UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 3 June, 2019 9:00 am to 12:00 pm MAT0

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

Prove that two elements of S_n are conjugate if and only if they have the same cycle type.

Describe a condition on the centraliser (in S_n) of a permutation $\sigma \in A_n$ that ensures the conjugacy class of σ in A_n is the same as the conjugacy class of σ in S_n. Justify your answer.

How many distinct conjugacy classes are there in A_5?

2D Groups

What is the orthogonal group $\mathrm{O}(n)$? What is the special orthogonal group $\mathrm{SO}(n)$?

Show that every element of $\mathrm{SO}(3)$ has an eigenvector with eigenvalue 1.

Is it true that every element of $\mathrm{O}(3)$ is either a rotation or a reflection? Justify your answer.

3B Vector Calculus

Apply the divergence theorem to the vector field $\mathbf{u}(\mathbf{x}) = \mathbf{a}\phi(\mathbf{x})$ where $\mathbf{a}$ is an arbitrary constant vector and ϕ is a scalar field, to show that

$$\int_V \boldsymbol{\nabla}\phi \, dV = \int_S \phi \, d\mathbf{S},$$

where V is a volume bounded by the surface S and $d\mathbf{S}$ is the outward pointing surface element.

Verify that this result holds when $\phi = x + y$ and V is the spherical volume $x^2 + y^2 + z^2 \leqslant a^2$. [You may use the result that $d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where θ and ϕ are the usual angular coordinates in spherical polars and the components of $d\mathbf{S}$ are with respect to standard Cartesian axes.]

4B Vector Calculus

Let

$$\begin{aligned} u &= (2x + x^2 z + z^3)\ \exp((x+y)z) \\ v &= (x^2 z + z^3)\ \exp((x+y)z) \\ w &= (2z + x^3 + x^2 y + x z^2 + y z^2)\ \exp((x+y)z) \end{aligned}$$

Show that $u\,dx + v\,dy + w\,dz$ is an *exact differential*, clearly stating any criteria that you use.

Show that for any path between $(-1,0,1)$ and $(1,0,1)$

$$\int_{(-1,0,1)}^{(1,0,1)} (u\,dx + v\,dy + w\,dz) = 4\sinh 1.$$

SECTION II

5D Groups

Let H and K be subgroups of a group G satisfying the following two properties.

(i) All elements of G can be written in the form hk for some $h \in H$ and some $k \in K$.

(ii) $H \cap K = \{e\}$.

Prove that H and K are normal subgroups of G if and only if all elements of H commute with all elements of K.

State and prove Cauchy's Theorem.

Let p and q be distinct primes. Prove that an abelian group of order pq is isomorphic to $C_p \times C_q$. Is it true that all abelian groups of order p^2 are isomorphic to $C_p \times C_p$?

6D Groups

State and prove Lagrange's Theorem.

Hence show that if G is a finite group and $g \in G$ then the order of g divides the order of G.

How many elements are there of order 3 in the following groups? Justify your answers.

(a) $C_3 \times C_9$, where C_n denotes the cyclic group of order n.

(b) D_{2n} the dihedral group of order $2n$.

(c) S_7 the symmetric group of degree 7.

(d) A_7 the alternating group of degree 7.

7D Groups

State and prove the first isomorphism theorem. [You may assume that kernels of homomorphisms are normal subgroups and images are subgroups.]

Let G be a group with subgroup H and normal subgroup N. Prove that $NH = \{nh : n \in N, h \in H\}$ is a subgroup of G and $N \cap H$ is a normal subgroup of H. Further, show that N is a normal subgroup of NH.

Prove that $\frac{H}{N \cap H}$ is isomorphic to $\frac{NH}{N}$.

If K and H are both normal subgroups of G must KH be a normal subgroup of G?

If K and H are subgroups of G, but not normal subgroups, must KH be a subgroup of G?

Justify your answers.

8D Groups

Let $\mathcal{M}$ be the group of Möbius transformations of $\mathbb{C} \cup \{\infty\}$ and let $\mathrm{SL}_2(\mathbb{C})$ be the group of all 2×2 complex matrices of determinant 1.

Show that the map $\theta : \mathrm{SL}_2(\mathbb{C}) \to \mathcal{M}$ given by

$$\theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az+b}{cz+d}$$

is a surjective homomorphism. Find its kernel.

Show that any $T \in \mathcal{M}$ not equal to the identity is conjugate to a Möbius map S where either $Sz = \mu z$ with $\mu \neq 0, 1$ or $Sz = z + 1$. [You may use results about matrices in $\mathrm{SL}_2(\mathbb{C})$ as long as they are clearly stated.]

Show that any non-identity Möbius map has one or two fixed points. Also show that if T is a Möbius map with just one fixed point z_0 then $T^n z \to z_0$ as $n \to \infty$ for any $z \in \mathbb{C} \cup \{\infty\}$. [You may assume that Möbius maps are continuous.]

9B Vector Calculus

Define the *Jacobian*, J, of the one-to-one transformation

$$(x, y, z) \to (u, v, w).$$

Give a careful explanation of the result

$$\int_D f(x, y, z)\, dx\, dy\, dz = \int_\Delta |J|\, \phi(u, v, w)\, du\, dv\, dw,$$

where

$$\phi(u, v, w) = f(x(u, v, w), y(u, v, w), z(u, v, w))$$

and the region D maps under the transformation to the region Δ.

Consider the region D defined by

$$x^2 + \frac{y^2}{k^2} + z^2 \leqslant 1$$

and

$$\frac{x^2}{\alpha^2} + \frac{y^2}{k^2\alpha^2} - \frac{z^2}{\gamma^2} \geqslant 1,$$

where α, γ and k are positive constants.

Let $\tilde{D}$ be the intersection of D with the plane $y = 0$. Write down the conditions for $\tilde{D}$ to be non-empty. Sketch the geometry of $\tilde{D}$ in $x > 0$, clearly specifying the curves that define its boundaries and points that correspond to minimum and maximum values of x and of z on the boundaries.

Use a suitable change of variables to evaluate the volume of the region D, clearly explaining the steps in your calculation.

10B Vector Calculus

For a given set of coordinate axes the components of a 2nd rank tensor T are given by T_{ij}.

(a) Show that if λ is an eigenvalue of the matrix with elements T_{ij} then it is also an eigenvalue of the matrix of the components of T in any other coordinate frame.

Show that if T is a symmetric tensor then the multiplicity of the eigenvalues of the matrix of components of T is independent of coordinate frame.

A symmetric tensor T in three dimensions has eigenvalues λ, λ, μ, with $\mu \neq \lambda$.

Show that the components of T can be written in the form

$$T_{ij} = \alpha\delta_{ij} + \beta n_i n_j \tag{1}$$

where n_i are the components of a unit vector.

(b) The tensor T is defined by

$$T_{ij}(\mathbf{y}) = \int_S x_i\, x_j\, \exp(-c|\mathbf{y} - \mathbf{x}|^2)\, dA(\mathbf{x}),$$

where S is the surface of the unit sphere, $\mathbf{y}$ is the position vector of a point on S, and c is a constant.

Deduce, with brief reasoning, that the components of T can be written in the form (1) with $n_i = y_i$. [You may quote any results derived in part (a).]

Using suitable spherical polar coordinates evaluate T_{kk} and $T_{ij}y_iy_j$.

Explain how to deduce the values of α and β from T_{kk} and $T_{ij}y_iy_j$. [You do not need to write out the detailed formulae for these quantities.]

11B Vector Calculus

Show that for a vector field $\mathbf{A}$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla}^2 \mathbf{A}.$$

Hence find an $\mathbf{A}(\mathbf{x})$, with $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, such that $\mathbf{u} = (y^2, z^2, x^2) = \boldsymbol{\nabla} \times \mathbf{A}$. *[Hint: Note that* $\mathbf{A}(\mathbf{x})$ *is not defined uniquely. Choose your expression for* $\mathbf{A}(\mathbf{x})$ *to be as simple as possible.]*

Now consider the cone $x^2 + y^2 \leqslant z^2 \tan^2 \alpha$, $0 \leqslant z \leqslant h$. Let S_1 be the curved part of the surface of the cone ($x^2 + y^2 = z^2 \tan^2 \alpha$, $0 \leqslant z \leqslant h$) and S_2 be the flat part of the surface of the cone ($x^2 + y^2 \leqslant h^2 \tan^2 \alpha$, $z = h$).

Using the variables z and ϕ as used in cylindrical polars (r, ϕ, z) to describe points on S_1, give an expression for the surface element $d\mathbf{S}$ in terms of dz and $d\phi$.

Evaluate $\int_{S_1} \mathbf{u} \cdot d\mathbf{S}$.

What does the divergence theorem predict about the two surface integrals $\int_{S_1} \mathbf{u} \cdot d\mathbf{S}$ and $\int_{S_2} \mathbf{u} \cdot d\mathbf{S}$ where in each case the vector $d\mathbf{S}$ is taken outwards from the cone?

What does Stokes theorem predict about the integrals $\int_{S_1} \mathbf{u} \cdot d\mathbf{S}$ and $\int_{S_2} \mathbf{u} \cdot d\mathbf{S}$ (defined as in the previous paragraph) and the line integral $\int_C \mathbf{A} \cdot d\mathbf{l}$ where C is the circle $x^2 + y^2 = h^2 \tan^2 \alpha$, $z = h$ and the integral is taken in the anticlockwise sense, looking from the positive z direction?

Evaluate $\int_{S_2} \mathbf{u} \cdot d\mathbf{S}$ and $\int_C \mathbf{A} \cdot d\mathbf{l}$, making your method clear and verify that each of these predictions holds.

12B Vector Calculus

(a) The function u satisfies $\nabla^2 u = 0$ in the volume V and $u = 0$ on S, the surface bounding V.

Show that $u = 0$ everywhere in V.

The function v satisfies $\nabla^2 v = 0$ in V and v is specified on S. Show that for all functions w such that $w = v$ on S

$$\int_V \nabla v \cdot \nabla w \, dV = \int_V |\nabla v|^2 \, dV.$$

Hence show that

$$\int_V |\nabla w|^2 \, dV = \int_V \{|\nabla v|^2 + |\nabla(w - v)|^2\} \, dV \geqslant \int_V |\nabla v|^2 \, dV.$$

(b) The function ϕ satisfies $\nabla^2 \phi = \rho(\mathbf{x})$ in the spherical region $|\mathbf{x}| < a$, with $\phi = 0$ on $|\mathbf{x}| = a$. The function $\rho(\mathbf{x})$ is spherically symmetric, i.e. $\rho(\mathbf{x}) = \rho(|\mathbf{x}|) = \rho(r)$.

Suppose that the equation and boundary conditions are satisfied by a spherically symmetric function $\Phi(r)$. Show that

$$4\pi r^2 \Phi'(r) = 4\pi \int_0^r s^2 \rho(s) \, ds.$$

Hence find the function $\Phi(r)$ when $\rho(r)$ is given by $\rho(r) = \begin{cases} \rho_0 & \text{if } 0 \leqslant r \leqslant b \\ 0 & \text{if } b < r \leqslant a \end{cases}$, with ρ_0 constant.

Explain how the results obtained in part (a) of the question imply that $\Phi(r)$ is the only solution of $\nabla^2 \phi = \rho(r)$ which satisfies the specified boundary condition on $|\mathbf{x}| = a$.

Use your solution and the results obtained in part (a) of the question to show that, for any function w such that $w = 1$ on $r = b$ and $w = 0$ on $r = a$,

$$\int_{U(b,a)} |\nabla w|^2 \, dV \geqslant \frac{4\pi ab}{a - b},$$

where $U(b, a)$ is the region $b < r < a$.

END OF PAPER