UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 7 June, 2019 1:30 pm to 4:30 pm MAT1

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

What is an *eigenvalue* of a matrix A? What is the *eigenspace* corresponding to an eigenvalue λ of A?

Consider the matrix

$$A = \begin{pmatrix} aa & ab & ac & ad \\ ba & bb & bc & bd \\ ca & cb & cc & cd \\ da & db & dc & dd \end{pmatrix}$$

for $(a, b, c, d) \in \mathbb{R}^4$ a non-zero vector. Show that A has rank 1. Find the eigenvalues of A and describe the corresponding eigenspaces. Is A diagonalisable?

2G Groups, Rings and Modules

Let G be a group and P a subgroup.

(a) Define the *normaliser* $N_G(P)$.

(b) Suppose that $K \lhd G$ and P is a Sylow p-subgroup of K. Using Sylow's second theorem, prove that $G = N_G(P)K$.

3E Analysis II

Let $A \subset \mathbb{R}$. What does it mean to say that a sequence of real-valued functions on A is *uniformly convergent*?

(i) If a sequence (f_n) of real-valued functions on A converges uniformly to f, and each f_n is continuous, must f also be continuous?

(ii) Let $f_n(x) = e^{-nx}$. Does the sequence (f_n) converge uniformly on $[0, 1]$?

(iii) If a sequence (f_n) of real-valued functions on $[-1, 1]$ converges uniformly to f, and each f_n is differentiable, must f also be differentiable?

Give a proof or counterexample in each case.

4F Complex Analysis

State the Cauchy Integral Formula for a disc. If $f : D(z_0; r) \to \mathbb{C}$ is a holomorphic function such that $|f(z)| \leqslant |f(z_0)|$ for all $z \in D(z_0; r)$, show using the Cauchy Integral Formula that f is constant.

5D Methods

Let

$$g_\epsilon(x) = \frac{-2\epsilon x}{\pi(\epsilon^2 + x^2)^2}\,.$$

By considering the integral $\int_{-\infty}^{\infty} \phi(x)\, g_\epsilon(x)\, dx$, where ϕ is a smooth, bounded function that vanishes sufficiently rapidly as $|x| \to \infty$, identify $\lim_{\epsilon\to 0} g_\epsilon(x)$ in terms of a generalized function.

6B Quantum Mechanics

(a) Define the probability density ρ and probability current j for the wavefunction $\Psi(x,t)$ of a particle of mass m. Show that

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0\,,$$

and deduce that $j = 0$ for a normalizable, stationary state wavefunction. Give an example of a non-normalizable, stationary state wavefunction for which j is non-zero, and calculate the value of j.

(b) A particle has the instantaneous, normalized wavefunction

$$\Psi(x,0) = \left(\frac{2\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2 + ikx}\,,$$

where α is positive and k is real. Calculate the expectation value of the momentum for this wavefunction.

7A Electromagnetism

Write down Maxwell's Equations for electric and magnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ in the absence of charges and currents. Show that there are solutions of the form

$$\mathbf{E}(\mathbf{x},t) \;=\; \mathrm{Re}\{\, \mathbf{E}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,\}\,, \quad \mathbf{B}(\mathbf{x},t) \;=\; \mathrm{Re}\{\, \mathbf{B}_0\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\,\}$$

if $\mathbf{E}_0$ and $\mathbf{k}$ satisfy a constraint and if $\mathbf{B}_0$ and ω are then chosen appropriately.

Find the solution with $\mathbf{E}_0 = E(1, i, 0)$, where E is real, and $\mathbf{k} = k(0,0,1)$. Compute the Poynting vector and state its physical significance.

8C Numerical Analysis

Calculate the LU factorization of the matrix

$$A = \begin{pmatrix} 3 & 2 & -3 & -3 \\ 6 & 3 & -7 & -8 \\ 3 & 1 & -6 & -4 \\ -6 & -3 & 9 & 6 \end{pmatrix}.$$

Use this to evaluate $\det(A)$ and to solve the equation

$$A\mathbf{x} = \mathbf{b}$$

with

$$\mathbf{b} = \begin{pmatrix} 3 \\ 3 \\ -1 \\ -3 \end{pmatrix}.$$

9H Markov Chains

For a Markov chain X on a state space S with $u, v \in S$, we let $p_{uv}(n)$ for $n \in \{0, 1, \ldots\}$ be the probability that $X_n = v$ when $X_0 = u$.

(a) Let X be a Markov chain. Prove that if X is recurrent at a state v, then $\sum_{n=0}^{\infty} p_{vv}(n) = \infty$. [You may use without proof that the number of returns of a Markov chain to a state v when starting from v has the geometric distribution.]

(b) Let X and Y be independent simple symmetric random walks on $\mathbb{Z}^2$ starting from the origin 0. Let $Z = \sum_{n=0}^{\infty} \mathbf{1}_{\{X_n = Y_n\}}$. Prove that $\mathbb{E}[Z] = \sum_{n=0}^{\infty} p_{00}(2n)$ and deduce that $\mathbb{E}[Z] = \infty$. [You may use without proof that $p_{xy}(n) = p_{yx}(n)$ for all $x, y \in \mathbb{Z}^2$ and $n \in \mathbb{N}$, and that X is recurrent at 0.]

SECTION II

10F Linear Algebra

If U is a finite-dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$, prove that the linear map $\phi : U \to U^*$ given by $\phi(u)(u') = \langle u, u' \rangle$ is an isomorphism. [You do not need to show that it is linear.]

If V and W are inner product spaces and $\alpha : V \to W$ is a linear map, what is meant by the *adjoint* α^* of α? If $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis for V, $\{f_1, f_2, \ldots, f_m\}$ is an orthonormal basis for W, and A is the matrix representing α in these bases, derive a formula for the matrix representing α^* in these bases.

Prove that $\mathrm{Im}(\alpha) = \mathrm{Ker}(\alpha^*)^{\perp}$.

If $w_0 \notin \mathrm{Im}(\alpha)$ then the linear equation $\alpha(v) = w_0$ has no solution, but we may instead search for a $v_0 \in V$ minimising $||\alpha(v) - w_0||^2$, known as a least-squares solution. Show that v_0 is such a least-squares solution if and only if it satisfies $\alpha^*\alpha(v_0) = \alpha^*(w_0)$. Hence find a least-squares solution to the linear equation

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

11G Groups, Rings and Modules

(a) Define the Smith Normal Form of a matrix. When is it guaranteed to exist?

(b) Deduce the classification of finitely generated abelian groups.

(c) How many conjugacy classes of matrices are there in $GL_{10}(\mathbb{Q})$ with minimal polynomial $X^7 - 4X^3$?

12E Analysis II

(a) (i) Show that a compact metric space must be complete.

(ii) If a metric space is complete and bounded, must it be compact? Give a proof or counterexample.

(b) A metric space (X, d) is said to be totally bounded if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ and $\{x_1, \ldots, x_N\} \subset X$ such that $X = \bigcup_{i=1}^{N} B_\epsilon(x_i)$.

(i) Show that a compact metric space is totally bounded.

(ii) Show that a complete, totally bounded metric space is compact.

[*Hint: If* (x_n) *is Cauchy, then there is a subsequence* (x_{n_j}) *such that*

$$\sum_j d(x_{n_{j+1}}, x_{n_j}) < \infty\,.]$$

(iii) Consider the space $C[0,1]$ of continuous functions $f : [0,1] \to \mathbb{R}$, with the metric

$$d(f,g) = \min\left\{\int_0^1 |f(t) - g(t)|dt,\ 1\right\}.$$

Is this space compact? Justify your answer.

13G Metric and Topological Spaces

(a) Define the *subspace*, *quotient* and *product topologies*.

(b) Let X be a compact topological space and Y a Hausdorff topological space. Prove that a continuous bijection $f : X \to Y$ is a homeomorphism.

(c) Let $S = [0,1] \times [0,1]$, equipped with the product topology. Let $\sim$ be the smallest equivalence relation on S such that $(s,0) \sim (s,1)$ and $(0,t) \sim (1,t)$, for all $s,t \in [0,1]$. Let

$$T = \{(x,y,z) \in \mathbb{R}^3 \mid (\sqrt{x^2+y^2} - 2)^2 + z^2 = 1\}$$

equipped with the subspace topology from $\mathbb{R}^3$. Prove that $S/{\sim}$ and T are homeomorphic.

[*You may assume without proof that* S *is compact.*]

14D Complex Methods

(a) Using the Bromwich contour integral, find the inverse Laplace transform of $1/s^2$.

The temperature $u(r,t)$ of mercury in a spherical thermometer bulb $r \leqslant a$ obeys the radial heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru)$$

with unit diffusion constant. At $t=0$ the mercury is at a uniform temperature u_0 equal to that of the surrounding air. For $t>0$ the surrounding air temperature lowers such that at the edge of the thermometer bulb

$$\frac{1}{k}\frac{\partial u}{\partial r}\bigg|_{r=a} = u_0 - u(a,t) - t\,,$$

where k is a constant.

(b) Find an explicit expression for $U(r,s) = \int_0^\infty e^{-st}\,u(r,t)\,dt$.

(c) Show that the temperature of the mercury at the centre of the thermometer bulb at late times is

$$u(0,t) \approx u_0 - t + \frac{a}{3k} + \frac{a^2}{6}\,.$$

[*You may assume that the late time behaviour of* $u(r,t)$ *is determined by the singular part of* $U(r,s)$ *at* $s=0$.]

15E Geometry

Let $H = \{x+iy \,|\, x,y \in \mathbb{R}, y>0\}$ be the upper-half plane with hyperbolic metric $\frac{dx^2+dy^2}{y^2}$. Define the group $PSL(2,\mathbb{R})$, and show that it acts by isometries on H. [If you use a generation statement you must carefully state it.]

(a) Prove that $PSL(2,\mathbb{R})$ acts transitively on the collection of pairs (l,P), where l is a hyperbolic line in H and $P \in l$.

(b) Let $l^+ \subset H$ be the imaginary half-axis. Find the isometries of H which fix l^+ pointwise. Hence or otherwise find all isometries of H.

(c) Describe without proof the collection of all hyperbolic lines which meet l^+ with (signed) angle α, $0<\alpha<\pi$. Explain why there exists a hyperbolic triangle with angles α,β and γ whenever $\alpha+\beta+\gamma<\pi$.

(d) Is this triangle unique up to isometry? Justify your answer. [You may use without proof the fact that Möbius maps preserve angles.]

16A Variational Principles

Consider the functional

$$I[y] = \int_{-\infty}^{\infty} \left(\tfrac{1}{2}\, y'^{\,2} + \tfrac{1}{2}\, U(y)^2 \right) dx\,,$$

where $y(x)$ is subject to boundary conditions $y(x) \to a_{\pm}$ as $x \to \pm\infty$ with $U(a_{\pm}) = 0$. [You may assume the integral converges.]

(a) Find expressions for the first-order and second-order variations δI and $\delta^2 I$ resulting from a variation δy that respects the boundary conditions.

(b) If $a_{\pm} = a$, show that $I[y] = 0$ if and only if $y(x) = a$ for all x. Explain briefly how this is consistent with your results for δI and $\delta^2 I$ in part (a).

(c) Now suppose that $U(y) = c^2 - y^2$ with $a_{\pm} = \pm c$ ($c > 0$). By considering an integral of $U(y)y'$, show that

$$I[y] \geqslant \frac{4c^3}{3}\,,$$

with equality if and only if y satisfies a first-order differential equation. Deduce that global minima of $I[y]$ with the specified boundary conditions occur precisely for

$$y(x) = c \tanh\{\, c(x - x_0)\,\}\,,$$

where x_0 is a constant. How is the first-order differential equation that appears in this case related to your general result for δI in part (a)?

17B Methods

(a) Show that the operator

$$\frac{d^4}{dx^4} + p\frac{d^2}{dx^2} + q\frac{d}{dx} + r\,,$$

where $p(x)$, $q(x)$ and $r(x)$ are real functions, is self-adjoint (for suitable boundary conditions which you need not state) if and only if

$$q = \frac{dp}{dx}\,.$$

(b) Consider the eigenvalue problem

$$\frac{d^4y}{dx^4} + p\frac{d^2y}{dx^2} + \frac{dp}{dx}\frac{dy}{dx} = \lambda y \qquad (*)$$

on the interval $[a, b]$ with boundary conditions

$$y(a) = \frac{dy}{dx}(a) = y(b) = \frac{dy}{dx}(b) = 0\,.$$

Assuming that $p(x)$ is everywhere negative, show that all eigenvalues λ are positive.

(c) Assume now that $p \equiv 0$ and that the eigenvalue problem $(*)$ is on the interval $[-c, c]$ with $c > 0$. Show that $\lambda = 1$ is an eigenvalue provided that

$$\cos c \sinh c \pm \sin c \cosh c = 0$$

and show graphically that this condition has just one solution in the range $0 < c < \pi$.

[*You may assume that all eigenfunctions are either symmetric or antisymmetric about* $x = 0$.]

18C Fluid Dynamics

The linear shallow-water equations governing the motion of a fluid layer in the neighbourhood of a point on the Earth's surface in the northern hemisphere are

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}\,,$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}\,,$$
$$\frac{\partial \eta}{\partial t} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\,,$$

where $u(x,y,t)$ and $v(x,y,t)$ are the horizontal velocity components and $\eta(x,y,t)$ is the perturbation of the height of the free surface.

(a) Explain the meaning of the three positive constants f, g and h appearing in the equations above and outline the assumptions made in deriving these equations.

(b) Show that ζ, the z-component of vorticity, satisfies

$$\frac{\partial \zeta}{\partial t} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\,,$$

and deduce that the potential vorticity

$$q = \zeta - \frac{f}{h}\eta$$

satisfies

$$\frac{\partial q}{\partial t} = 0\,.$$

(c) Consider a steady geostrophic flow that is uniform in the latitudinal (y) direction. Show that

$$\frac{d^2\eta}{dx^2} - \frac{f^2}{gh}\eta = \frac{f}{g}q\,.$$

Given that the potential vorticity has the piecewise constant profile

$$q = \begin{cases} q_1\,, & x < 0\,, \\ q_2\,, & x > 0\,, \end{cases}$$

where q_1 and q_2 are constants, and that $v \to 0$ as $x \to \pm\infty$, solve for $\eta(x)$ and $v(x)$ in terms of the Rossby radius $R = \sqrt{gh}/f$. Sketch the functions $\eta(x)$ and $v(x)$ in the case $q_1 > q_2$.

19H Statistics

Consider the linear model

$$Y_i = \beta x_i + \epsilon_i \quad \text{for} \quad i = 1, \ldots, n$$

where $x_1, \ldots, x_n$ are known and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. $N(0, \sigma^2)$. We assume that the parameters β and σ^2 are unknown.

(a) Find the MLE $\widehat{\beta}$ of β. Explain why $\widehat{\beta}$ is the same as the least squares estimator of β.

(b) State and prove the Gauss–Markov theorem for this model.

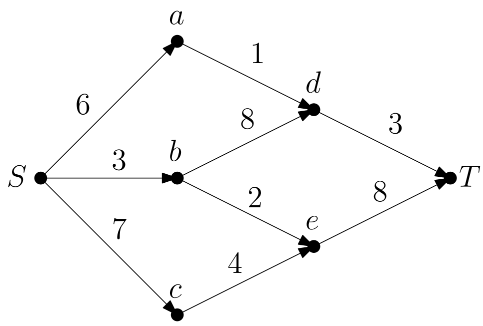
(c) For each value of $\theta \in \mathbb{R}$ with $\theta \neq 0$, determine the unbiased linear estimator $\widetilde{\beta}$ of β which minimizes

$$\mathbb{E}_{\beta,\sigma^2}[\exp(\theta(\widetilde{\beta} - \beta))] .$$

20H Optimisation

(a) State and prove the max-flow min-cut theorem.

(b) (i) Apply the Ford–Fulkerson algorithm to find the maximum flow of the network illustrated below, where S is the source and T is the sink.



(ii) Verify the optimality of your solution using the max-flow min-cut theorem.

(iii) Is there a unique flow which attains the maximum? Explain your answer.

(c) Prove that the Ford–Fulkerson algorithm always terminates when the network is finite, the capacities are integers, and the algorithm is initialised where the initial flow is 0 across all edges. Prove also in this case that the flow across each edge is an integer.

END OF PAPER