UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday, 7 June, 2019 9:00 am to 12:00 pm MAT2

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *. . . ,* ***J*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet <u>must</u> bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet
Script paper
Rough paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Show that the product

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{-1}$$

and the series

$$\sum_{p \text{ prime}} \frac{1}{p}$$

are both divergent.

2H Topics in Analysis

Show that π is irrational. [*Hint: consider the functions* $f_n : [0, \pi] \to \mathbb{R}$ *given by* $f_n(x) = x^n(\pi - x)^n \sin x$.]

3G Coding and Cryptography

(a) Describe *Diffie-Hellman key exchange.* Why is it believed to be a secure system?

(b) Consider the following authentication procedure. Alice chooses public key N for the Rabin–Williams cryptosystem. To be sure we are in communication with Alice we send her a 'random item' $r \equiv m^2 \bmod N$. On receiving r, Alice proceeds to decode using her knowledge of the factorisation of N and finds a square root m_1 of r. She returns m_1 to us and we check $r \equiv m_1^2 \bmod N$. Is this authentication procedure secure? Justify your answer.

4H Automata and Formal Languages

(a) Which of the following are regular languages? Justify your answers.

(i) $\{w^n \mid w \in \{a,b\}^*,\ n \geqslant 2\}$.

(ii) $\{w \in \{a,b,c\}^* \mid w \text{ contains an odd number of } b\text{'s and an even number of } c\text{'s}\}$.

(iii) $\{w \in \{0,1\}^* \mid w \text{ contains no more than 7 consecutive 0's}\}$.

(b) Consider the language L over alphabet $\{a,b\}$ defined via

$$L := \{wab^n \mid w \in \{a,b\}^*,\ n \in \mathbb{K}\} \cup \{b\}^*.$$

Show that L satisfies the pumping lemma for regular languages but is not a regular language itself.

5J Statistical Modelling

In a normal linear model with design matrix $X \in \mathbb{R}^{n\times p}$, output variables $y \in \mathbb{R}^n$ and parameters $\beta \in \mathbb{R}^p$ and $\sigma^2 > 0$, define a $(1-\alpha)$-*level prediction interval* for a new observation with input variables $x^* \in \mathbb{R}^p$. Derive an explicit formula for the interval, proving that it satisfies the properties required by the definition. [*You may assume that the maximum likelihood estimator $\hat{\beta}$ is independent of $\sigma^{-2}\|y - X\hat{\beta}\|_2^2$, which has a χ^2_{n-p} distribution.*]

6C Mathematical Biology

(a) A variant of the classic logistic population model is given by:

$$\frac{dx(t)}{dt} = \alpha\left[x(t) - x(t-T)^2\right]$$

where $\alpha, T > 0$.

Show that for small T, the constant solution $x(t) = 1$ is stable.

Allow T to increase. Express in terms of α the value of T at which the constant solution $x(t) = 1$ loses stability.

(b) Another variant of the logistic model is given by this equation:

$$\frac{dx(t)}{dt} = \alpha\, x(t-T)\,[1 - x(t)]$$

where $\alpha, T > 0$. When is the constant solution $x(t) = 1$ stable for this model?

7A Further Complex Methods

A single-valued function $\text{Arcsin}(z)$ can be defined, for $0 \leqslant \arg z < 2\pi$, by means of an integral as:

$$\text{Arcsin}(z) = \int_0^z \frac{dt}{(1-t^2)^{1/2}}\,. \qquad (\dagger)$$

(a) Choose a suitable branch-cut with the integrand taking a value $+1$ at the origin on the upper side of the cut, i.e. at $t = 0^+$, and describe suitable paths of integration in the two cases $0 \leqslant \arg z \leqslant \pi$ and $\pi < \arg z < 2\pi$.

(b) Construct the multivalued function $\arcsin(z)$ by analytic continuation.

(c) Express $\arcsin\left(e^{2\pi i} z\right)$ in terms of $\text{Arcsin}(z)$ and deduce the periodicity property of $\sin(z)$.

8E Classical Dynamics

(a) The angular momentum of a rigid body about its centre of mass is conserved. Derive Euler's equations,

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \,, \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \,, \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \,, \end{aligned}$$

explaining the meaning of the quantities appearing in the equations.

(b) Show that there are two independent conserved quantities that are quadratic functions of $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, and give a physical interpretation of them.

(c) Derive a linear approximation to Euler's equations that applies when $|\omega_1| \ll |\omega_3|$ and $|\omega_2| \ll |\omega_3|$. Use this to determine the stability of rotation about each of the three principal axes of an asymmetric top.

9B Cosmology

Derive the relation between the neutrino temperature T_ν and the photon temperature T_γ at a time long after electrons and positrons have become non-relativistic.

[*In this question you may work in units of the speed of light, so that $c = 1$. You may also use without derivation the following formulae. The energy density ϵ_a and pressure P_a for a single relativistic species a with a number g_a of degenerate states at temperature T are given by*

$$\epsilon_a = \frac{4\pi g_a}{h^3}\int \frac{p^3 dp}{e^{p/(k_B T)} \mp 1}\,, \qquad P_a = \frac{4\pi g_a}{3h^3}\int \frac{p^3 dp}{e^{p/(k_B T)} \mp 1}\,,$$

where k_B is Boltzmann's constant, h is Planck's constant, and the minus or plus depends on whether the particle is a boson or a fermion respectively. For each species a, the entropy density s_a at temperature T_a is given by,

$$s_a = \frac{\epsilon_a + P_a}{k_B T_a}\,.$$

The effective total number g_* of relativistic species is defined in terms of the numbers of bosonic and fermionic particles in the theory as,

$$g_* = \sum_{bosons} g_{bosons} + \frac{7}{8}\sum_{fermions} g_{fermions}\,,$$

with the specific values $g_\gamma = g_{e^+} = g_{e^-} = 2$ for photons, positrons and electrons.]

10D Quantum Information and Computation

(a) Define the *order* of α mod N for coprime integers α and N with $\alpha < N$. Explain briefly how knowledge of this order can be used to provide a factor of N, stating conditions on α and its order that must be satisfied.

(b) Shor's algorithm for factoring N starts by choosing $\alpha < N$ coprime. Describe the subsequent steps of a single run of Shor's algorithm that computes the order of α mod N with probability $O(1/\log\log N)$.

[*Any significant theorems that you invoke to justify the algorithm should be clearly stated (but proofs are not required). In addition you may use without proof the following two technical results.*

Theorem FT: For positive integers t and M with $M \geqslant t^2$, and any $0 \leqslant x_0 < t$, let K be the largest integer such that $x_0 + (K-1)t < M$. Let QFT denote the quantum Fourier transform mod M. Suppose we measure $QFT\left(\frac{1}{\sqrt{K}}\sum_{k=0}^{K-1}|x_0 + kt\rangle\right)$ to obtain an integer c with $0 \leqslant c < M$. Then with probability $O(1/\log\log t)$, c will be an integer closest to a multiple $j(M/t)$ of M/t for which the value of j (between 0 and t) is coprime to t.

Theorem CF: For any rational number a/b with $0 < a/b < 1$ and with integers a and b having at most n digits each, let p/q with p and q coprime, be any rational number satisfying

$$\left|\frac{a}{b} - \frac{p}{q}\right| \leqslant \frac{1}{2q^2}.$$

Then p/q is one of the $O(n)$ convergents of the continued fraction of a/b and all the convergents can be classically computed from a/b in time $O(n^3)$.]

SECTION II

11I Number Theory

(a) Let $a_0, a_1, \ldots$ be positive integers, and $\beta > 0$ a positive real number. Show that for every $n \geqslant 0$, if $\theta_n = [a_0, \ldots, a_n, \beta]$, then $\theta_n = (\beta p_n + p_{n-1})/(\beta q_n + q_{n-1})$, where (p_n), (q_n) $(n \geqslant -1)$ are sequences of integers satisfying

$$p_0 = a_0,\ q_0 = 1, \quad p_{-1} = 1,\ q_{-1} = 0 \quad \text{and}$$
$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \quad (n \geqslant 1).$$

Show that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$, and that θ_n lies between p_n/q_n and p_{n-1}/q_{n-1}.

(b) Show that if $[a_0, a_1, \ldots]$ is the continued fraction expansion of a positive irrational θ, then $p_n/q_n \to \theta$ as $n \to \infty$.

(c) Let the convergents of the continued fraction $[a_0, a_1, \ldots, a_n]$ be p_j/q_j $(0 \leqslant j \leqslant n)$. Using part (a) or otherwise, show that the n-th and $(n-1)$-th convergents of $[a_n, a_{n-1}, \ldots, a_0]$ are p_n/p_{n-1} and q_n/q_{n-1} respectively.

(d) Show that if $\theta = [\,\overline{a_0, a_1, \ldots, a_n}\,]$ is a purely periodic continued fraction with convergents p_j/q_j, then $f(\theta) = 0$, where $f(X) = q_n X^2 + (q_{n-1} - p_n)X - p_{n-1}$. Deduce that if θ' is the other root of $f(X)$, then $-1/\theta' = [\,\overline{a_n, a_{n-1}, \ldots, a_0}\,]$.

12H Topics in Analysis

(a) Suppose that $K \subset \mathbb{C}$ is a non-empty subset of the square $\{x + iy : x, y \in (-1, 1)\}$ and f is analytic in the larger square $\{x + iy : x, y \in (-1-\delta, 1+\delta)\}$ for some $\delta > 0$. Show that f can be uniformly approximated on K by polynomials.

(b) Let K be a closed non-empty proper subset of $\mathbb{C}$. Let Λ be the set of $\lambda \in \mathbb{C} \setminus K$ such that $g_\lambda(z) = (z - \lambda)^{-1}$ can be approximated uniformly on K by polynomials and let $\Gamma = \mathbb{C} \setminus (K \cup \Lambda)$. Show that Λ and Γ are open. Is it always true that Λ is non-empty? Is it always true that, if K is bounded, then Γ is empty? Give reasons.

[*No form of Runge's theorem may be used without proof.*]

13J Statistical Modelling

A sociologist collects a dataset on friendships among m Cambridge graduates. Let $y_{i,j} = 1$ if persons i and j are friends 3 years after graduation, and $y_{i,j} = 0$ otherwise. Let z_i be a categorical variable for person i's college, taking values in the set $\{1, 2, \ldots, C\}$. Consider logistic regression models,

$$\mathbb{P}(y_{i,j} = 1) = \frac{e^{\theta_{i,j}}}{1 + e^{\theta_{i,j}}}, \quad 1 \leqslant i < j \leqslant m,$$

with parameters either

1. $\theta_{i,j} = \beta_{z_i, z_j}$; or,
2. $\theta_{i,j} = \beta_{z_i} + \beta_{z_j}$; or,
3. $\theta_{i,j} = \beta_{z_i} + \beta_{z_j} + \beta_0 \delta_{z_i, z_j}$, where $\delta_{z_i, z_j} = 1$ if $z_i = z_j$ and 0 otherwise.

(a) Write the likelihood of the models.

(b) Show that the three models are nested and specify the order. Suggest a statistic to compare models 1 and 3, give its definition and specify its asymptotic distribution under the null hypothesis, citing any necessary theorems.

(c) Suppose persons i and j are in the same college k; consider the number of friendships, M_i and M_j, that each of them has with people in college $\ell \neq k$ (ℓ and k fixed). In each of the models above, compare the distribution of these two random variables. Explain why this might lead to a poor quality of fit.

(d) Find a minimal sufficient statistic for model 3. [*You may use the following characterisation of a minimal sufficient statistic: let* $f(\beta; y)$ *be the likelihood in this model, where* $\beta = (\beta_k)_{k=0,1,\ldots,C}$ *and* $y = (y_{i,j})_{i,j=1,\ldots,m}$*; suppose* $T = t(y)$ *is a statistic such that* $f(\beta; y)/f(\beta; y')$ *is constant in* β *if and only if* $t(y) = t(y')$*; then,* T *is a minimal sufficient statistic for* β.]

14C Mathematical Biology

A model of an infectious disease in a plant population is given by

$$\dot{S} = (S+I) - (S+I)S - \beta IS\,, \quad (1)$$

$$\dot{I} = -(S+I)I + \beta IS \quad (2)$$

where $S(t)$ is the density of healthy plants and $I(t)$ is the density of diseased plants at time t and β is a positive constant.

(a) Give an interpretation of what each of the terms in equations (1) and (2) represents in terms of the dynamics of the plants. What does the coefficient β represent? What can you deduce from the equations about the effect of the disease on the plants?

(b) By finding all fixed points for $S \geqslant 0$ and $I \geqslant 0$ and analysing their stability, explain what will happen to a healthy plant population if the disease is introduced. Sketch the phase diagram, treating the cases $\beta < 1$ and $\beta > 1$ separately.

(c) Define new variables $N(t)$ for the total plant population density and $\theta(t)$ for the proportion of the population that is diseased. Starting from equations (1) and (2) above, derive equations for $\dot{N}$ and $\dot{\theta}$ purely in terms of N, θ and β. Without carrying out a full fixed point analysis, explain how this system can be used directly to show the same results you had in part (b). [*Hint: start by considering the dynamics of* $N(t)$ *alone.*]

(d) Suppose now that in an attempt to control disease, plants are culled at a rate k per capita, independently of whether the plants are healthy or diseased. Write down the modified versions of equations (1) and (2). Use these to build updated equations for $\dot{N}$ and $\dot{\theta}$. Without carrying out a detailed fixed point analysis, what can you deduce about the effect of culling? Give the range of k for which culling can effectively control the disease.

15E Classical Dynamics

(a) Explain what is meant by a *Lagrange top*. You may assume that such a top has the Lagrangian

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos \theta \right)^2 - Mgl \cos \theta$$

in terms of the Euler angles (θ, ϕ, ψ). State the meaning of the quantities I_1, I_3, M and l appearing in this expression.

Explain why the quantity

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}}$$

is conserved, and give two other independent integrals of motion.

Show that steady precession, with a constant value of $\theta \in (0, \frac{\pi}{2})$, is possible if

$$p_\psi^2 \geqslant 4MglI_1 \cos \theta \, .$$

(b) A rigid body of mass M is of uniform density and its surface is defined by

$$x_1^2 + x_2^2 = x_3^2 - \frac{x_3^3}{h} \, ,$$

where h is a positive constant and (x_1, x_2, x_3) are Cartesian coordinates in the body frame.

Calculate the values of I_1, I_3 and l for this symmetric top, when it rotates about the sharp point at the origin of this coordinate system.

16I Logic and Set Theory

Define the cardinals $\aleph_\alpha$, and explain briefly why every infinite set has cardinality an $\aleph_\alpha$.

Show that if κ is an infinite cardinal then $\kappa^2 = \kappa$.

Let $X_1, X_2, \ldots, X_n$ be infinite sets. Show that $X_1 \cup X_2 \cup \cdots \cup X_n$ must have the same cardinality as X_i for some i.

Let $X_1, X_2, \ldots$ be infinite sets, no two of the same cardinality. Is it possible that $X_1 \cup X_2 \cup \ldots$ has the same cardinality as some X_i? Justify your answer.

17G Graph Theory

State and prove *Hall's theorem.*

Let n be an even positive integer. Let $X = \{A : A \subset [n]\}$ be the power set of $[n] = \{1, 2, \ldots, n\}$. For $1 \leqslant i \leqslant n$, let $X_i = \{A \in X : |A| = i\}$. Let Q be the graph with vertex set X where A, $B \in X$ are adjacent if and only if $|A \triangle B| = 1$. [Here, $A \triangle B$ denotes the *symmetric difference* of A and B, given by $A \triangle B := (A \cup B) \setminus (A \cap B)$.]

Let $1 \leqslant i \leqslant \frac{n}{2}$. Why is the induced subgraph $Q[X_i \cup X_{i-1}]$ bipartite? Show that it contains a matching from X_{i-1} to X_i.

A *chain* in X is a subset $\mathcal{C} \subset X$ such that whenever A, $B \in \mathcal{C}$ we have $A \subset B$ or $B \subset A$. What is the least positive integer k such that X can be partitioned into k pairwise disjoint chains? Justify your answer.

18F Galois Theory

State (without proof) a result concerning uniqueness of splitting fields of a polynomial.

Given a polynomial $f \in \mathbb{Q}[X]$ with distinct roots, what is meant by its *Galois group* $\mathrm{Gal}_{\mathbb{Q}}(f)$? Show that f is irreducible over $\mathbb{Q}$ if and only if $\mathrm{Gal}_{\mathbb{Q}}(f)$ acts transitively on the roots of f.

Now consider an irreducible quartic of the form $g(X) = X^4 + bX^2 + c \in \mathbb{Q}[X]$. If $\alpha \in \mathbb{C}$ denotes a root of g, show that the splitting field $K \subset \mathbb{C}$ is $\mathbb{Q}(\alpha, \sqrt{c})$. Give an explicit description of $\mathrm{Gal}(K/\mathbb{Q})$ in the cases:

(i) $\sqrt{c} \in \mathbb{Q}(\alpha)$, and

(ii) $\sqrt{c} \notin \mathbb{Q}(\alpha)$.

If c is a square in $\mathbb{Q}$, deduce that $\mathrm{Gal}_{\mathbb{Q}}(g) \cong C_2 \times C_2$. Conversely, if $\mathrm{Gal}_{\mathbb{Q}}(g) \cong C_2 \times C_2$, show that $\sqrt{c}$ is invariant under at least two elements of order two in the Galois group, and deduce that c is a square in $\mathbb{Q}$.

19I Representation Theory

(a) What is meant by a *compact topological group?* Explain why $\mathrm{SU}(n)$ is an example of such a group.

[In the following the existence of a Haar measure for any compact Hausdorff topological group may be assumed, if required.]

(b) Let G be any compact Hausdorff topological group. Show that there is a continuous group homomorphism $\rho : G \to \mathrm{O}(n)$ if and only if G has an n-dimensional representation over $\mathbb{R}$. [Here $\mathrm{O}(n)$ denotes the subgroup of $\mathrm{GL}_n(\mathbb{R})$ preserving the standard (positive-definite) symmetric bilinear form.]

(c) Explicitly construct such a representation $\rho : \mathrm{SU}(2) \to \mathrm{SO}(3)$ by showing that $\mathrm{SU}(2)$ acts on the following vector space of matrices,

$$\left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{M}_2(\mathbb{C}) : A + \overline{A^t} = 0 \right\}$$

by conjugation.

Show that

(i) this subspace is isomorphic to $\mathbb{R}^3$;

(ii) the trace map $(A, B) \mapsto -\mathrm{tr}(AB)$ induces an invariant positive definite symmetric bilinear form;

(iii) ρ is surjective with kernel $\{\pm I_2\}$. [You may assume, without proof, that $\mathrm{SU}(2)$ is connected.]

20G Number Fields

(a) Let L be a number field, and suppose there exists $\alpha \in \mathcal{O}_L$ such that $\mathcal{O}_L = \mathbb{Z}[\alpha]$. Let $f(X) \in \mathbb{Z}[X]$ denote the minimal polynomial of α, and let p be a prime. Let $\overline{f}(X) \in (\mathbb{Z}/p\mathbb{Z})[X]$ denote the reduction modulo p of $f(X)$, and let

$$\overline{f}(X) = \overline{g}_1(X)^{e_1} \cdots \overline{g}_r(X)^{e_r}$$

denote the factorisation of $\overline{f}(X)$ in $(\mathbb{Z}/p\mathbb{Z})[X]$ as a product of powers of distinct monic irreducible polynomials $\overline{g}_1(X), \ldots, \overline{g}_r(X)$, where $e_1, \ldots, e_r$ are all positive integers.

For each $i = 1, \ldots, r$, let $g_i(X) \in \mathbb{Z}[X]$ be any polynomial with reduction modulo p equal to $\overline{g}_i(X)$, and let $P_i = (p, g_i(\alpha)) \subset \mathcal{O}_L$. Show that $P_1, \ldots, P_r$ are distinct, non-zero prime ideals of $\mathcal{O}_L$, and that there is a factorisation

$$p\mathcal{O}_L = P_1^{e_1} \cdots P_r^{e_r},$$

and that $N(P_i) = p^{\deg \overline{g}_i(X)}$.

(b) Let K be a number field of degree $n = [K : \mathbb{Q}]$, and let p be a prime. Suppose that there is a factorisation

$$p\mathcal{O}_K = Q_1 \cdots Q_s,$$

where $Q_1, \ldots, Q_s$ are distinct, non-zero prime ideals of $\mathcal{O}_K$ with $N(Q_i) = p$ for each $i = 1, \ldots, s$. Use the result of part (a) to show that if $n > p$ then there is no $\alpha \in \mathcal{O}_K$ such that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

21F Algebraic Topology

State the *Lefschetz fixed point theorem.*

Let $n \geqslant 2$ be an integer, and $x_0 \in S^2$ a choice of base point. Define a space

$$X := (S^2 \times \mathbb{Z}/n\mathbb{Z})/\sim$$

where $\mathbb{Z}/n\mathbb{Z}$ is discrete and $\sim$ is the smallest equivalence relation such that $(x_0, i) \sim (-x_0, i+1)$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Let $\phi : X \to X$ be a homeomorphism without fixed points. Use the Lefschetz fixed point theorem to prove the following facts.

(i) If $\phi^3 = \mathrm{Id}_X$ then n is divisible by 3.

(ii) If $\phi^2 = \mathrm{Id}_X$ then n is even.

22H Linear Analysis

(a) State and prove the *Riesz representation theorem* for a real Hilbert space H.

[You may use that if H is a real Hilbert space and $Y \subset H$ is a closed subspace, then $H = Y \oplus Y^{\perp}$.]

(b) Let H be a real Hilbert space and $T : H \to H$ a bounded linear operator. Show that T is invertible if and only if both T and T^* are bounded below. [Recall that an operator $S : H \to H$ is bounded below if there is $c > 0$ such that $\|Sx\| \geqslant c\|x\|$ for all $x \in H$.]

(c) Consider the complex Hilbert space of two-sided sequences,

$$X \quad = \quad \{(x_n)_{n\in\mathbb{Z}} : x_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} |x_n|^2 < \infty\}$$

with norm $\|x\| = (\sum_n |x_n|^2)^{1/2}$. Define $T : X \to X$ by $(Tx)_n = x_{n+1}$. Show that T is unitary and find the point spectrum and the approximate point spectrum of T.

23H Analysis of Functions

(a) Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear map. If B is continuous prove that there is an $M > 0$ such that $|B(u, v)| \leqslant M\|u\|\|v\|$ for all $u, v \in \mathcal{H}$. [You may use any form of the Banach–Steinhaus theorem as long as you state it clearly.]

(b) Now suppose that B defined as above is bilinear and continuous, and assume also that it is coercive: i.e. there is a $C > 0$ such that $B(u, u) \geqslant C\|u\|^2$ for all $u \in \mathcal{H}$. Prove that for any $f \in \mathcal{H}$, there exists a unique $v_f \in \mathcal{H}$ such that $B(u, v_f) = \langle u, f\rangle$ for all $u \in \mathcal{H}$.

[*Hint: show that there is a bounded invertible linear operator L with bounded inverse so that $B(u, v) = \langle u, Lv\rangle$ for all $u, v \in \mathcal{H}$. You may use any form of the Riesz representation theorem as long as you state it clearly.*]

(c) Define the *Sobolev space* $H^1_0(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is open and bounded.

(d) Suppose $f \in L^2(\Omega)$ and $A \in \mathbb{R}^d$ with $|A|_2 < 2$, where $|\cdot|_2$ is the Euclidean norm on $\mathbb{R}^d$. Consider the Dirichlet problem

$$-\Delta v + v + A \cdot \nabla v = f \quad \text{in } \Omega, \qquad v = 0 \quad \text{in } \partial\Omega.$$

Using the result of part (b), prove there is a unique weak solution $v \in H^1_0(\Omega)$.

(e) Now assume that Ω is the open unit disk in $\mathbb{R}^2$ and g is a smooth function on $\mathbb{S}^1$. Sketch how you would solve the following variant:

$$-\Delta v + v + A \cdot \nabla v = 0 \quad \text{in } \Omega, \qquad v = g \quad \text{in } \partial\Omega.$$

[*Hint: Reduce to the result of part (d).*]

24F Algebraic Geometry

(a) Let $X \subseteq \mathbb{P}^2$ be a smooth projective plane curve, defined by a homogeneous polynomial $F(x, y, z)$ of degree d over the complex numbers $\mathbb{C}$.

(i) Define the divisor $[X \cap H]$, where H is a hyperplane in $\mathbb{P}^2$ not contained in X, and prove that it has degree d.

(ii) Give (without proof) an expression for the degree of $\mathcal{K}_X$ in terms of d.

(iii) Show that X does not have genus 2.

(b) Let X be a smooth projective curve of genus g over the complex numbers $\mathbb{C}$. For $p \in X$ let

$$G(p) = \{n \in \mathbb{N} \mid \text{ there is } \textit{no } f \in k(X) \text{ with } v_p(f) = n, \text{ and } v_q(f) \leqslant 0 \text{ for all } q \neq p\}.$$

(i) Define $\ell(D)$, for a divisor D.

(ii) Show that for all $p \in X$,

$$\ell(np) = \begin{cases} \ell((n-1)p) & \text{for } n \in G(p) \\ \ell((n-1)p) + 1 & \text{otherwise.} \end{cases}$$

(iii) Show that $G(p)$ has exactly g elements. [*Hint: What happens for large n?*]

(iv) Now suppose that X has genus 2. Show that $G(p) = \{1, 2\}$ or $G(p) = \{1, 3\}$.

[*In this question $\mathbb{N}$ denotes the set of positive integers.*]

25H Differential Geometry

(a) Let $\gamma : (a,b) \to \mathbb{R}^2$ be a regular curve without self-intersection given by $\gamma(v) = (f(v), g(v))$ with $f(v) > 0$ for $v \in (a,b)$ and let S be the surface of revolution defined globally by the parametrisation

$$\phi : (0, 2\pi) \times (a,b) \to \mathbb{R}^3,$$

where $\phi(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$, i.e. $S = \phi((0,2\pi)\times(a,b))$. Compute its mean curvature H and its Gaussian curvature K.

(b) Define what it means for a regular surface $S \subset \mathbb{R}^3$ to be *minimal*. Give an example of a minimal surface which is not locally isometric to a cone, cylinder or plane. Justify your answer.

(c) Let S be a regular surface such that $K \equiv 1$. Is it necessarily the case that given any $p \in S$, there exists an open neighbourhood $\mathcal{U} \subset S$ of p such that $\mathcal{U}$ lies on some sphere in $\mathbb{R}^3$? Justify your answer.

26K Probability and Measure

(a) Let $(X_n)_{n\geqslant 1}$ and X be real random variables with finite second moment on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. Assume that X_n converges to X almost surely. Show that the following assertions are equivalent:

(i) $X_n \to X$ in $\mathbf{L}^2$ as $n\to\infty$,

(ii) $\mathbb{E}(X_n^2)\to\mathbb{E}(X^2)$ as $n\to\infty$.

(b) Suppose now that $\Omega=(0,1)$, $\mathcal{F}$ is the Borel σ-algebra of $(0,1)$ and $\mathbb{P}$ is Lebesgue measure. Given a Borel probability measure μ on $\mathbb{R}$ we set

$$X_\mu(\omega)=\inf\{x\in\mathbb{R}|F_\mu(x)\geqslant\omega\},$$

where $F_\mu(x):=\mu((-\infty,x])$ is the distribution function of μ and $\omega\in\Omega$.

(i) Show that X_μ is a random variable on $(\Omega,\mathcal{F},\mathbb{P})$ with law μ.

(ii) Let $(\mu_n)_{n\geqslant 1}$ and ν be Borel probability measures on $\mathbb{R}$ with finite second moments. Show that

$$\mathbb{E}((X_{\mu_n}-X_\nu)^2)\to 0 \text{ as } n\to\infty$$

if and only if μ_n converges weakly to ν and $\int x^2 d\mu_n(x)$ converges to $\int x^2 d\nu(x)$ as $n\to\infty$.

[*You may use any theorem proven in lectures as long as it is clearly stated. Furthermore, you may use without proof the fact that μ_n converges weakly to ν as $n\to\infty$ if and only if X_{μ_n} converges to X_ν almost surely.*]

27K Applied Probability

(a) Let $\lambda : \mathbb{R}^d \to [0,\infty)$ be such that $\Lambda(A) := \int_A \lambda(\mathbf{x})\,d\mathbf{x}$ is finite for any bounded measurable set $A \subseteq \mathbb{R}^d$. State the properties which define a (non-homogeneous) Poisson process Π on $\mathbb{R}^d$ with intensity function λ.

(b) Let Π be a Poisson process on $\mathbb{R}^d$ with intensity function λ, and let $f : \mathbb{R}^d \to \mathbb{R}^s$ be a given function. Give a clear statement of the necessary conditions on the pair Λ, f subject to which $f(\Pi)$ is a Poisson process on $\mathbb{R}^s$. When these conditions hold, express the mean measure of $f(\Pi)$ in terms of Λ and f.

(c) Let Π be a homogeneous Poisson process on $\mathbb{R}^2$ with constant intensity 1, and let $f : \mathbb{R}^2 \to [0,\infty)$ be given by $f(x_1,x_2) = x_1^2 + x_2^2$. Show that $f(\Pi)$ is a homogeneous Poisson process on $[0,\infty)$ with constant intensity π.

Let $R_1, R_2, \ldots$ be an increasing sequence of positive random variables such that the points of $f(\Pi)$ are $R_1^2, R_2^2, \ldots$. Show that R_k has density function

$$h_k(r) = \frac{1}{(k-1)!} 2\pi r (\pi r^2)^{k-1} e^{-\pi r^2}, \qquad r > 0.$$

28J Principles of Statistics

We consider a statistical model $\{f(\cdot,\theta) : \theta \in \Theta\}$.

(a) Define the maximum likelihood estimator (MLE) and the Fisher information $I(\theta)$.

(b) Let $\Theta = \mathbb{R}$ and assume there exist a continuous one-to-one function $\mu : \mathbb{R} \to \mathbb{R}$ and a real-valued function h such that

$$\mathbb{E}_\theta[h(X)] = \mu(\theta) \qquad \forall \theta \in \mathbb{R}.$$

(i) For $X_1, \ldots, X_n$ i.i.d. from the model for some $\theta_0 \in \mathbb{R}$, give the limit in almost sure sense of

$$\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^n h(X_i)\,.$$

Give a consistent estimator $\hat{\theta}_n$ of θ_0 in terms of $\hat{\mu}_n$.

(ii) Assume further that $\mathbb{E}_{\theta_0}[h(X)^2] < \infty$ and that μ is continuously differentiable and strictly monotone. What is the limit in distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$? Assume too that the statistical model satisfies the usual regularity assumptions. Do you necessarily expect $\mathrm{Var}(\hat{\theta}_n) \geqslant (nI(\theta_0))^{-1}$ for all n? Why?

(iii) Propose an alternative estimator for θ_0 with smaller bias than $\hat{\theta}_n$ if $B_n(\theta_0) = \mathbb{E}_{\theta_0}[\hat{\theta}_n] - \theta_0 = \frac{a}{n} + \frac{b}{n^2} + O(\frac{1}{n^3})$ for some $a, b \in \mathbb{R}$ with $a \neq 0$.

(iv) Further to all the assumptions in iii), assume that the MLE for θ_0 is of the form

$$\hat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^n h(X_i).$$

What is the link between the Fisher information at θ_0 and the variance of $h(X)$? What does this mean in terms of the precision of the estimator and why?

[*You may use results from the course, provided you state them clearly.*]

29K Stochastic Financial Models

(a) Describe the *(Cox-Ross-Rubinstein) binomial model*. What are the necessary and sufficient conditions on the model parameters for it to be arbitrage-free? How is the equivalent martingale measure $\mathbb{Q}$ characterised in this case?

(b) Consider a discounted claim H of the form $H = h(S_0^1, S_1^1, \ldots, S_T^1)$ for some function h. Show that the value process of H is of the form

$$V_t(\omega) = v_t\big(S_0^1, S_1^1(\omega), \ldots, S_t^1(\omega)\big),$$

for $t \in \{0, \ldots, T\}$, where the function v_t is given by

$$v_t(x_0, \ldots, x_t) = \mathbb{E}_{\mathbb{Q}}\left[h\Big(x_0, \ldots, x_t, x_t \cdot \frac{S_1^1}{S_0^1}, \ldots, x_t \cdot \frac{S_{T-t}^1}{S_0^1}\Big)\right].$$

You may use any property of conditional expectations without proof.

(c) Suppose that $H = h(S_T^1)$ only depends on the terminal value S_T^1 of the stock price. Derive an explicit formula for the value of H at time $t \in \{0, \ldots, T\}$.

(d) Suppose that H is of the form $H = h(S_T^1, M_T)$, where $M_t := \max_{s\in\{0,\ldots,t\}} S_s^1$. Show that the value process of H is of the form

$$V_t(\omega) = v_t\big(S_t^1(\omega), M_t(\omega)\big),$$

for $t \in \{0, \ldots, T\}$, where the function v_t is given by

$$v_t(x, m) = \mathbb{E}_{\mathbb{Q}}\left[g(x, m, S_0^1, S_{T-t}^1, M_{T-t})\right]$$

for a function g to be determined.

30A Asymptotic Methods
Consider, for small ϵ, the equation

$$\epsilon^2 \frac{d^2\psi}{dx^2} - q(x)\psi = 0. \qquad (*)$$

Assume that $(*)$ has bounded solutions with two turning points a, b where $b > a$, $q'(b) > 0$ and $q'(a) < 0$.

(a) Use the WKB approximation to derive the relationship

$$\frac{1}{\epsilon}\int_a^b |q(\xi)|^{1/2} d\xi = \left(n + \frac{1}{2}\right)\pi \quad \text{with} \quad n = 0, 1, 2, \cdots . \qquad (**)$$

[You may quote without proof any standard results or formulae from WKB theory.]

(b) In suitable units, the radial Schrödinger equation for a spherically symmetric potential given by $V(r) = -V_0/r$, for constant V_0, can be recast in the standard form $(*)$ as:

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + e^{2x}\left[\lambda - V(e^x) - \frac{\hbar^2}{2m}\left(l + \frac{1}{2}\right)^2 e^{-2x}\right]\psi = 0,$$

where $r = e^x$ and $\epsilon = \hbar/\sqrt{2m}$ is a small parameter.

Use result $(**)$ to show that the energies of the bound states (i.e $\lambda = -|\lambda| < 0$) are approximated by the expression:

$$E = -|\lambda| = -\frac{m}{2\hbar^2}\frac{V_0^2}{(n + l + 1)^2}.$$

[*You may use the result*

$$\int_a^b \frac{1}{r}\sqrt{(r-a)(b-r)}\; dr = (\pi/2)\left[\sqrt{b} - \sqrt{a}\right]^2 .]$$

31E Dynamical Systems

Consider the dynamical system

$$\begin{aligned}\dot{x} &= x + y^2 - a\,,\\ \dot{y} &= y(4x - x^2 - a)\,,\end{aligned}$$

for $(x, y) \in \mathbb{R}^2$, $a \in \mathbb{R}$.

Find all fixed points of this system. Find the three different values of a at which bifurcations appear. For each such value give the location (x, y) of all bifurcations. For each of these, what types of bifurcation are suggested from this analysis?

Use centre manifold theory to analyse these bifurcations. In particular, for each bifurcation derive an equation for the dynamics on the extended centre manifold and hence classify the bifurcation.

32B Principles of Quantum Mechanics

Define the *spin raising* and *spin lowering* operators S_+ and S_-. Show that

$$S_\pm|s,\sigma\rangle = \hbar\sqrt{s(s+1) - \sigma(\sigma \pm 1)}\,|s, \sigma \pm 1\rangle\,,$$

where $S_z|s,\sigma\rangle = \hbar\sigma|s,\sigma\rangle$ and $S^2|s,\sigma\rangle = s(s+1)\hbar^2|s,\sigma\rangle$.

Two spin-$\frac{1}{2}$ particles, with spin operators $\mathbf{S}^{(1)}$ and $\mathbf{S}^{(2)}$, have a Hamiltonian

$$H = \alpha\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} + \mathbf{B} \cdot (\mathbf{S}^{(1)} - \mathbf{S}^{(2)})\,,$$

where α and $\mathbf{B} = (0, 0, B)$ are constants. Express H in terms of the two particles' spin raising and spin lowering operators $S_\pm^{(1)}$, $S_\pm^{(2)}$ and the corresponding z-components $S_z^{(1)}$, $S_z^{(2)}$. Hence find the eigenvalues of H. Show that there is a unique groundstate in the limit $B \to 0$ and that the first excited state is triply degenerate in this limit. Explain this degeneracy by considering the action of the combined spin operator $\mathbf{S}^{(1)} + \mathbf{S}^{(2)}$ on the energy eigenstates.

33B Applications of Quantum Mechanics

(a) A classical beam of particles scatters off a spherically symmetric potential $V(r)$. Draw a diagram to illustrate the differential cross-section $d\sigma/d\Omega$, and use this to derive an expression for $d\sigma/d\Omega$ in terms of the impact parameter b and the scattering angle θ.

A quantum beam of particles of mass m and momentum $p = \hbar k$ is incident along the z-axis and scatters off a spherically symmetric potential $V(r)$. Write down the asymptotic form of the wavefunction ψ in terms of the scattering amplitude $f(\theta)$. By considering the probability current $\mathbf{J} = -i(\hbar/2m)\left(\psi^\star\nabla\psi - (\nabla\psi^\star)\psi\right)$, derive an expression for the differential cross-section $d\sigma/d\Omega$ in terms of $f(\theta)$.

(b) The solution $\psi(\mathbf{r})$ of the radial Schrödinger equation for a particle of mass m and wave number k moving in a spherically symmetric potential $V(r)$ has the asymptotic form

$$\psi(\mathbf{r}) \quad \sim \quad \sum_{l=0}^{\infty}\left[A_l(k)\frac{\sin\left(kr - \frac{l\pi}{2}\right)}{kr} \; - \; B_l(k)\frac{\cos\left(kr - \frac{l\pi}{2}\right)}{kr}\right] P_l\left(\cos\theta\right) ,$$

valid for $kr \gg 1$, where $A_l(k)$ and $B_l(k)$ are constants and P_l denotes the l'th Legendre polynomial. Define the S-matrix element S_l and the corresponding phase shift δ_l for the partial wave of angular momentum l, in terms of $A_l(k)$ and $B_l(k)$. Define also the scattering length a_s for the potential V.

Outside some core region, $r > r_0$, the Schrödinger equation for some such potential is solved by the s-wave (i.e. $l = 0$) wavefunction $\psi(\mathbf{r}) = \psi(r)$ with,

$$\psi(r) = \frac{e^{-ikr}}{r} + \frac{k + i\lambda\tanh(\lambda r)}{k - i\lambda}\frac{e^{ikr}}{r}$$

where $\lambda > 0$ is a constant. Extract the S-matrix element S_0, the phase shift δ_0 and the scattering length a_s. Deduce that the potential $V(r)$ has a bound state of zero angular momentum and compute its energy. Give the form of the (un-normalised) bound state wavefunction in the region $r > r_0$.

34D Statistical Physics

Give an outline of the Landau theory of phase transitions for a system with one real order parameter ϕ. Describe the phase transitions that can be modelled by the Landau potentials

(i) $$G = \frac{1}{4}\phi^4 + \frac{1}{2}\varepsilon\phi^2,$$

(ii) $$G = \frac{1}{6}\phi^6 + \frac{1}{4}g\phi^4 + \frac{1}{2}\varepsilon\phi^2,$$

where ε and g are control parameters that depend on the temperature and pressure.

In case (ii), find the curve of first-order phase transitions in the (g, ε) plane. Find the region where it is possible for superheating to occur. Find also the region where it is possible for supercooling to occur.

35E Electrodynamics

Consider a medium in which the electric displacement $\mathbf{D}(t, \mathbf{x})$ and magnetising field $\mathbf{H}(t, \mathbf{x})$ are linearly related to the electric and magnetic fields respectively with corresponding polarisation constants ε and μ;

$$\mathbf{D} = \varepsilon\, \mathbf{E}, \qquad \mathbf{B} = \mu\, \mathbf{H}.$$

Write down Maxwell's equations for $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$ in the absence of free charges and currents.

Consider EM waves of the form,

$$\begin{aligned} \mathbf{E}(t, \mathbf{x}) &= \mathbf{E}_0 \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)\,, \\ \mathbf{B}(t, \mathbf{x}) &= \mathbf{B}_0 \sin(\mathbf{k} \cdot \mathbf{x} - \omega t)\,. \end{aligned}$$

Find conditions on the electric and magnetic polarisation vectors $\mathbf{E}_0$ and $\mathbf{B}_0$, wave-vector $\mathbf{k}$ and angular frequency ω such that these fields satisfy Maxwell's equations for the medium described above. At what speed do the waves propagate?

Consider two media, filling the regions $x < 0$ and $x > 0$ in three dimensional space, and having two different values ε_- and ε_+ of the electric polarisation constant. Suppose an electromagnetic wave is incident from the region $x < 0$ resulting in a transmitted wave in the region $x > 0$ and also a reflected wave for $x < 0$. The angles of incidence, reflection and transmission are denoted θ_I, θ_R and θ_T respectively. By constructing a corresponding solution of Maxwell's equations, derive the *law of reflection* $\theta_I = \theta_R$ and *Snell's law of refraction*, $n_- \sin\theta_I = n_+ \sin\theta_T$ where $n_\pm = c\sqrt{\varepsilon_\pm \mu}$ are the indices of refraction of the two media.

Consider the special case in which the electric polarisation vectors $\mathbf{E}_I$, $\mathbf{E}_R$ and $\mathbf{E}_T$ of the incident, reflected and transmitted waves are all normal to the plane of incidence (i.e. the plane containing the corresponding wave-vectors). By imposing appropriate boundary conditions for $\mathbf{E}$ and $\mathbf{H}$ at $x = 0$, show that,

$$\frac{|\mathbf{E}_R|}{|\mathbf{E}_T|} = \frac{1}{2}\left(1 - \frac{\tan\theta_R}{\tan\theta_T}\right).$$

36D General Relativity

(a) Consider the spherically symmetric spacetime metric

$$ds^2 = -\lambda^2 dt^2 + \mu^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \,, \qquad (\dagger)$$

where λ and μ are functions of t and r. Use the Euler-Lagrange equations for the geodesics of the spacetime to compute all non-vanishing Christoffel symbols for this metric.

(b) Consider the static limit of the line element ($\dagger$) where λ and μ are functions of the radius r only, and let the matter coupled to gravity be a spherically symmetric fluid with energy momentum tensor

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu} \,, \qquad u^\mu = [\lambda^{-1}, 0, 0, 0] \,,$$

where the pressure P and energy density ρ are also functions of the radius r. For these Tolman-Oppenheimer-Volkoff stellar models, the Einstein and matter equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ and $\nabla_\mu T^\mu{}_\nu = 0$ reduce to

$$\begin{aligned}
\frac{\partial_r \lambda}{\lambda} &= \frac{\mu^2 - 1}{2r} + 4\pi r \mu^2 P \,, \\
\partial_r m &= 4\pi r^2 \rho \,, \qquad \text{where} \qquad m(r) = \frac{r}{2}\left(1 - \frac{1}{\mu^2}\right) \,, \\
\partial_r P &= -(\rho + P)\left(\frac{\mu^2 - 1}{2r} + 4\pi r \mu^2 P\right) \,.
\end{aligned}$$

Consider now a constant density solution to the above Einstein and matter equations, where ρ takes the non-zero constant value ρ_0 out to a radius R and $\rho = 0$ for $r > R$. Show that for such a star,

$$\partial_r P = \frac{4\pi r}{1 - \frac{8}{3}\pi\rho_0 r^2}\left(P + \frac{1}{3}\rho_0\right)(P + \rho_0) \,,$$

and that the pressure at the centre of the star is

$$P(0) = -\rho_0 \frac{1 - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - 1} \,, \qquad \text{with} \qquad M = \frac{4}{3}\pi\rho_0 R^3 \,.$$

Show that $P(0)$ diverges if $M = 4R/9$. [*Hint: at the surface of the star the pressure vanishes:* $P(R) = 0$ *.*]

37A Fluid Dynamics

(a) Show that the Stokes flow around a rigid moving sphere has the minimum viscous dissipation rate of all incompressible flows which satisfy the no-slip boundary conditions on the sphere.

(b) Let $\boldsymbol{u} = \boldsymbol{\nabla}(\boldsymbol{x} \cdot \boldsymbol{\Phi} + \chi) - 2\boldsymbol{\Phi}$, where $\boldsymbol{\Phi}$ and χ are solutions of Laplace's equation, i.e. $\nabla^2 \boldsymbol{\Phi} = \mathbf{0}$ and $\nabla^2 \chi = 0$.

(i) Show that $\boldsymbol{u}$ is incompressible.

(ii) Show that $\boldsymbol{u}$ satisfies Stokes equation if the pressure $p = 2\mu \boldsymbol{\nabla} \cdot \boldsymbol{\Phi}$.

(c) Consider a rigid sphere moving with velocity $\boldsymbol{U}$. The Stokes flow around the sphere is given by

$$\boldsymbol{\Phi} = \alpha \frac{\boldsymbol{U}}{r} \quad \text{and} \quad \chi = \beta \boldsymbol{U} \cdot \boldsymbol{\nabla} \left(\frac{1}{r} \right),$$

where the origin is chosen to be at the centre of the sphere. Find the values for α and β which ensure no-slip conditions are satisfied on the sphere.

38A Waves

(a) Assuming a slowly-varying two-dimensional wave pattern of the form

$$\varphi(\mathbf{x},t) = A(\mathbf{x},t;\varepsilon)\exp\left[\frac{i}{\varepsilon}\theta(\mathbf{x},t)\right],$$

where $0 < \varepsilon \ll 1$, and a local dispersion relation $\omega = \Omega(\mathbf{k};\mathbf{x},t)$, derive the ray tracing equations,

$$\frac{dx_i}{dt} = \frac{\partial\Omega}{\partial k_i}, \quad \frac{d\omega}{dt} = \frac{\partial\Omega}{\partial t}, \quad \frac{dk_i}{dt} = -\frac{\partial\Omega}{\partial x_i}, \quad \frac{1}{\varepsilon}\frac{d\theta}{dt} = -\omega + k_j\frac{\partial\Omega}{\partial k_j},$$

for $i,j = 1,2$, explaining carefully the meaning of the notation used.

(b) For a homogeneous, time-independent (but not necessarily isotropic) medium, show that all rays are straight lines. When the waves have zero frequency, deduce that if the point $\mathbf{x}$ lies on a ray emanating from the origin in the direction given by a unit vector $\widehat{\mathbf{c}}_{\mathbf{g}}$, then

$$\theta(\mathbf{x}) = \theta(\mathbf{0}) + \widehat{\mathbf{c}}_{\mathbf{g}}\cdot\mathbf{k}\,|\mathbf{x}|\,.$$

(c) Consider a stationary obstacle in a steadily moving homogeneous medium which has the dispersion relation

$$\Omega = \alpha\left(k_1^2 + k_2^2\right)^{1/4} - Vk_1\,,$$

where $(V,0)$ is the velocity of the medium and $\alpha > 0$ is a constant. The obstacle generates a steady wave system. Writing $(k_1,k_2) = \kappa(\cos\phi,\sin\phi)$, with $\kappa > 0$, show that the wave satisfies

$$\kappa = \frac{\alpha^2}{V^2\cos^2\phi}, \quad \widehat{\mathbf{c}}_{\mathbf{g}} = (\cos\psi,\sin\psi)\,,$$

where ψ is defined by

$$\tan\psi = -\frac{\tan\phi}{1+2\tan^2\phi}$$

with $\frac{1}{2}\pi < \psi < \frac{3}{2}\pi$ and $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$. Deduce that the wave pattern occupies a wedge of semi-angle $\tan^{-1}\left(2^{-3/2}\right)$, extending in the negative x_1-direction.

39C Numerical Analysis

For a 2-periodic analytic function f, its Fourier expansion is given by the formula

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}_n e^{i\pi n x}, \qquad \widehat{f}_n = \frac{1}{2}\int_{-1}^{1} f(t)e^{-i\pi n t}\,dt\,.$$

(a) Consider the two-point boundary value problem

$$-\frac{1}{\pi^2}(1+2\cos \pi x)u'' + u = 1 + \sum_{n=1}^{\infty} \frac{2}{n^2+1}\cos \pi n x, \qquad -1 \leqslant x \leqslant 1\,,$$

with periodic boundary conditions $u(-1) = u(1)$. Construct explicitly the infinite dimensional linear algebraic system that arises from the application of the Fourier spectral method to the above equation, and explain how to truncate the system to a finite-dimensional one.

(b) A rectangle rule is applied to computing the integral of a 2-periodic analytic function h:

$$\int_{-1}^{1} h(t)\,dt \approx \frac{2}{N} \sum_{k=-N/2+1}^{N/2} h\left(\frac{2k}{N}\right). \qquad (*)$$

Find an expression for the error $e_N(h) :=$ LHS $-$ RHS of $(*)$, in terms of $\widehat{h}_n$, and show that $e_N(h)$ has a spectral rate of decay as $N \to \infty$. [In the last part, you may quote a relevant theorem about $\widehat{h}_n$.]

END OF PAPER