MATHEMATICAL TRIPOS   Part II

Thursday, 7 June, 2018   9:00 am to 12:00 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

*Candidates may obtain credit from attempts on **at most six questions** from Section I and from any number of questions from Section II.*

*Write on **one side** of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

*Tie up your answers in bundles, marked **A**, **B**, **C**, ..., **K** according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet <u>must</u> bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

> **You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

SECTION I

1G Number Theory

What is a *multiplicative function*? Show that if $f(n)$ is a multiplicative function, then so is $g(n) = \sum_{d|n} f(d)$.

Define the *Möbius function* $\mu(n)$, and show that it is multiplicative. Deduce that

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

and that

$$f(n) = \sum_{e|n} \mu(e) g\left(\frac{n}{e}\right).$$

What is $g(n)$ if $f(n) = n$? What is $f(n)$ if $g(n) = n$?

2F Topics in Analysis

State a version of the Baire category theorem and use it to prove the following result:

If $f : \mathbb{C} \to \mathbb{C}$ is analytic, but not a polynomial, then there exists a point $z_0 \in \mathbb{C}$ such that each coefficient of the Taylor series of f at z_0 is non-zero.

3H Coding & Cryptography

Compute the rank and minimum distance of the cyclic code with generator polynomial $g(X) = X^3 + X^2 + 1$ and parity check polynomial $h(X) = X^4 + X^3 + X^2 + 1$. Now let α be a root of $g(X)$ in the field with 8 elements. We receive the word $r(X) = X^2 + X + 1$ (mod $X^7 - 1$). Verify that $r(\alpha) = \alpha^4$, and hence decode $r(X)$ using minimum-distance decoding.

4G   Automata and Formal Languages

(a) Define what it means for a context-free grammar (CFG) to be in *Chomsky normal form* (CNF).

(b) Give an algorithm for converting a CFG G into a corresponding CFG G_{Chom} in CNF satisfying $\mathcal{L}(G_{\text{Chom}}) = \mathcal{L}(G) - \{\epsilon\}$. [You need only outline the steps, without proof.]

(c) Convert the following CFG G:

$$S \to ASc \mid B \ , \ \ A \to a \ , \ \ B \to b \ \ ,$$

into a grammar in CNF.

5J Statistical Modelling

The data frame `Cases.of.flu` contains a list of cases of flu recorded in 3 London hospitals during each month of 2017. Consider the following R code and its output.

```
> table(Cases.of.flu)
         Hospital
Month         A   B   C
  April      10  40  27
  August      9  34  19
  December   24 129  81
  February   49 134  74
  January    45 138  78
  July        5  47  35
  June       11  36  22
  March      20  82  41
  May         5  43  23
  November   17  82  62
  October     6  26  19
  September   6  40  21
> Cases.of.flu.table = as.data.frame(table(Cases.of.flu))
> head(Cases.of.flu.table)
     Month Hospital Freq
1    April        A   10
2   August        A    9
3 December        A   24
4 February        A   49
5  January        A   45
6     July        A    5
> mod1 = glm(Freq ~., data=Cases.of.flu.table, family=poisson)
> mod1$dev
[1] 28.51836
> levels(Cases.of.flu$Month)
 [1] "April"     "August"    "December"  "February"  "January"   "July"
 [7] "June"      "March"     "May"       "November"  "October"   "September"
> levels(Cases.of.flu$Month) <- c("Q2","Q3","Q4","Q1","Q1","Q3",
+                                 "Q2","Q1","Q2","Q4","Q4","Q3")
> Cases.of.flu.table = as.data.frame(table(Cases.of.flu))
> mod2 = glm(Freq ~., data=Cases.of.flu.table, family=poisson)
> mod2$dev
[1] 17.9181
```

Describe a test for the null hypothesis of independence between the variables `Month` and `Hospital` using the deviance statistic. State the assumptions of the test.

Perform the test at the 1% level for each of the two different models shown above. You may use the table below showing 99th percentiles of the χ^2_p distribution with a range of degrees of freedom p. How would you explain the discrepancy between their conclusions?

Degrees of freedom	99th percentile	Degrees of freedom	99th percentile
1	6.63	21	38.93
2	9.21	22	40.29
3	11.34	23	41.64
4	13.28	24	42.98
5	15.09	25	44.31
6	16.81	26	45.64
7	18.48	27	46.96
8	20.09	28	48.28
9	21.67	29	49.59
10	23.21	30	50.89
11	24.72	31	52.19
12	26.22	32	53.49
13	27.69	33	54.78
14	29.14	34	56.06
15	30.58	35	57.34
16	32.00	36	58.62
17	33.41	37	59.89
18	34.81	38	61.16
19	36.19	39	62.43
20	37.57	40	63.69

6C   Mathematical Biology

Consider a nonlinear model for the axisymmetric dispersal of a population in two spatial dimensions whose density, $n(r,t)$, obeys

$$\frac{\partial n}{\partial t} = D\boldsymbol{\nabla} \cdot (n\boldsymbol{\nabla}n)\,,$$

where D is a positive constant, r is a radial polar coordinate, and t is time.

Show that

$$2\pi \int_0^\infty n(r,t)r dr = N$$

is constant. Interpret this condition.

Show that a similarity solution of the form

$$n(r,t) = \left(\frac{N}{Dt}\right)^{1/2} f\left(\frac{r}{(NDt)^{1/4}}\right)$$

is valid for $t > 0$ provided that the scaling function $f(x)$ satisfies

$$\frac{d}{dx}\left(xf\frac{df}{dx} + \frac{1}{4}x^2 f\right) = 0\,.$$

Show that there exists a value x_0 (which need not be evaluated) such that $f(x) > 0$ for $x < x_0$ but $f(x) = 0$ for $x > x_0$. Determine the area within which $n(r,t) > 0$ at time t in terms of x_0.

$\Big[$*Hint: The gradient and divergence operators in cylindrical polar coordinates act on radial functions f and g as*

$$\boldsymbol{\nabla}f(r) = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} \quad , \quad \boldsymbol{\nabla}\cdot[g(r)\hat{\boldsymbol{r}}] = \frac{1}{r}\frac{\partial}{\partial r}(rg(r)). \quad \Big]$$

7B   Further Complex Methods

Using a suitable branch cut, show that

$$\int_{-a}^{a}(a^2 - x^2)^{1/2}dx = \frac{a^2\pi}{2},$$

where $a > 0$.

8B Classical Dynamics

Three particles of unit mass move along a line in a potential

$$V = \frac{1}{2}\Big((x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_3 - x_2)^2 + x_1^2 + x_2^2 + x_3^2 \Big),$$

where x_i is the coordinate of the i'th particle, $i = 1, 2, 3$.

Write the Lagrangian in the form

$$\mathcal{L} = \frac{1}{2} T_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} V_{ij} x_i x_j,$$

and specify the matrices T_{ij} and V_{ij}.

Find the normal frequencies and normal modes for this system.

9B **Cosmology**

The energy density of a particle species is defined by

$$\epsilon = \int_0^\infty E(p)n(p)dp\,,$$

where $E(p) = c\sqrt{p^2 + m^2c^2}$ is the energy, and $n(p)$ the distribution function, of a particle with momentum p. Here c is the speed of light and m is the rest mass of the particle. If the particle species is in thermal equilibrium then the distribution function takes the form

$$n(p) = \frac{4\pi}{h^3}g\frac{p^2}{\exp((E(p) - \mu)/kT) \mp 1}\,,$$

where g is the number of degrees of freedom of the particle, T is the temperature, h and k are constants and $-$ is for bosons and $+$ is for fermions.

(a) Stating any assumptions you require, show that in the very early universe the energy density of a given particle species i is

$$\epsilon_i = \frac{4\pi g_i}{(hc)^3}\left(kT\right)^4 \int_0^\infty \frac{y^3}{e^y \mp 1}dy\,.$$

(b) Show that the total energy density in the very early universe is

$$\epsilon = \frac{4\pi^5}{15(hc)^3}g^* \left(kT\right)^4\,,$$

where g^* is defined by

$$g^* \equiv \sum_{\text{Bosons}} g_i + \frac{7}{8} \sum_{\text{Fermions}} g_i.$$

[*Hint: You may use the fact that* $\int_0^\infty y^3(e^y - 1)^{-1}dy = \pi^4/15$.]

10D Quantum Information and Computation

Let B_n denote the set of all n-bit strings. For any Boolean function on 2 bits $f : B_2 \to B_1$ consider the linear operation on 3 qubits defined by

$$U_f \ket{x} \ket{y} = \ket{x} \ket{y \oplus f(x)}$$

for all $x \in B_2$, $y \in B_1$ and $\oplus$ denoting addition of bits modulo 2. Here the first register is a 2-qubit register and the second is a 1-qubit register. We are able to apply only the 1-qubit Pauli X and Hadamard H gates to any desired qubits, as well as the 3-qubit gate U_f to any three qubits. We can also perform measurements in the computational basis.

(a) Describe how we can construct the state

$$\ket{f} = \frac{1}{2} \sum_{x \in B_2} (-1)^{f(x)} \ket{x}$$

starting from the standard 3-qubit state $\ket{0} \ket{0} \ket{0}$.

(b) Suppose now that the gate U_f is given to us but f is not specified. However f is promised to be one of two following cases:

    (i) f is a constant function (i.e. $f(x) = 0$ for all x, or $f(x) = 1$ for all x),

    (ii) for any 2-bit string $x = b_1 b_2$ we have $f(b_1 b_2) = b_1 \oplus b_2$ (with $\oplus$ as above).

Show how we may determine with certainty which of the two cases (i) or (ii) applies, using only a *single* application of U_f.

SECTION II

11G Number Theory

What does it mean to say that a positive definite binary quadratic form is *reduced*? What does it mean to say that two binary quadratic forms are *equivalent*? Show that every positive definite binary quadratic form is equivalent to some reduced form.

Show that the reduced positive definite binary quadratic forms of discriminant -35 are $f_1 = x^2 + xy + 9y^2$ and $f_2 = 3x^2 + xy + 3y^2$. Show also that a prime $p > 7$ is represented by f_i if and only if

$$\left(\frac{p}{5}\right) = \left(\frac{p}{7}\right) = \begin{cases} +1 & (i = 1) \\ -1 & (i = 2). \end{cases}$$

12G Automata and Formal Languages

(a) State and prove the pumping lemma for regular languages.

(b) Let D be a minimal deterministic finite-state automaton whose language $\mathcal{L}(D)$ is finite. Let Γ_D be the transition diagram of D, and suppose there exists a non-empty closed path γ in Γ_D starting and ending at state p.

    (i) Show that there is no path in Γ_D from p to any accept state of D.

    (ii) Show that there is no path in Γ_D from p to any other state of D.

13C Mathematical Biology

Consider fluctuations of a population described by the vector $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. The probability of the state $\mathbf{x}$ at time t, $P(\mathbf{x}, t)$, obeys the multivariate Fokker–Planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x_i}\big(A_i(\mathbf{x})P\big) + \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j}\big(B_{ij}(\mathbf{x})P\big),$$

where $P = P(\mathbf{x}, t)$, A_i is a *drift* vector and B_{ij} is a symmetric positive-definite *diffusion* matrix, and the summation convention is used throughout.

(a) Show that the Fokker–Planck equation can be expressed as a continuity equation

$$\frac{\partial P}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0,$$

for some choice of probability flux $\mathbf{J}$ which you should determine explicitly. Here, $\boldsymbol{\nabla} = \big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_N}\big)$ denotes the gradient operator.

(b) Show that the above implies that an initially normalised probability distribution remains normalised,

$$\int P(\mathbf{x}, t)dV = 1,$$

at all times, where the volume element $dV = dx_1 dx_2 \ldots dx_N$.

(c) Show that the first two moments of the probability distribution obey

$$\frac{d}{dt}\langle x_k \rangle = \langle A_k \rangle$$
$$\frac{d}{dt}\langle x_k x_l \rangle = \langle x_l A_k + x_k A_l + B_{kl} \rangle.$$

(d) Now consider small fluctuations with zero mean, and assume that it is possible to linearise the drift vector and the diffusion matrix as $A_i(\mathbf{x}) = a_{ij}x_j$ and $B_{ij}(\mathbf{x}) = b_{ij}$ where a_{ij} has real negative eigenvalues and b_{ij} is a symmetric positive-definite matrix. Express the probability flux in terms of the matrices a_{ij} and b_{ij} and assume that it vanishes in the stationary state.

(e) Hence show that the multivariate normal distribution,

$$P(\mathbf{x}) = \frac{1}{Z}\exp(-\frac{1}{2}D_{ij}x_i x_j),$$

where Z is a normalisation and D_{ij} is symmetric, is a solution of the linearised Fokker–Planck equation in the stationary state, and obtain an equation that relates D_{ij} to the matrices a_{ij} and b_{ij}.

(f) Show that the inverse of the matrix D_{ij} is the matrix of covariances $C_{ij} = \langle x_i x_j \rangle$ and obtain an equation relating C_{ij} to the matrices a_{ij} and b_{ij}.

14B Cosmology

The pressure support equation for stars is

$$\frac{1}{r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -4\pi G\rho\,,$$

where ρ is the density, P is the pressure, r is the radial distance, and G is Newton's constant.

(a) What two boundary conditions should we impose on the above equation for it to describe a star?

(b) By assuming a polytropic equation of state,

$$P(r) = K\rho^{1+\frac{1}{n}}(r)\,,$$

where K is a constant, derive the Lane–Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n,$$

where $\rho = \rho_c\theta^n$, with ρ_c the density at the centre of the star, and $r = a\xi$, for some a that you should determine.

(c) Show that the mass of a polytropic star is

$$M = \frac{1}{2\sqrt{\pi}}\left(\frac{(n+1)\,K}{G}\right)^{\frac{3}{2}}\rho_c^{\frac{3-n}{2n}}Y_n,$$

where $Y_n \equiv -\left.\xi_1^2\frac{d\theta}{d\xi}\right|_{\xi=\xi_1}$ and ξ_1 is the value of ξ at the surface of the star.

(d) Derive the following relation between the mass, M, and radius, R, of a polytropic star

$$M = A_n K^{\frac{n}{n-1}}R^{\frac{3-n}{1-n}},$$

where you should determine the constant A_n. What type of star does the $n = 3$ polytrope represent and what is the significance of the mass being constant for this star?

15D Quantum Information and Computation

In this question you may assume the following fact about the quantum Fourier transform QFT mod N: if $N = Ar$ and $0 \leqslant x_0 < r$, where $A, r, x_0 \in \mathbb{Z}$, then

$$QFT \frac{1}{\sqrt{A}} \sum_{k=0}^{A-1} |\, x_0 + kr \rangle = \frac{1}{\sqrt{r}} \sum_{l=0}^{r-1} \omega^{x_0 lA} |\, lA \rangle$$

where $\omega = e^{2\pi i/N}$.

(a) Let $\mathbb{Z}_N$ denote the integers modulo N. Let $f : \mathbb{Z}_N \to \mathbb{Z}$ be a periodic function with period r and with the property that f is one-to-one within each period. We have one instance of the quantum state

$$|\, f \rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |\, x \rangle |\, f(x) \rangle$$

and the ability to calculate the function f on at most two x values of our choice.

Describe a procedure that may be used to determine the period r with success probability $O(1/\log \log N)$. As a further requirement, at the end of the procedure we should know if it has been successful, or not, in outputting the correct period value. [You may assume that the number of integers less than N that are coprime to N is $O(N/\log \log N)$].

(b) Consider the function $f : \mathbb{Z}_{12} \to \mathbb{Z}_{10}$ defined by $f(x) = 3^x \mod 10$.

(i) Show that f is periodic and find the period.

(ii) Suppose we are given the state $|\, f \rangle = \frac{1}{\sqrt{12}} \sum_{x=0}^{11} |\, x \rangle |\, f(x) \rangle$ and we measure the second register. What are the possible resulting measurement values y and their probabilities?

(iii) Suppose the measurement result was $y = 3$. Find the resulting state $|\, \alpha \rangle$ of the first register after the measurement.

(iv) Suppose we measure the state $QFT |\, \alpha \rangle$ (with $|\, \alpha \rangle$ from part (iii)). What is the probability of each outcome $0 \leqslant c \leqslant 11$?

16G Logic and Set Theory

State and prove the Compactness Theorem for first-order predicate logic. State and prove the Upward Löwenheim–Skolem Theorem.

[You may assume the Completeness Theorem for first-order predicate logic.]

For each of the following theories, either give axioms (in the specified language) for the theory or prove that the theory is not axiomatisable.

(i) The theory of finite groups (in the language of groups).

(ii) The theory of groups in which every non-identity element has infinite order (in the language of groups).

(iii) The theory of total orders (in the language of posets).

(iv) The theory of well-orderings (in the language of posets).

If a theory is axiomatisable by a set S of sentences, and also by a finite set T of sentences, does it follow that the theory is axiomatisable by some finite subset of S? Justify your answer.

17I   Graph Theory

What does it mean to say that a graph G has a *k-colouring*? What are the *chromatic number* $\chi(G)$ and the *independence number* $\alpha(G)$ of a graph G? For each $r \geqslant 3$, give an example of a graph G such that $\chi(G) > r$ but $K_r \not\subset G$.

Let $g, k \geqslant 3$. Show that there exists a graph G containing no cycle of length $\leqslant g$ with $\chi(G) \geqslant k$.

Show also that if n is sufficiently large then there is a triangle-free G of order n with $\alpha(G) < n^{0.7}$.

18I Galois Theory

Let L be a finite field extension of a field K, and let G be a finite group of K-automorphisms of L. Denote by L^G the field of elements of L fixed by the action of G.

(a) Prove that the degree of L over L^G is equal to the order of the group G.

(b) For any $\alpha \in L$ write $f(t, \alpha) = \Pi_{g \in G}(t - g(\alpha))$.

    (i) Suppose that $L = K(\alpha)$. Prove that the coefficients of $f(t, \alpha)$ generate L^G over K.

    (ii) Suppose that $L = K(\alpha_1, \alpha_2)$. Prove that the coefficients of $f(t, \alpha_1)$ and $f(t, \alpha_2)$ lie in L^G. By considering the case $L = K(a_1^{1/2}, a_2^{1/2})$ with a_1 and a_2 in K, or otherwise, show that they need not generate L^G over K.

19I Representation Theory

State the row orthogonality relations. Prove that if χ is an irreducible character of the finite group G, then $\chi(1)$ divides the order of G.

Stating clearly any additional results you use, deduce the following statements:

(i) Groups of order p^2, where p is prime, are abelian.

(ii) If G is a group of order $2p$, where p is prime, then either the degrees of the irreducible characters of G are all 1, or they are

$$1, 1, 2, \ldots, 2 \ (\text{with } (p-1)/2 \text{ of degree 2}).$$

(iii) No simple group has an irreducible character of degree 2.

(iv) Let p and q be prime numbers with $p > q$, and let G be a non-abelian group of order pq. Then q divides $p - 1$ and G has $q + ((p-1)/q)$ conjugacy classes.

20H   Algebraic Topology

(a) State a version of the Seifert–van Kampen theorem for a cell complex X written as the union of two subcomplexes Y, Z.

(b) Let

$$X_n = \underbrace{S^1 \vee \ldots \vee S^1}_{n} \vee \mathbb{R}P^2$$

for $n \geqslant 1$, and take any $x_0 \in X_n$. Write down a presentation for $\pi_1(X_n, x_0)$.

(c) By computing a homology group of a suitable four-sheeted covering space of X_n, prove that X_n is not homotopy equivalent to a compact, connected surface whenever $n \geqslant 1$.

21F   Linear Analysis

(a) Let X be a normed vector space and let Y be a Banach space. Show that the space of bounded linear operators $\mathcal{B}(X, Y)$ is a Banach space.

(b) Let X and Y be Banach spaces, and let $D \subset X$ be a dense linear subspace. Prove that a bounded linear map $T : D \to Y$ can be extended uniquely to a bounded linear map $T : X \to Y$ with the same operator norm. Is the claim also true if one of X and Y is not complete?

(c) Let X be a normed vector space. Let (x_n) be a sequence in X such that

$$\sum_{n=1}^{\infty} |f(x_n)| < \infty \qquad \forall f \in X^*.$$

Prove that there is a constant C such that

$$\sum_{n=1}^{\infty} |f(x_n)| \leqslant C\|f\| \qquad \forall f \in X^*.$$

22F Analysis of Functions

(a) Let $(X, \mathcal{A}, \mu)$ be a measure space. Define the spaces $L^p(X)$ for $p \in [1, \infty]$. Prove that if $\mu(X) < \infty$ then $L^q(X) \subset L^p(X)$ for all $1 \leqslant p < q \leqslant \infty$.

(b) Now let $X = \mathbb{R}^n$ endowed with Borel sets and Lebesgue measure. Describe the dual spaces of $L^p(X)$ for $p \in [1, \infty)$. Define *reflexivity* and say which $L^p(X)$ are reflexive. Prove that $L^1(X)$ is not the dual space of $L^\infty(X)$.

(c) Now let $X \subset \mathbb{R}^n$ be a Borel subset and consider the measure space $(X, \mathcal{A}, \mu)$ induced from Borel sets and Lebesgue measure on $\mathbb{R}^n$.

    (i) Given any $p \in [1, \infty]$, prove that any sequence (f_n) in $L^p(X)$ converging in $L^p(X)$ to some $f \in L^p(X)$ admits a subsequence converging almost everywhere to f.

    (ii) Prove that if $L^q(X) \subset L^p(X)$ for $1 \leqslant p < q \leqslant \infty$ then $\mu(X) < \infty$. [*Hint: You might want to prove first that the inclusion is continuous with the help of one of the corollaries of Baire's category theorem.*]

23F Riemann Surfaces

Define the *degree* of an analytic map of compact Riemann surfaces, and state the Riemann–Hurwitz formula.

Let Λ be a lattice in $\mathbb{C}$ and $E = \mathbb{C}/\Lambda$ the associated complex torus. Show that the map
$$\psi : z + \Lambda \mapsto -z + \Lambda$$
is biholomorphic with four fixed points in E.

Let $S = E/\sim$ be the quotient surface (the topological surface obtained by identifying $z + \Lambda$ and $\psi(z + \Lambda)$), and let $p : E \to S$ be the associated projection map. Denote by E' the complement of the four fixed points of ψ, and let $S' = p(E')$. Describe briefly a family of charts making S' into a Riemann surface, so that $p : E' \to S'$ is a holomorphic map.

Now assume that, by adding finitely many points, it is possible to compactify S' to a Riemann surface S so that p extends to a regular map $E \to S$. Find the genus of S.

24I Algebraic Geometry

(a) State the Riemann–Roch theorem.

(b) Let E be a smooth projective curve of genus 1 over an algebraically closed field k, with char $k \neq 2, 3$. Show that there exists an isomorphism from E to the plane cubic in $\mathbf{P}^2$ defined by the equation

$$y^2 = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3),$$

for some distinct $\lambda_1, \lambda_2, \lambda_3 \in k$.

(c) Let Q be the point at infinity on E. Show that the map $E \to Cl^0(E)$, $P \mapsto [P-Q]$ is an isomorphism.

Describe how this defines a group structure on E. Denote addition by $\boxplus$. Determine all the points $P \in E$ with $P \boxplus P = Q$ in terms of the equation of the plane curve in part (b).

25I Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface.

(a) Define the *Gaussian curvature* K of S in terms of the coefficients of the first and second fundamental forms, computed with respect to a local parametrization $\phi(u, v)$ of S.

Prove the *Theorema Egregium*, i.e. show that the Gaussian curvature can be expressed entirely in terms of the coefficients of the first fundamental form and their first and second derivatives with respect to u and v.

(b) State the global Gauss–Bonnet theorem for a compact orientable surface S.

(c) Now assume that S is non-compact and diffeomorphic to $\mathbb{S}^2 \setminus \{(1, 0, 0)\}$ but that there is a point $p \in \mathbb{R}^3$ such that $S \cup \{p\}$ is a compact subset of $\mathbb{R}^3$. Is it necessarily the case that $\int_S K dA = 4/\pi$? Justify your answer.

26J   Probability and Measure

Let m be the Lebesgue measure on the real line. Recall that if $E \subseteq \mathbb{R}$ is a Borel subset, then

$$m(E) = \inf \left\{ \sum_{n \geqslant 1} |I_n|,\ E \subseteq \bigcup_{n \geqslant 1} I_n \right\},$$

where the infimum is taken over all covers of E by countably many intervals, and $|I|$ denotes the length of an interval I.

(a) State the definition of a *Borel subset* of $\mathbb{R}$.

(b) State a definition of a *Lebesgue measurable subset* of $\mathbb{R}$.

(c) Explain why the following sets are Borel and compute their Lebesgue measure:

$$\mathbb{Q}, \quad \mathbb{R} \setminus \mathbb{Q}, \quad \bigcap_{n \geqslant 2} \left[\frac{1}{n}, n \right].$$

(d) State the definition of a *Borel measurable function* $f : \mathbb{R} \to \mathbb{R}$.

(e) Let f be a Borel measurable function $f : \mathbb{R} \to \mathbb{R}$. Is it true that the subset of all $x \in \mathbb{R}$ where f is continuous at x is a Borel subset? Justify your answer.

(f) Let $E \subseteq [0,1]$ be a Borel subset with $m(E) = 1/2 + \alpha$, $\alpha > 0$. Show that

$$E - E := \{x - y : x,\, y \in E\}$$

contains the interval $(-2\alpha, 2\alpha)$.

(g) Let $E \subseteq \mathbb{R}$ be a Borel subset such that $m(E) > 0$. Show that for every $\varepsilon > 0$, there exists $a < b$ in $\mathbb{R}$ such that

$$m(E \cap (a,b)) > (1 - \varepsilon) m((a,b)).$$

Deduce that $E - E$ contains an open interval around 0.

27J Applied Probability

Individuals arrive in a shop in the manner of a Poisson process with intensity λ, and they await service in the order of their arrival. Their service times are independent, identically distributed random variables $S_1, S_2, \ldots$. For $n \geqslant 1$, let Q_n be the number remaining in the shop immediately after the nth departure. Show that

$$Q_{n+1} = A_n + Q_n - h(Q_n),$$

where A_n is the number of arrivals during the $(n+1)$th service period, and $h(x) = \min\{1, x\}$.

Show that
$$\mathbb{E}(A_n) = \rho, \quad \mathbb{E}(A_n^2) = \rho + \lambda^2 \mathbb{E}(S^2),$$

where S is a typical service period, and $\rho = \lambda \mathbb{E}(S)$ is the *traffic intensity* of the queue.

Suppose $\rho < 1$, and the queue is in equilibrium in the sense that Q_n and Q_{n+1} have the same distribution for all n. Express $\mathbb{E}(Q_n)$ in terms of λ, $\mathbb{E}(S)$, $\mathbb{E}(S^2)$. Deduce that the mean waiting time (prior to service) of a typical individual is $\frac{1}{2}\lambda \mathbb{E}(S^2)/(1 - \rho)$.

28K Principles of Statistics

In the model $\{\mathcal{N}(\theta, I_p), \theta \in \mathbb{R}^p\}$ of a Gaussian distribution in dimension p, with unknown mean θ and known identity covariance matrix I_p, we estimate θ based on a sample of i.i.d. observations $X_1, \ldots, X_n$ drawn from $\mathcal{N}(\theta_0, I_p)$.

(a) Define the *Fisher information* $I(\theta_0)$, and compute it in this model.

(b) We recall that the *observed Fisher information* $i_n(\theta)$ is given by

$$i_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \nabla_\theta \log f(X_i, \theta) \, \nabla_\theta \log f(X_i, \theta)^\top.$$

Find the limit of $\hat{i}_n = i_n(\hat{\theta}_{MLE})$, where $\hat{\theta}_{MLE}$ is the maximum likelihood estimator of θ in this model.

(c) Define the *Wald statistic* $W_n(\theta)$ and compute it. Give the limiting distribution of $W_n(\theta_0)$ and explain how it can be used to design a confidence interval for θ_0.

[*You may use results from the course provided that you state them clearly.*]

29K Stochastic Financial Models

Consider a multi-period model with asset prices $\bar{S}_t = (S_t^0, \ldots, S_t^d)$, $t \in \{0, \ldots, T\}$, modelled on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to a filtration $(\mathcal{F}_t)_{t \in \{0, \ldots, T\}}$. Assume that $\mathcal{F}_0$ is $\mathbb{P}$-trivial, i.e. $\mathbb{P}[A] \in \{0, 1\}$ for all $A \in \mathcal{F}_0$, and assume that S^0 is a $\mathbb{P}$-a.s. strictly positive numéraire, i.e. $S_t^0 > 0$ $\mathbb{P}$-a.s. for all $t \in \{0, \ldots, T\}$. Further, let $X_t = (X_t^1, \ldots, X_t^d)$ denote the discounted price process defined by $X_t^i := S_t^i / S_t^0$, $t \in \{0, \ldots, T\}$, $i \in \{1, \ldots, d\}$.

(a) What does it mean to say that a self-financing strategy $\bar{\theta}$ is an *arbitrage*?

(b) State the fundamental theorem of asset pricing.

(c) Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F})$ which is equivalent to $\mathbb{P}$ and for which $\mathbb{E}_{\mathbb{Q}}[\|X_t\|] < \infty$ for all t. Show that the following are equivalent:

    (i) $\mathbb{Q}$ is a martingale measure.

    (ii) If $\bar{\theta} = (\theta^0, \theta)$ is self-financing and θ is bounded, i.e. $\max_{t=1,\ldots,T} |\theta_t| \leqslant c < \infty$ for a suitable $c > 0$, then the value process V of $\bar{\theta}$ is a $\mathbb{Q}$-martingale.

    (iii) If $\bar{\theta} = (\theta^0, \theta)$ is self-financing and θ is bounded, then the value process V of $\bar{\theta}$ satisfies

$$\mathbb{E}_{\mathbb{Q}}[V_T] = V_0.$$

[*Hint: To show that (iii) implies (i) you might find it useful to consider self-financing strategies $\bar{\theta} = (\theta^0, \theta)$ with θ of the form*

$$\theta_s := \begin{cases} \mathbf{1}_A & \text{if } s = t, \\ 0 & \text{otherwise,} \end{cases}$$

for any $A \in \mathcal{F}_{t-1}$ and any $t \in \{1, \ldots, T\}$.]

(d) Prove that if there exists a martingale measure $\mathbb{Q}$ satisfying the conditions in (c) then there is no arbitrage.

30K Optimisation and Control

The scalars x_t, y_t, u_t are related by the equations

$$x_t = x_{t-1} + u_{t-1}, \quad y_t = x_{t-1} + \eta_{t-1}, \quad t = 1, 2, \ldots, T,$$

where the initial state x_0 is normally distributed with mean $\hat{x}_0$ and variance 1 and $\{\eta_t\}$ is a sequence of independent random variables each normally distributed with mean 0 and variance 1. The control variable u_t is to be chosen at time t on the basis of information W_t, where $W_0 = (\hat{x}_0)$ and

$$W_t = (\hat{x}_0, u_0, \ldots, u_{t-1}, y_1, \ldots, y_t), \quad t = 1, 2, \ldots, T.$$

(a) Let $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_T$ be the Kalman filter estimates of $x_1, x_2, \ldots, x_T$, i.e.

$$\hat{x}_t = \hat{x}_{t-1} + u_{t-1} + h_t(y_t - \hat{x}_{t-1})$$

where h_t is chosen to minimise $\mathbb{E}((\hat{x}_t - x_t)^2 \mid W_t)$. Calculate h_t and show that, conditional on W_t, x_t is normally distributed with mean $\hat{x}_t$ and variance $V_t = 1/(1+t)$.

(b) Define

$$F(W_T) = \mathbb{E}\left(x_T^2 \mid W_T\right), \quad \text{and}$$

$$F(W_t) = \inf_{u_t, \ldots, u_{T-1}} \mathbb{E}\left(x_T^2 + \sum_{j=t}^{T-1} u_j^2 \,\Big|\, W_t \right), \quad t = 0, \ldots, T-1.$$

Show that $F(W_t) = \hat{x}_t^2 P_t + d_t$, where $P_t = 1/(T - t + 1), d_T = 1/(1 + T)$ and $d_{t-1} = V_{t-1} V_t P_t + d_t$.

(c) Show that the minimising control u_t can be expressed in the form $u_t = -K_t \hat{x}_t$ and find K_t. How would the expression for K_t be altered if x_0 or $\{\eta_t\}$ had variances other than 1?

31B Asymptotic Methods

(a) Find the curves of steepest descent emanating from $t = 0$ for the integral

$$J_x(x) \;=\; \frac{1}{2\pi i} \int_C e^{x(\sinh t - t)} \, dt \,,$$

for $x > 0$ and determine the angles at which they meet at $t = 0$, and their asymptotes at infinity.

(b) An integral representation for the Bessel function $K_\nu(x)$ for real $x > 0$ is

$$K_\nu(x) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{\nu h(t)} \, dt \quad , \qquad h(t) = t - \left(\frac{x}{\nu}\right) \cosh t \;.$$

Show that, as $\nu \to +\infty$, with x fixed,

$$K_\nu(x) \sim \left(\frac{\pi}{2\nu}\right)^{\frac{1}{2}} \left(\frac{2\nu}{ex}\right)^{\nu} \;.$$

32E Dynamical Systems

Consider the system

$$\dot{x} = y \,, \qquad \dot{y} = \mu_1 x + \mu_2 y - (x + y)^3 \,,$$

where μ_1 and μ_2 are parameters.

By considering a function of the form $V(x, y) = f(x + y) + \frac{1}{2}y^2$, show that when $\mu_1 = \mu_2 = 0$ the origin is globally asymptotically stable. Sketch the phase plane for this case.

Find the fixed points for the general case. Find the values of μ_1 and μ_2 for which the fixed points have (i) a stationary bifurcation and (ii) oscillatory (Hopf) bifurcations. Sketch these bifurcation values in the (μ_1, μ_2)-plane.

For the case $\mu_2 = -1$, find the leading-order approximation to the extended centre manifold of the bifurcation as μ_1 varies, assuming that $\mu_1 = O(x^2)$. Find also the evolution equation on the extended centre manifold to leading order. Deduce the type of bifurcation, and sketch the bifurcation diagram in the (μ_1, x)-plane.

33A Integrable Systems

Suppose $\psi^s : (x, u) \mapsto (\tilde{x}, \tilde{u})$ is a smooth one-parameter group of transformations acting on $\mathbb{R}^2$.

(a) Define the *generator* of the transformation,

$$V = \xi(x, u)\frac{\partial}{\partial x} + \eta(x, u)\frac{\partial}{\partial u},$$

where you should specify ξ and η in terms of ψ^s.

(b) Define the nth *prolongation* of V, $\mathrm{Pr}^{(n)} V$ and explicitly compute $\mathrm{Pr}^{(1)} V$ in terms of ξ, η.

Recall that if ψ^s is a Lie point symmetry of the ordinary differential equation:

$$\Delta\left(x, u, \frac{du}{dx}, \ldots, \frac{d^n u}{dx^n}\right) = 0,$$

then it follows that $\mathrm{Pr}^{(n)} V [\Delta] = 0$ whenever $\Delta = 0$.

(c) Consider the ordinary differential equation:

$$\frac{du}{dx} = F(x, u),$$

for F a smooth function. Show that if V generates a Lie point symmetry of this equation, then:

$$0 = \eta_x + (\eta_u - \xi_x - F\xi_u) F - \xi F_x - \eta F_u.$$

(d) Find all the Lie point symmetries of the equation:

$$\frac{du}{dx} = xG\left(\frac{u}{x^2}\right),$$

where G is an arbitrary smooth function.

34D Principles of Quantum Mechanics

A quantum system is prepared in the ground state $|0\rangle$ at time $t = 0$. It is subjected to a time-varying Hamiltonian $H = H_0 + \Delta(t)$. Show that, to first order in $\Delta(t)$, the system evolves as

$$|\psi(t)\rangle = \sum_k c_k(t)\, e^{-iE_k t/\hbar}|k\rangle\,,$$

where $H_0|k\rangle = E_k|k\rangle$ and

$$c_k(t) = \frac{1}{i\hbar}\int_0^t \langle k|\Delta(t')|0\rangle\, e^{i(E_k - E_0)t'/\hbar}\, \mathrm{d}t'\,.$$

A large number of hydrogen atoms, each in the ground state, are subjected to an electric field

$$\mathbf{E}(t) = \begin{cases} 0 & \text{for}\quad t < 0 \\ \hat{\mathbf{z}}\,\mathcal{E}_0 \exp(-t/\tau) & \text{for}\quad t > 0\,, \end{cases}$$

where $\mathcal{E}_0$ is a constant. Show that the fraction of atoms found in the state $|n,\ell,m\rangle = |2,1,0\rangle$ is, after a long time and to lowest non-trivial order in $\mathcal{E}_0$,

$$\frac{2^{15}}{3^{10}}\frac{a_0^2 e^2 \mathcal{E}_0^2}{\hbar^2(\omega^2 + 1/\tau^2)}\,,$$

where $\hbar\omega$ is the energy difference between the $|2,1,0\rangle$ and $|1,0,0\rangle$ states, and e is the electron charge and a_0 the Bohr radius. What fraction of atoms lie in the $|2,0,0\rangle$ state?

[*Hint: You may assume the hydrogenic wavefunctions*

$$\langle \mathbf{r}|1,0,0\rangle = \frac{2}{\sqrt{4\pi}}\frac{1}{a_0^{3/2}}\exp\left(-\frac{r}{a_0}\right) \quad and \quad \langle \mathbf{r}|2,1,0\rangle = \frac{1}{\sqrt{4\pi}}\frac{1}{(2a_0)^{3/2}}\frac{r}{a_0}\cos\theta\,\exp\left(-\frac{r}{2a_0}\right)$$

and the integral

$$\int_0^\infty r^m e^{-\alpha r}\, \mathrm{d}r = \frac{m!}{\alpha^{m+1}}$$

for m *a positive integer.*]

35A Applications of Quantum Mechanics

A beam of particles of mass m and momentum $p = \hbar k$ is incident along the z-axis. The beam scatters off a spherically symmetric potential $V(r)$. Write down the asymptotic form of the wavefunction in terms of the scattering amplitude $f(\theta)$.

The incoming plane wave and the scattering amplitude can be expanded in partial waves as,

$$e^{ikr\cos\theta} \sim \frac{1}{2ikr}\sum_{l=0}^{\infty}(2l+1)\left(e^{ikr}-(-1)^l e^{-ikr}\right)P_l(\cos\theta)$$

$$f(\theta) = \sum_{l=0}^{\infty}\frac{2l+1}{k}f_l\,P_l(\cos\theta)$$

where P_l are Legendre polynomials. Define the *S-matrix*. Assuming that the S-matrix is unitary, explain why we can write

$$f_l = e^{i\delta_l}\sin\delta_l$$

for some real phase shifts δ_l. Obtain an expression for the total cross-section σ_T in terms of the phase shifts δ_l.

[*Hint: You may use the orthogonality of Legendre polynomials:*

$$\int_{-1}^{+1} dw\ P_l(w)P_{l'}(w) = \frac{2}{2l+1}\delta_{ll'}\ .\ ]$$

Consider the repulsive, spherical potential

$$V(r) = \begin{cases} +V_0 & r < a \\ 0 & r > a \end{cases}$$

where $V_0 = \hbar^2\gamma^2/2m$. By considering the s-wave solution to the Schrödinger equation, show that

$$\frac{\tan(ka+\delta_0)}{ka} = \frac{\tanh(\sqrt{\gamma^2-k^2}a)}{\sqrt{\gamma^2-k^2}a}\ .$$

For low momenta, $ka \ll 1$, compute the s-wave contribution to the total cross-section. Comment on the physical interpretation of your result in the limit $\gamma a \to \infty$.

36A Statistical Physics

(a) A system of non-interacting bosons has single particle states $|i\rangle$ with energies $\epsilon_i \geqslant 0$. Show that the grand canonical partition function is

$$\log \mathcal{Z} = -\sum_i \log\left(1 - e^{-\beta(\epsilon_i - \mu)}\right)$$

where $\beta = 1/(kT)$, k is Boltzmann's constant, and μ is the chemical potential. What is the maximum possible value for μ?

(b) A system of $N \gg 1$ bosons has one energy level with zero energy and $M \gg 1$ energy levels with energy $\epsilon > 0$. The number of particles with energies 0, ϵ is N_0, N_ϵ respectively.

(i) Write down expressions for $\langle N_0 \rangle$ and $\langle N_\epsilon \rangle$ in terms of μ and β.

(ii) At temperature T what is the maximum possible number $N_\epsilon^{\max}$ of bosons in the state with energy ϵ? What happens for $N > N_\epsilon^{\max}$?

(iii) Calculate the temperature T_B at which Bose condensation occurs.

(iv) For $T > T_B$, show that $\mu = \epsilon(T_B - T)/T_B$. For $T < T_B$ show that

$$\mu \approx -\frac{kT}{N}\frac{e^{\epsilon/(kT)} - 1}{e^{\epsilon/(kT)} - e^{\epsilon/(kT_B)}} \ .$$

(v) Calculate the mean energy $\langle E \rangle$ for $T > T_B$ and for $T < T_B$. Hence show that the heat capacity of the system is

$$C \approx \begin{cases} \frac{1}{kT^2}\frac{M\epsilon^2}{(e^{\beta\epsilon}-1)^2} & T < T_B \\ 0 & T > T_B \end{cases} \ .$$

37D Electrodynamics

Starting from the covariant form of the Maxwell equations and making a suitable choice of gauge which you should specify, show that the 4-vector potential due to an arbitrary 4-current $J^\mu(x)$ obeys the wave equation,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A^\mu \;=\; -\mu_0 J^\mu,$$

where $x^\mu = (ct, \mathbf{x})$.

Use the method of Green's functions to show that, for a localised current distribution, this equation is solved by

$$A^\mu(t, \mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{J^\mu(t_{\text{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|},$$

for some t_{ret} that you should specify.

A point particle, of charge q, moving along a worldline $y^\mu(\tau)$ parameterised by proper time τ, produces a 4-vector potential

$$A^\mu(x) = \frac{\mu_0 q c}{4\pi} \frac{\dot{y}^\mu(\tau_\star)}{|R^\nu(\tau_\star)\dot{y}_\nu(\tau_\star)|},$$

where $R^\mu(\tau) = x^\mu - y^\mu(\tau)$. Define $\tau_\star(x)$ and draw a spacetime diagram to illustrate its physical significance.

Suppose the particle follows a circular trajectory,

$$\mathbf{y}(t) \;=\; (R\cos(\omega\, t), R\sin(\omega\, t), 0)$$

(with $y^0 = ct$), in some inertial frame with coordinates (ct, x, y, z). Evaluate the resulting 4-vector potential at a point on the z-axis as a function of z and t.

38E General Relativity

The Schwarzschild metric in isotropic coordinates $\bar{x}^{\bar{\alpha}} = (\bar{t}, \bar{x}, \bar{y}, \bar{z})$, $\bar{\alpha} = 0, \ldots, 3$, is given by:

$$ds^2 = \bar{g}_{\bar{\alpha}\bar{\beta}} d\bar{x}^{\bar{\alpha}} d\bar{x}^{\bar{\beta}} = -\frac{(1-A)^2}{(1+A)^2} d\bar{t}^2 + (1+A)^4 (d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2)$$

where

$$A = \frac{m}{2\bar{r}}, \quad \bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2},$$

and m is the mass of the black hole.

(a) Let $x^{\mu} = (t, x, y, z)$, $\mu = 0, \ldots, 3$, denote a coordinate system related to $\bar{x}^{\bar{\alpha}}$ by

$$\bar{t} = \gamma(t - vx), \quad \bar{x} = \gamma(x - vt), \quad \bar{y} = y, \quad \bar{z} = z,$$

where $\gamma = 1/\sqrt{1-v^2}$ and $-1 < v < 1$. Write down the transformation matrix $\partial \bar{x}^{\bar{\alpha}}/\partial x^{\mu}$, briefly explain its physical meaning and show that the inverse transformation is of the same form, but with $v \to -v$.

(b) Using the coordinate transformation matrix of part (a), or otherwise, show that the components $g_{\mu\nu}$ of the metric in coordinates x^{μ} are given by

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = f(A)(-dt^2 + dx^2 + dy^2 + dz^2) + \gamma^2 g(A)(dt - v\,dx)^2,$$

where f and g are functions of A that you should determine. You should also express A in terms of the coordinates (t, x, y, z).

(c) Consider the limit $v \to 1$ with $p = m\gamma$ held constant. Show that for points $x \neq t$ the function $A \to 0$, while $\gamma^2 A$ tends to a finite value, which you should determine. Hence determine the metric components $g_{\mu\nu}$ at points $x \neq t$ in this limit.

39C Fluid Dynamics II

For two Stokes flows $\mathbf{u}^{(1)}(\mathbf{x})$ and $\mathbf{u}^{(2)}(\mathbf{x})$ inside the same volume V with different boundary conditions on its boundary S, prove the reciprocal theorem

$$\int_S u_i^{(1)} \sigma_{ij}^{(2)} n_j \, dS = \int_S u_i^{(2)} \sigma_{ij}^{(1)} n_j \, dS,$$

where $\sigma^{(1)}$ and $\sigma^{(2)}$ are the stress tensors associated with the flows.

Stating clearly any properties of Stokes flow that you require, use the reciprocal theorem to prove that the drag $\mathbf{F}$ on a body translating with uniform velocity $\mathbf{U}$ is given by

$$F_i = A_{ij} U_j,$$

where $\mathbf{A}$ is a symmetric second-rank tensor that depends only on the geometry of the body.

A slender rod falls slowly through very viscous fluid with its axis inclined to the vertical. Explain why the rod does not rotate, stating any properties of Stokes flow that you use.

When the axis of the rod is inclined at an angle θ to the vertical, the centre of mass of the rod travels at an angle ϕ to the vertical. Given that the rod falls twice as quickly when its axis is vertical as when its axis is horizontal, show that

$$\tan \phi = \frac{\sin \theta \cos \theta}{1 + \cos^2 \theta}.$$

40C Waves

Derive the ray-tracing equations

$$\frac{dx_i}{dt} = \frac{\partial \Omega}{\partial k_i}, \quad \frac{dk_i}{dt} = -\frac{\partial \Omega}{\partial x_i}, \quad \frac{d\omega}{dt} = \frac{\partial \Omega}{\partial t},$$

for wave propagation through a slowly-varying medium with local dispersion relation $\omega = \Omega(\mathbf{k}; \mathbf{x}, t)$, where ω and $\mathbf{k}$ are the frequency and wavevector respectively, t is time and $\mathbf{x} = (x, y, z)$ are spatial coordinates. The meaning of the notation d/dt should be carefully explained.

A slowly-varying medium has a dispersion relation $\Omega(\mathbf{k}; \mathbf{x}, t) = kc(z)$, where $k = |\mathbf{k}|$. State and prove Snell's law relating the angle ψ between a ray and the z-axis to c.

Consider the case of a medium with wavespeed $c = c_0(1 + \beta^2 z^2)$, where β and c_0 are positive constants. Show that a ray that passes through the origin with wavevector $k(\cos\phi, 0, \sin\phi)$, remains in the region

$$|z| \leqslant z_m \equiv \frac{1}{\beta} \left[\frac{1}{|\cos\phi|} - 1 \right]^{1/2}.$$

By considering an approximation to the equation for a ray in the region $|z_m - z| \ll \beta^{-1}$, or otherwise, determine the path of a ray near z_m, and hence sketch rays passing through the origin for a few sample values of ϕ in the range $0 < \phi < \pi/2$.

41E Numerical Analysis

The diffusion equation for $u(x, t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in \mathbb{R}, \quad t \geqslant 0,$$

is solved numerically by the difference scheme

$$u_m^{n+1} = u_m^n + \tfrac{3}{2}\mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n) - \tfrac{1}{2}\mu(u_{m-1}^{n-1} - 2u_m^{n-1} + u_{m+1}^{n-1}).$$

Here $\mu = \frac{k}{h^2}$ is the Courant number, with $k = \Delta t$, $h = \Delta x$, and $u_m^n \approx u(mh, nk)$.

(a) Prove that, as $k \to 0$ with constant μ, the local error of the method is $\mathcal{O}(k^2)$.

(b) Applying the Fourier stability analysis, show that the method is stable if and only if $\mu \leqslant \frac{1}{4}$. [*Hint: If a polynomial $p(x) = x^2 - 2\alpha x + \beta$ has real roots, then those roots lie in $[a, b]$ if and only if $p(a)p(b) \geqslant 0$ and $\alpha \in [a, b]$.*]

(c) Prove that, for the same equation, the leapfrog scheme

$$u_m^{n+1} = u_m^{n-1} + 2\mu(u_{m-1}^n - 2u_m^n + u_{m+1}^n)$$

is unstable for any choice of $\mu > 0$.

END OF PAPER