UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 8 June, 2018 1:30pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

Define a *quadratic form* on a finite dimensional real vector space. What does it mean for a quadratic form to be *positive definite*?

Find a basis with respect to which the quadratic form

$$x^2 + 2xy + 2y^2 + 2yz + 3z^2$$

is diagonal. Is this quadratic form positive definite?

2G Groups, Rings and Modules

(a) Show that every automorphism α of the dihedral group D_6 is equal to conjugation by an element of D_6; that is, there is an $h \in D_6$ such that

$$\alpha(g) = hgh^{-1}$$

for all $g \in D_6$.

(b) Give an example of a non-abelian group G with an automorphism which is not equal to conjugation by an element of G.

3F Analysis II

State the Bolzano–Weierstrass theorem in $\mathbb{R}$. Use it to deduce the Bolzano–Weierstrass theorem in $\mathbb{R}^n$.

Let D be a closed, bounded subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a function. Let $\mathcal{S}$ be the set of points in D where f is discontinuous. For $\rho > 0$ and $z \in \mathbb{R}^n$, let $B_\rho(z)$ denote the ball $\{x \in \mathbb{R}^n : \|x - z\| < \rho\}$. Prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x \in D$, $y \in D \setminus \cup_{z \in \mathcal{S}} B_\epsilon(z)$ and $\|x - y\| < \delta$.

(If you use the fact that a continuous function on a compact metric space is uniformly continuous, you must prove it.)

4F Complex Analysis

(a) Let $\Omega \subset \mathbb{C}$ be open, $a \in \Omega$ and suppose that $D_\rho(a) = \{z \in \mathbb{C} : |z-a| \leqslant \rho\} \subset \Omega$. Let $f : \Omega \to \mathbb{C}$ be analytic.

State the Cauchy integral formula expressing $f(a)$ as a contour integral over $C = \partial D_\rho(a)$. Give, without proof, a similar expression for $f'(a)$.

If additionally $\Omega = \mathbb{C}$ and f is bounded, deduce that f must be constant.

(b) If $g = u + iv : \mathbb{C} \to \mathbb{C}$ is analytic where u, v are real, and if $u^2(z) - u(z) \geqslant v^2(z)$ for all $z \in \mathbb{C}$, show that g is constant.

5A Methods

By using separation of variables, solve Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 < x < 1, \quad 0 < y < 1,$$

subject to

$$\begin{aligned} u(0,y) &= 0 & 0 \leqslant y \leqslant 1, \\ u(1,y) &= 0 & 0 \leqslant y \leqslant 1, \\ u(x,0) &= 0 & 0 \leqslant x \leqslant 1, \\ u(x,1) &= 2\sin(3\pi x) & 0 \leqslant x \leqslant 1. \end{aligned}$$

6B Quantum Mechanics

A particle moving in one space dimension with wavefunction $\Psi(x,t)$ obeys the time-dependent Schrödinger equation. Write down the probability density ρ and current density j in terms of the wavefunction and show that they obey the equation

$$\frac{\partial j}{\partial x} + \frac{\partial \rho}{\partial t} = 0\,.$$

Evaluate $j(x,t)$ in the case that

$$\Psi(x,t) = \left(Ae^{ikx} + Be^{-ikx}\right) e^{-iEt/\hbar}\,,$$

where $E = \hbar^2 k^2/2m$, and A and B are constants, which may be complex.

7C Electromagnetism

Show that Maxwell's equations imply the conservation of charge.

A conducting medium has $\mathbf{J} = \sigma \mathbf{E}$ where σ is a constant. Show that any charge density decays exponentially in time, at a rate to be determined.

8D Numerical Analysis

Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 5 & 5 & 6 \\ 1 & 5 & 13 & 14 \\ 2 & 6 & 14 & \lambda \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 3 \\ 7 \\ \mu \end{bmatrix},$$

where λ and μ are real parameters. Find the LU factorisation of the matrix A. For what values of λ does the equation $Ax = b$ have a unique solution for x?

For $\lambda = 20$, use the LU decomposition with forward and backward substitution to determine a value for μ for which a solution to $Ax = b$ exists. Find the most general solution to the equation in this case.

9H Markov Chains

Let $P = (p_{ij})_{i,j \in S}$ be the transition matrix for an irreducible Markov chain on the finite state space S.

(a) What does it mean to say that a distribution π is the *invariant distribution* for the chain?

(b) What does it mean to say that the chain is *in detailed balance* with respect to a distribution π? Show that if the chain is in detailed balance with respect to a distribution π then π is the invariant distribution for the chain.

(c) A symmetric random walk on a connected finite graph is the Markov chain whose state space is the set of vertices of the graph and whose transition probabilities are

$$p_{ij} = \begin{cases} 1/D_i & \text{if } j \text{ is adjacent to } i \\ 0 & \text{otherwise} \end{cases}$$

where D_i is the number of vertices adjacent to vertex i. Show that the random walk is in detailed balance with respect to its invariant distribution.

SECTION II

10E Linear Algebra

Let V be a finite dimensional inner-product space over $\mathbb{C}$. What does it mean to say that an endomorphism of V is *self-adjoint*? Prove that a self-adjoint endomorphism has real eigenvalues and may be diagonalised.

An endomorphism $\alpha : V \to V$ is called *positive definite* if it is self-adjoint and satisfies $\langle \alpha(x), x \rangle > 0$ for all non-zero $x \in V$; it is called *negative definite* if $-\alpha$ is positive definite. Characterise the property of being positive definite in terms of eigenvalues, and show that the sum of two positive definite endomorphisms is positive definite.

Show that a self-adjoint endomorphism $\alpha : V \to V$ has all eigenvalues in the interval $[a, b]$ if and only if $\alpha - \lambda I$ is positive definite for all $\lambda < a$ and negative definite for all $\lambda > b$.

Let $\alpha, \beta : V \to V$ be self-adjoint endomorphisms whose eigenvalues lie in the intervals $[a, b]$ and $[c, d]$ respectively. Show that all of the eigenvalues of $\alpha + \beta$ lie in the interval $[a + c, b + d]$.

11G Groups, Rings and Modules

(a) State the classification theorem for finitely generated modules over a Euclidean domain.

(b) Deduce the existence of the rational canonical form for an $n \times n$ matrix A over a field F.

(c) Compute the rational canonical form of the matrix

$$A = \begin{pmatrix} 3/2 & 1 & 0 \\ -1 & -1/2 & 0 \\ 2 & 2 & 1/2 \end{pmatrix}$$

12F Analysis II

(a) Define what it means for a metric space (X, d) to be *complete*. Give a metric d on the interval $I = (0, 1]$ such that (I, d) is complete and such that a subset of I is open with respect to d if and only if it is open with respect to the Euclidean metric on I. Be sure to prove that d has the required properties.

(b) Let (X, d) be a complete metric space.

(i) If $Y \subset X$, show that Y taken with the subspace metric is complete if and only if Y is closed in X.

(ii) Let $f : X \to X$ and suppose that there is a number $\lambda \in (0, 1)$ such that $d(f(x), f(y)) \leqslant \lambda d(x, y)$ for every $x, y \in X$. Show that there is a unique point $x_0 \in X$ such that $f(x_0) = x_0$.

Deduce that if (a_n) is a sequence of points in X converging to a point $a \neq x_0$, then there are integers ℓ and $m \geqslant \ell$ such that $f(a_m) \neq a_n$ for every $n \geqslant \ell$.

13E Metric and Topological Spaces

Let $X = \{2, 3, 4, 5, 6, 7, 8, \ldots\}$ and for each $n \in X$ let

$$U_n = \{d \in X \mid d \text{ divides } n\}.$$

Prove that the set of unions of the sets U_n forms a topology on X.

Prove or disprove each of the following:

(i) X is Hausdorff;

(ii) X is compact.

If Y and Z are topological spaces, Y is the union of closed subspaces A and B, and $f : Y \to Z$ is a function such that both $f|_A : A \to Z$ and $f|_B : B \to Z$ are continuous, show that f is continuous. Hence show that X is path-connected.

14A Complex Methods

(a) Find the Laplace transform of

$$y(t) = \frac{e^{-a^2/4t}}{\sqrt{\pi t}},$$

for $a \in \mathbb{R}$, $a \neq 0$.

[*You may use without proof that*

$$\int_0^\infty \exp\left(-c^2x^2 - \frac{c^2}{x^2}\right) dx = \frac{\sqrt{\pi}}{2|c|}e^{-2c^2}.]$$

(b) By using the Laplace transform, show that the solution to

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t} \qquad -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), \\ u(x,t) &\quad \text{bounded}, \end{aligned}$$

can be written as

$$u(x,t) = \int_{-\infty}^{\infty} K(|x-\xi|, t) f(\xi) d\xi$$

for some $K(|x-\xi|, t)$ to be determined.

[*You may use without proof that a particular solution to*

$$y''(x) - sy(x) + f(x) = 0$$

is given by

$$y(x) = \frac{e^{-\sqrt{s}\,x}}{2\sqrt{s}} \int_0^x e^{\sqrt{s}\,\xi} f(\xi)\, d\xi - \frac{e^{\sqrt{s}\,x}}{2\sqrt{s}} \int_0^x e^{-\sqrt{s}\,\xi} f(\xi)\, d\xi.]$$

15G Geometry

A Möbius strip in $\mathbb{R}^3$ is parametrized by

$$\sigma(u,v) = \big(Q(u,v)\sin u, Q(u,v)\cos u, v\cos(u/2)\big)$$

for $(u,v) \in U = (0, 2\pi) \times \mathbb{R}$, where $Q \equiv Q(u,v) = 2 - v\sin(u/2)$. Show that the Gaussian curvature is

$$K = \frac{-1}{(v^2/4 + Q^2)^2}$$

at $(u,v) \in U$.

16B Variational Principles

(a) A two-dimensional oscillator has action

$$S = \int_{t_0}^{t_1} \left\{ \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}\omega^2 x^2 - \frac{1}{2}\omega^2 y^2 \right\} dt \,.$$

Find the equations of motion as the Euler-Lagrange equations associated with S, and use them to show that

$$J = \dot{x}y - \dot{y}x$$

is conserved. Write down the general solution of the equations of motion in terms of $\sin \omega t$ and $\cos \omega t$, and evaluate J in terms of the coefficients that arise in the general solution.

(b) Another kind of oscillator has action

$$\widetilde{S} = \int_{t_0}^{t_1} \left\{ \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{4}\alpha x^4 - \frac{1}{2}\beta x^2 y^2 - \frac{1}{4}\alpha y^4 \right\} dt \,,$$

where α and β are real constants. Find the equations of motion and use these to show that in general $J = \dot{x}y - \dot{y}x$ is not conserved. Find the special value of the ratio β/α for which J is conserved. Explain what is special about the action $\widetilde{S}$ in this case, and state the interpretation of J.

17C Methods

Let Ω be a bounded region in the plane, with smooth boundary $\partial\Omega$. Green's second identity states that for any smooth functions u, v on Ω

$$\int_{\Omega} \left(u\,\nabla^2 v - v\,\nabla^2 u\right)\,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial\Omega} u\,(\mathbf{n}\cdot\boldsymbol{\nabla}v) - v\,(\mathbf{n}\cdot\boldsymbol{\nabla}u)\,\mathrm{d}s\,,$$

where $\mathbf{n}$ is the outward pointing normal to $\partial\Omega$. Using this identity with v replaced by

$$G_0(\mathbf{x};\mathbf{x}_0) = \frac{1}{2\pi}\ln\left(\|\mathbf{x}-\mathbf{x}_0\|\right) = \frac{1}{4\pi}\ln\left((x-x_0)^2+(y-y_0)^2\right)$$

and taking care of the singular point $(x,y)=(x_0,y_0)$, show that if u solves the Poisson equation $\nabla^2 u = -\rho$ then

$$u(\mathbf{x}) = -\int_{\Omega} G_0(\mathbf{x};\mathbf{x}_0)\,\rho(\mathbf{x}_0)\,\mathrm{d}x_0\,\mathrm{d}y_0 + \oint_{\partial\Omega}\Big(u(\mathbf{x}_0)\,\mathbf{n}\cdot\boldsymbol{\nabla}G_0(\mathbf{x};\mathbf{x}_0) - G_0(\mathbf{x};\mathbf{x}_0)\,\mathbf{n}\cdot\boldsymbol{\nabla}u(\mathbf{x}_0)\Big)\,\mathrm{d}s$$

at any $\mathbf{x}=(x,y)\in\Omega$, where all derivatives are taken with respect to $\mathbf{x}_0=(x_0,y_0)$.

In the case that Ω is the unit disc $\|\mathbf{x}\|\leqslant 1$, use the method of images to show that the solution to Laplace's equation $\nabla^2 u = 0$ inside Ω, subject to the boundary condition

$$u(1,\theta) = \delta(\theta-\alpha),$$

is

$$u(r,\theta) = \frac{1}{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\alpha)}\,,$$

where (r,θ) are polar coordinates in the disc and α is a constant.

[*Hint: The image of a point* $\mathbf{x}_0\in\Omega$ *is the point* $\mathbf{y}_0=\mathbf{x}_0/\|\mathbf{x}_0\|^2$*, and then*

$$\|\mathbf{x}-\mathbf{x}_0\| = \|\mathbf{x}_0\|\;\|\mathbf{x}-\mathbf{y}_0\|$$

for all $\mathbf{x}\in\partial\Omega$.]

18D Fluid Dynamics

A deep layer of inviscid fluid is initially confined to the region $0 < x < a$, $0 < y < a$, $z < 0$ in Cartesian coordinates, with z directed vertically upwards. An irrotational disturbance is caused to the fluid so that its upper surface takes position $z = \eta(x, y, t)$. Determine the linear normal modes of the system and the dispersion relation between the frequencies of the normal modes and their wavenumbers.

If the interface is initially displaced to position $z = \epsilon \cos \dfrac{3\pi x}{a} \cos \dfrac{4\pi y}{a}$ and released from rest, where ϵ is a small constant, determine its position for subsequent times. How far below the surface will the velocity have decayed to $1/e$ times its surface value?

19H Statistics

There is widespread agreement amongst the managers of the Reliable Motor Company that the number X of faulty cars produced in a month has a binomial distribution

$$P(X = s) = \binom{n}{s} p^s (1-p)^{n-s} \quad (s = 0, 1, \ldots, n; \quad 0 \leqslant p \leqslant 1),$$

where n is the total number of cars produced in a month. There is, however, some dispute about the parameter p. The general manager has a prior distribution for p which is uniform, while the more pessimistic production manager has a prior distribution with density $2p$, both on the interval $[0, 1]$.

In a particular month, s faulty cars are produced. Show that if the general manager's loss function is $(\hat{p} - p)^2$, where $\hat{p}$ is her estimate and p the true value, then her best estimate of p is

$$\hat{p} = \frac{s+1}{n+2}.$$

The production manager has responsibilities different from those of the general manager, and a different loss function given by $(1-p)(\hat{p}-p)^2$. Find his best estimate of p and show that it is greater than that of the general manager unless $s \geqslant \frac{1}{2}n$.

[*You may use the fact that for non-negative integers* α, β,

$$\int_0^1 p^\alpha (1-p)^\beta \, dp = \frac{\alpha! \beta!}{(\alpha + \beta + 1)!}. \quad]$$

20H Optimisation

Given a network with a source A, a sink B, and capacities on directed edges, define a *cut*. What is meant by the *capacity* of a cut? State the max-flow min-cut theorem. If the capacities of edges are integral, what can be said about the maximum flow?

Consider an $m \times n$ matrix A in which each entry is either 0 or 1. We say that a set of lines (rows or columns of the matrix) *covers* the matrix if each 1 belongs to some line of the set. We say that a set of 1's is *independent* if no pair of 1's of the set lie in the same line. Use the max-flow min-cut theorem to show that the maximal number of independent 1's equals the minimum number of lines that cover the matrix.

END OF PAPER