UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 30 May, 2019 9:00 am to 12:00 pm MAT0

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Vectors and Matrices

(a) If

$$x + iy = \sum_{a=0}^{200} i^a + \prod_{b=1}^{50} i^b,$$

where $x, y \in \mathbb{R}$, what is the value of xy?

(b) Evaluate

$$\frac{(1+i)^{2019}}{(1-i)^{2017}}.$$

(c) Find a complex number z such that

$$i^{i^z} = 2.$$

(d) Interpret geometrically the curve defined by the set of points satisfying

$$\log z = i \log \overline{z}$$

in the complex z-plane.

2A Vectors and Matrices

If A is an n by n matrix, define its *determinant* $\det A$.

Find the following in terms of $\det A$ and a scalar λ, clearly showing your argument:

(i) $\det B$, where B is obtained from A by multiplying one row by λ.

(ii) $\det(\lambda A)$.

(iii) $\det C$, where C is obtained from A by switching row k and row l ($k \neq l$).

(iv) $\det D$, where D is obtained from A by adding λ times column l to column k ($k \neq l$).

3E Analysis I

State the Bolzano-Weierstrass theorem.

Let (a_n) be a sequence of non-zero real numbers. Which of the following conditions is sufficient to ensure that $(1/a_n)$ converges? Give a proof or counter-example as appropriate.

(i) $a_n \to \ell$ for some real number ℓ.

(ii) $a_n \to \ell$ for some non-zero real number ℓ.

(iii) (a_n) has no convergent subsequence.

4F Analysis I

Let $\sum_{n=1}^{\infty} a_n x^n$ be a real power series that diverges for at least one value of x. Show that there exists a non-negative real number R such that $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely whenever $|x| < R$ and diverges whenever $|x| > R$.

Find, with justification, such a number R for each of the following real power series:

(i) $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$;

(ii) $\sum_{n=1}^{\infty} x^n \left(1 + \frac{1}{n}\right)^n$.

SECTION II

5C Vectors and Matrices

(a) Use index notation to prove $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.
Hence simplify

(i) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$,

(ii) $(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})]$.

(b) Give the general solution for $\mathbf{x}$ and $\mathbf{y}$ of the simultaneous equations

$$\mathbf{x} + \mathbf{y} = 2\mathbf{a}, \qquad \mathbf{x} \cdot \mathbf{y} = c \qquad (c < \mathbf{a} \cdot \mathbf{a}).$$

Show in particular that $\mathbf{x}$ and $\mathbf{y}$ must lie at opposite ends of a diameter of a sphere whose centre and radius should be found.

(c) If two pairs of opposite edges of a tetrahedron are perpendicular, show that the third pair are also perpendicular to each other. Show also that the sum of the lengths squared of two opposite edges is the same for each pair.

6B Vectors and Matrices

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the standard basis vectors of $\mathbb{R}^3$. A second set of vectors $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ are defined with respect to the standard basis by

$$\mathbf{f}_j = \sum_{i=1}^{3} P_{ij}\mathbf{e}_i, \qquad j = 1, 2, 3.$$

The P_{ij} are the elements of the 3×3 matrix P. State the condition on P under which the set $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ forms a basis of $\mathbb{R}^3$.

Define the matrix A that, for a given linear transformation α, gives the relation between the components of any vector $\mathbf{v}$ and those of the corresponding $\alpha(\mathbf{v})$, with the components specified with respect to the standard basis.

Show that the relation between the matrix A and the matrix $\tilde{A}$ of the same transformation with respect to the second basis $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is

$$\tilde{A} = P^{-1}AP.$$

Consider the matrix

$$A = \begin{pmatrix} 2 & 6 & 2 \\ 0 & -1 & -1 \\ 0 & 6 & 4 \end{pmatrix}.$$

Find a matrix P such that $B = P^{-1}AP$ is diagonal. Give the elements of B and demonstrate explicitly that the relation between A and B holds.

Give the elements of A^nP for any positive integer n.

7B Vectors and Matrices

(a) Let A be an $n \times n$ matrix. Define the characteristic polynomial $\chi_A(z)$ of A. [Choose a sign convention such that the coefficient of z^n in the polynomial is equal to $(-1)^n$.] State and justify the relation between the characteristic polynomial and the eigenvalues of A. Why does A have at least one eigenvalue?

(b) Assume that A has n distinct eigenvalues. Show that $\chi_A(A) = 0$. [Each term $c_r z^r$ in $\chi_A(z)$ corresponds to a term $c_r A^r$ in $\chi_A(A)$.]

(c) For a general $n \times n$ matrix B and integer $m \geqslant 1$, show that $\chi_{B^m}(z^m) = \prod_{l=1}^{m} \chi_B(\omega_l z)$, where $\omega_l = e^{2\pi i l/m}$, $(l = 1, \ldots, m)$. [*Hint: You may find it helpful to note the factorization of* $z^m - 1$.]

Prove that if B^m has an eigenvalue λ then B has an eigenvalue μ where $\mu^m = \lambda$.

8A Vectors and Matrices

The exponential of a square matrix M is defined as

$$\exp M = I + \sum_{n=1}^{\infty} \frac{M^n}{n!},$$

where I is the identity matrix. [You do not have to consider issues of convergence.]

(a) Calculate the elements of R and S, where

$$R = \exp \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}, \qquad S = \exp \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}$$

and θ is a real number.

(b) Show that $RR^T = I$ and that

$$SJS = J, \qquad \text{where} \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(c) Consider the matrices

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1/2 \\ 0 & 1/2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Calculate:
(i) $\exp(xA)$,
(ii) $\exp(xB)$.

(d) Defining

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

find the elements of the following matrices, where N is a natural number:
(i)

$$\sum_{n=1}^{N} \left(\exp(xA) C [\exp(xA)]^T\right)^n,$$

(ii)

$$\sum_{n=1}^{N} \left(\exp(xB) C \exp(xB)\right)^n.$$

[Your answers to parts (a), (c) and (d) should be in closed form, i.e. not given as series.]

9D Analysis I

Let $g : \mathbb{R} \to \mathbb{R}$ be a function that is continuous at at least one point $z \in \mathbb{R}$. Suppose further that g satisfies

$$g(x + y) = g(x) + g(y)$$

for all $x, y \in \mathbb{R}$. Show that g is continuous on $\mathbb{R}$.

Show that there exists a constant c such that $g(x) = cx$ for all $x \in \mathbb{R}$.

Suppose that $h : \mathbb{R} \to \mathbb{R}$ is a continuous function defined on $\mathbb{R}$ and that h satisfies the equation

$$h(x + y) = h(x)h(y)$$

for all $x, y \in \mathbb{R}$. Show that h is either identically zero or everywhere positive. What is the general form for h?

10D Analysis I

State and prove the Intermediate Value Theorem.

State the Mean Value Theorem.

Suppose that the function g is differentiable everywhere in some open interval containing $[a, b]$, and that $g'(a) < k < g'(b)$. By considering the functions h and f defined by

$$h(x) = \frac{g(x) - g(a)}{x - a} \quad (a < x \leqslant b), \quad h(a) = g'(a)$$

and

$$f(x) = \frac{g(b) - g(x)}{b - x} \quad (a \leqslant x < b), \quad f(b) = g'(b),$$

or otherwise, show that there is a subinterval $[\alpha, \beta] \subseteq [a, b]$ such that

$$\frac{g(\beta) - g(\alpha)}{\beta - \alpha} = k.$$

Deduce that there exists $c \in (a, b)$ with $g'(c) = k$.

11E Analysis I

Let (a_n) and (b_n) be sequences of positive real numbers. Let $s_n = \sum_{i=1}^{n} a_i$.

(a) Show that if $\sum a_n$ and $\sum b_n$ converge then so does $\sum \left(a_n^2 + b_n^2\right)^{1/2}$.

(b) Show that if $\sum a_n$ converges then $\sum \sqrt{a_n a_{n+1}}$ converges. Is the converse true?

(c) Show that if $\sum a_n$ diverges then $\sum \frac{a_n}{s_n}$ diverges. Is the converse true?

[*For part (c), it may help to show that for any* $N \in \mathbb{N}$ *there exist* $m \geqslant n \geqslant N$ *with*

$$\frac{a_{n+1}}{s_{n+1}} + \frac{a_{n+2}}{s_{n+2}} + \ldots + \frac{a_m}{s_m} \geqslant \frac{1}{2}.]$$

12F Analysis I

Let $f \colon [0,1] \to \mathbb{R}$ be a bounded function. Define the upper and lower integrals of f. What does it mean to say that f is *Riemann integrable*? If f is Riemann integrable, what is the *Riemann integral* $\int_0^1 f(x)\,dx$?

Which of the following functions $f \colon [0,1] \to \mathbb{R}$ are Riemann integrable? For those that are Riemann integrable, find $\int_0^1 f(x)\,dx$. Justify your answers.

(i) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$;

(ii) $f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$,
where $A = \{x \in [0,1] : x \text{ has a base-3 expansion containing a } 1\}$;

[Hint: You may find it helpful to note, for example, that $\frac{2}{3} \in A$ *as one of the base–3 expansions of* $\frac{2}{3}$ *is 0.1222... .]*

(iii) $f(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$,
where $B = \{x \in [0,1] : x \text{ has a base–3 expansion containing infinitely many 1s}\}$.

END OF PAPER