$. Let $\psi(x)$ be a wavefunction for the particle satisfying the time-independent Schrödinger equation with energy E.

Suppose ψ has the asymptotic behaviour

$$\psi(x) \sim Ae^{ikx} + Be^{-ikx} \quad (x \to -\infty)\,, \qquad \psi(x) \sim Ce^{ikx} \quad (x \to +\infty)\,,$$

where A, B, C are complex coefficients. Explain, in outline, how the probability current $j(x)$ is used in the interpretation of such a solution as a scattering process and how the transmission and reflection probabilities $P_{\rm tr}$ and $P_{\rm ref}$ are found.

Now suppose instead that $\psi(x)$ is a bound state solution. Write down the asymptotic behaviour in this case, relating an appropriate parameter to the energy E.

(b) Consider the potential

$$V(x) \;=\; -\frac{\hbar^2}{m}\,\frac{a^2}{\cosh^2 ax}$$

where a is a real, positive constant. Show that

$$\psi(x) \;=\; Ne^{ikx}(a\tanh ax \;-\; ik)\,,$$

where N is a complex coefficient, is a solution of the time-independent Schrödinger equation for any real k and find the energy E. Show that ψ represents a scattering process for which $P_{\rm ref} = 0$, and find $P_{\rm tr}$ explicitly.

Now let $k = i\lambda$ in the formula for ψ above. Show that this defines a bound state if a certain real positive value of λ is chosen and find the energy of this solution.

Paper 1, Section I

6H Statistics

Suppose $X_1, \ldots, X_n$ are independent with distribution $N(\mu, 1)$. Suppose a prior $\mu \sim N(\theta, \tau^{-2})$ is placed on the unknown parameter μ for some given deterministic $\theta \in \mathbb{R}$ and $\tau > 0$. Derive the posterior mean.

Find an expression for the mean squared error of this posterior mean when $\theta = 0$.

Paper 1, Section II

19H Statistics

Let $X_1, \ldots, X_n$ be i.i.d. $U[0, 2\theta]$ random variables, where $\theta > 0$ is unknown.

(a) Derive the maximum likelihood estimator $\hat{\theta}$ of θ.

(b) What is a *sufficient statistic*? What is a *minimal sufficient statistic*? Is $\hat{\theta}$ sufficient for θ? Is it minimal sufficient? Answer the same questions for the sample mean $\tilde{\theta} := \sum_{i=1}^{n} X_i/n$. Briefly justify your answers.

[You may use any result from the course provided it is stated clearly.]

(c) Show that the mean squared errors of $\hat{\theta}$ and $\tilde{\theta}$ are respectively

$$\frac{2\theta^2}{(n+1)(n+2)} \qquad \text{and} \qquad \frac{\theta^2}{3n}.$$

(d) Show that for each $t \in \mathbb{R}$, $\lim_{n\to\infty} \mathbb{P}(n(1 - \hat{\theta}/\theta) \geqslant t) = h(t)$ for a function h you should specify. Give, with justification, an approximate $1 - \alpha$ confidence interval for θ whose expected length is

$$\left(\frac{n\theta}{n+1}\right)\left(\frac{\log(1/\alpha)}{n - \log(1/\alpha)}\right).$$

[*Hint:* $\lim_{n\to\infty}(1 - \frac{t}{n})^n = e^{-t}$ *for all* $t \in \mathbb{R}$.]

Paper 2, Section II

18H Statistics

Consider the general linear model $Y = X\beta^0 + \varepsilon$ where X is a known $n \times p$ design matrix with $p \geqslant 2$, $\beta^0 \in \mathbb{R}^p$ is an unknown vector of parameters, and $\varepsilon \in \mathbb{R}^n$ is a vector of stochastic errors with $\mathbb{E}(\varepsilon_i) = 0$, $\operatorname{var}(\varepsilon_i) = \sigma^2 > 0$ and $\operatorname{cov}(\varepsilon_i, \varepsilon_j) = 0$ for all $i, j = 1, \ldots, n$ with $i \neq j$. Suppose X has full column rank.

(a) Write down the least squares estimate $\hat{\beta}$ of β^0 and show that it minimises the least squares objective $S(\beta) = \|Y - X\beta\|^2$ over $\beta \in \mathbb{R}^p$.

(b) Write down the variance–covariance matrix $\operatorname{cov}(\hat{\beta})$.

(c) Let $\tilde{\beta} \in \mathbb{R}^p$ minimise $S(\beta)$ over $\beta \in \mathbb{R}^p$ subject to $\beta_p = 0$. Let Z be the $n \times (p-1)$ submatrix of X that excludes the final column. Write down $\operatorname{cov}(\tilde{\beta})$.

(d) Let P and P_0 be $n \times n$ orthogonal projections onto the column spaces of X and Z respectively. Show that for all $u \in \mathbb{R}^n$, $u^T P u \geqslant u^T P_0 u$.

(e) Show that for all $x \in \mathbb{R}^p$,

$$\operatorname{var}(x^T \tilde{\beta}) \leqslant \operatorname{var}(x^T \hat{\beta}).$$

[*Hint: Argue that $x = X^T u$ for some $u \in \mathbb{R}^n$.*]

Paper 2, Section I

3D Variational Principles

Find the stationary points of the function $\phi = xyz$ subject to the constraint $x + a^2y^2 + z^2 = b^2$, with $a, b > 0$. What are the maximum and minimum values attained by ϕ, subject to this constraint, if we further restrict to $x \geqslant 0$?

Paper 1, Section II

13D Variational Principles

A motion sensor sits at the origin, in the middle of a field. The probability that you are detected as you sneak from one point to another along a path $\mathbf{x}(t) : 0 \leqslant t \leqslant T$ is

$$P[\mathbf{x}(t)] = \lambda \int_0^T \frac{v(t)}{r(t)}\, dt\,,$$

where λ is a positive constant, $r(t)$ is your distance to the sensor, and $v(t)$ is your speed. (If $P[\mathbf{x}(t)] \geqslant 1$ for some path then you are detected with certainty.)

You start at point $(x, y) = (A, 0)$, where $A > 0$. Your mission is to reach the point $(x, y) = (B\cos\alpha, B\sin\alpha)$, where $B > 0$. What path should you take to minimise the chance of detection? Should you tiptoe or should you run?

A new and improved sensor detects you with probability

$$\tilde{P}[\mathbf{x}(t)] = \lambda \int_0^T \frac{v(t)^2}{r(t)}\, dt\,.$$

Show that the optimal path now satisfies the equation

$$\left(\frac{dr}{dt}\right)^2 = Er - h^2$$

for some constants E and h that you should identify.