UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Wednesday, 6 June, 2018 9:00 am to 12:00 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

Let V be a real vector space. Define the *dual* vector space V^* of V. If U is a subspace of V, define the *annihilator* U^0 of U. If $x_1, x_2, \ldots, x_n$ is a basis for V, define its dual $x_1^*, x_2^*, \ldots, x_n^*$ and prove that it is a basis for V^*.

If V has basis x_1, x_2, x_3, x_4 and U is the subspace spanned by

$$x_1 + 2x_2 + 3x_3 + 4x_4 \quad \text{and} \quad 5x_1 + 6x_2 + 7x_3 + 8x_4,$$

give a basis for U^0 in terms of the dual basis $x_1^*, x_2^*, x_3^*, x_4^*$.

2G Groups, Rings and Modules

Let R be a principal ideal domain and x a non-zero element of R. We define a new ring R' as follows. We define an equivalence relation $\sim$ on $R \times \{x^n \mid n \in \mathbb{Z}_{\geqslant 0}\}$ by

$$(r, x^n) \sim (r', x^{n'})$$

if and only if $x^{n'} r = x^n r'$. The underlying set of R' is the set of $\sim$-equivalence classes. We define addition on R' by

$$[(r, x^n)] + [(r', x^{n'})] = [(x^{n'} r + x^n r', x^{n+n'})]$$

and multiplication by $[(r, x^n)][(r', x^{n'})] = [(rr', x^{n+n'})]$.

(a) Show that R' is a well defined ring.

(b) Prove that R' is a principal ideal domain.

3F Analysis II

Show that $\|f\|_1 = \int_0^1 |f(x)|\, dx$ defines a norm on the space $C([0,1])$ of continuous functions $f : [0,1] \to \mathbb{R}$.

Let $\mathcal{S}$ be the set of continuous functions $g : [0,1] \to \mathbb{R}$ with $g(0) = g(1) = 0$. Show that for each continuous function $f : [0,1] \to \mathbb{R}$, there is a sequence $g_n \in \mathcal{S}$ with $\sup_{x\in[0,1]} |g_n(x)| \leqslant \sup_{x\in[0,1]} |f(x)|$ such that $\|f - g_n\|_1 \to 0$ as $n \to \infty$.

Show that if $f : [0,1] \to \mathbb{R}$ is continuous and $\int_0^1 f(x)g(x)\, dx = 0$ for every $g \in \mathcal{S}$ then $f = 0$.

4E Metric and Topological Spaces

What does it mean to say that d is a *metric* on a set X? What does it mean to say that a subset of X is *open* with respect to the metric d? Show that the collection of subsets of X that are open with respect to d satisfies the axioms of a topology.

For $X = C[0,1]$, the set of continuous functions $f : [0,1] \to \mathbb{R}$, show that the metrics

$$d_1(f,g) = \int_0^1 |f(x) - g(x)| \, \mathrm{d}x$$
$$d_2(f,g) = \left[\int_0^1 |f(x) - g(x)|^2 \, \mathrm{d}x \right]^{1/2}$$

give different topologies.

5C Methods

Show that

$$a(x,y) \left(\frac{dy}{ds} \right)^2 - 2b(x,y) \frac{dx}{ds} \frac{dy}{ds} + c(x,y) \left(\frac{dx}{ds} \right)^2 = 0$$

along a characteristic curve $(x(s), y(s))$ of the 2^{nd}-order pde

$$a(x,y)\, u_{xx} + 2b(x,y)\, u_{xy} + c(x,y)\, u_{yy} = f(x,y)\,.$$

6C Electromagnetism

Derive the Biot–Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}V$$

from Maxwell's equations, where the time–independent current $\mathbf{j}(\mathbf{r})$ vanishes outside V. [*You may assume that the vector potential can be chosen to be divergence–free.*]

7D Fluid Dynamics

The Euler equations for steady fluid flow $\mathbf{u}$ in a rapidly rotating system can be written

$$\rho \mathbf{f} \times \mathbf{u} = -\nabla p + \rho \mathbf{g},$$

where ρ is the density of the fluid, p is its pressure, $\mathbf{g}$ is the acceleration due to gravity and $\mathbf{f} = (0, 0, f)$ is the constant Coriolis parameter in a Cartesian frame of reference (x, y, z), with z pointing vertically upwards.

Fluid occupies a layer of slowly-varying height $h(x, y)$. Given that the pressure $p = p_0$ is constant at $z = h$ and that the flow is approximately horizontal with components $\mathbf{u} = (u, v, 0)$, show that the contours of h are streamlines of the horizontal flow. What is the leading-order horizontal volume flux of fluid between two locations at which $h = h_0$ and $h = h_0 + \Delta h$, where $\Delta h \ll h_0$?

Identify the dimensions of all the quantities involved in your expression for the volume flux and show that your expression is dimensionally consistent.

8H Statistics

Define a *simple hypothesis*. Define the terms *size* and *power* for a test of one simple hypothesis against another. State the Neyman-Pearson lemma.

There is a single observation of a random variable X which has a probability density function $f(x)$. Construct a best test of size 0.05 for the null hypothesis

$$H_0 : \quad f(x) = \tfrac{1}{2}, \qquad -1 \leqslant x \leqslant 1,$$

against the alternative hypothesis

$$H_1 : \quad f(x) = \frac{3}{4}(1 - x^2), \quad -1 \leqslant x \leqslant 1.$$

Calculate the power of your test.

9H Optimisation

What does it mean to state that $f : \mathbb{R}^n \to \mathbb{R}$ is a *convex function*?

Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are convex functions, and for $b \in \mathbb{R}$ let

$$\phi(b) = \inf\{f(x) : g(x) \leqslant b\}.$$

Assuming $\phi(b)$ is finite for all $b \in \mathbb{R}$, prove that the function ϕ is convex.

SECTION II

10E Linear Algebra

If X is an $n \times m$ matrix over a field, show that there are invertible matrices P and Q such that

$$Q^{-1}XP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

for some $0 \leqslant r \leqslant \min(m,n)$, where I_r is the identity matrix of dimension r.

For a square matrix of the form $A = \begin{bmatrix} B & D \\ 0 & C \end{bmatrix}$ with B and C square matrices, prove that $\det(A) = \det(B)\det(C)$.

If $A \in M_{n\times n}(\mathbb{C})$ and $B \in M_{m\times m}(\mathbb{C})$ have no common eigenvalue, show that the linear map

$$\begin{aligned} L : M_{n\times m}(\mathbb{C}) &\longrightarrow M_{n\times m}(\mathbb{C}) \\ X &\longmapsto AX - XB \end{aligned}$$

is injective.

11G Groups, Rings and Modules

(a) Prove that every principal ideal domain is a unique factorization domain.

(b) Consider the ring $R = \{f(X) \in \mathbb{Q}[X] \mid f(0) \in \mathbb{Z}\}$.

(i) What are the units in R?

(ii) Let $f(X) \in R$ be irreducible. Prove that either $f(X) = \pm p$, for $p \in \mathbb{Z}$ a prime, or $\deg(f) \geqslant 1$ and $f(0) = \pm 1$.

(iii) Prove that $f(X) = X$ is not expressible as a product of irreducibles.

12F Analysis II

(a) Let (X, d) be a metric space, A a non-empty subset of X and $f : A \to \mathbb{R}$. Define what it means for f to be *Lipschitz*. If f is Lipschitz with Lipschitz constant L and if

$$F(x) = \inf_{y \in A} \left(f(y) + L d(x, y) \right)$$

for each $x \in X$, show that $F(x) = f(x)$ for each $x \in A$ and that $F : X \to \mathbb{R}$ is Lipschitz with Lipschitz constant L. (Be sure to justify that $F(x) \in \mathbb{R}$, i.e. that the infimum is finite for every $x \in X$.)

(b) What does it mean to say that two norms on a vector space are *Lipschitz equivalent*?

Let V be an n-dimensional real vector space equipped with a norm $\| \cdot \|$. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for V. Show that the map $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(x_1, x_2, \ldots, x_n) = \|x_1 e_1 + x_2 e_2 + \ldots + x_n e_n\|$ is continuous. Deduce that any two norms on V are Lipschitz equivalent.

(c) Prove that for each positive integer n and each $a \in (0, 1]$, there is a constant $C > 0$ with the following property: for every polynomial p of degree $\leqslant n$, there is a point $y \in [0, a]$ such that

$$\sup_{x \in [0,1]} |p'(x)| \leqslant C |p(y)|,$$

where p' is the derivative of p.

13A Complex Analysis or Complex Methods

(a) Let $f(z)$ be a complex function. Define the *Laurent series* of $f(z)$ about $z = z_0$, and give suitable formulae in terms of integrals for calculating the coefficients of the series.

(b) Calculate, by any means, the first 3 terms in the Laurent series about $z = 0$ for

$$f(z) = \frac{1}{e^{2z} - 1}.$$

Indicate the range of values of $|z|$ for which your series is valid.

(c) Let

$$g(z) = \frac{1}{2z} + \sum_{k=1}^{m} \frac{z}{z^2 + \pi^2 k^2}.$$

Classify the singularities of $F(z) = f(z) - g(z)$ for $|z| < (m+1)\pi$.

(d) By considering

$$\oint_{C_R} \frac{F(z)}{z^2} dz$$

where $C_R = \{|z| = R\}$ for some suitably chosen $R > 0$, show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

14G Geometry

For any matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}),$$

the corresponding Möbius transformation is

$$z \mapsto Az = \frac{az+b}{cz+d},$$

which acts on the upper half-plane $\mathbb{H}$, equipped with the hyperbolic metric ρ.

(a) Assuming that $|\mathrm{tr}\ A| > 2$, prove that A is conjugate in $SL(2,\mathbb{R})$ to a diagonal matrix B. Determine the relationship between $|\mathrm{tr}\ A|$ and $\rho(i, Bi)$.

(b) For a diagonal matrix B with $|\mathrm{tr}\ B| > 2$, prove that

$$\rho(x, Bx) > \rho(i, Bi)$$

for all $x \in \mathbb{H}$ not on the imaginary axis.

(c) Assume now that $|\mathrm{tr}\ A| < 2$. Prove that A fixes a point in $\mathbb{H}$.

(d) Give an example of a matrix A in $SL(2,\mathbb{R})$ that does not preserve any point or hyperbolic line in $\mathbb{H}$. Justify your answer.

15B Variational Principles

Derive the Euler-Lagrange equation for the integral

$$I[y] = \int_{x_0}^{x_1} f(y, y', y'', x)\, dx,$$

when $y(x)$ and $y'(x)$ take given values at the fixed endpoints.

Show that the only function $y(x)$ with $y(0) = 1$, $y'(0) = 2$ and $y(x) \to 0$ as $x \to \infty$ for which the integral

$$I[y] = \int_0^\infty \left(y^2 + (y')^2 + (y' + y'')^2\right) dx$$

is stationary is $(3x+1)e^{-x}$.

16A Methods

(a) Let $f(x)$ be a 2π-periodic function (i.e. $f(x)=f(x+2\pi)$ for all x) defined on $[-\pi,\pi]$ by

$$f(x)=\begin{cases} x & x\in[0,\pi] \\ -x & x\in[-\pi,0] \end{cases}$$

Find the Fourier series of $f(x)$ in the form

$$f(x)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(nx)+\sum_{n=1}^{\infty}b_n\sin(nx).$$

(b) Find the general solution to

$$y''+2y'+y=f(x)$$

where $f(x)$ is as given in part (a) and $y(x)$ is 2π-periodic.

17B Quantum Mechanics

For an electron in a hydrogen atom, the stationary-state wavefunctions are of the form $\psi(r,\theta,\phi)=R(r)Y_{lm}(\theta,\phi)$, where in suitable units R obeys the radial equation

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{l(l+1)}{r^2}R+2\left(E+\frac{1}{r}\right)R=0\,.$$

Explain briefly how the terms in this equation arise.

This radial equation has bound-state solutions of energy $E=E_n$, where $E_n=-\frac{1}{2n^2}$ $(n=1,2,3,\dots)$. Show that when $l=n-1$, there is a solution of the form $R(r)=r^{\alpha}e^{-r/n}$, and determine α. Find the expectation value $\langle r\rangle$ in this state.

Determine the total degeneracy of the energy level with energy E_n.

18C Electromagnetism

A plane with unit normal $\mathbf{n}$ supports a charge density and a current density that are each time–independent. Show that the tangential components of the electric field and the normal component of the magnetic field are continuous across the plane.

Albert moves with constant velocity $\mathbf{v}=v\mathbf{n}$ relative to the plane. Find the boundary conditions at the plane on the normal component of the magnetic field and the tangential components of the electric field as seen in Albert's frame.

19D Numerical Analysis

Show that the recurrence relation

$$p_0(x) = 1,$$
$$p_{n+1}(x) = q_{n+1}(x) - \sum_{k=0}^{n} \frac{\langle q_{n+1}, p_k \rangle}{\langle p_k, p_k \rangle} p_k(x),$$

where $\langle \cdot, \cdot \rangle$ is an inner product on real polynomials, produces a sequence of orthogonal, monic, real polynomials $p_n(x)$ of degree exactly n of the real variable x, provided that q_n is a monic, real polynomial of degree exactly n.

Show that the choice $q_{n+1}(x) = xp_n(x)$ leads to a three-term recurrence relation of the form

$$p_0(x) = 1,$$
$$p_1(x) = x - \alpha_0,$$
$$p_{n+1}(x) = (x - \alpha_n)p_n(x) - \beta_n p_{n-1}(x),$$

where α_n and β_n are constants that should be determined in terms of the inner products $\langle p_n, p_n \rangle$, $\langle p_{n-1}, p_{n-1} \rangle$ and $\langle p_n, xp_n \rangle$.

Use this recurrence relation to find the first four monic Legendre polynomials, which correspond to the inner product defined by

$$\langle p, q \rangle \equiv \int_{-1}^{1} p(x)q(x)dx.$$

20H Markov Chains

For a finite irreducible Markov chain, what is the relationship between the invariant probability distribution and the mean recurrence times of states?

A particle moves on the 2^n vertices of the hypercube, $\{0,1\}^n$, in the following way: at each step the particle is equally likely to move to each of the n adjacent vertices, independently of its past motion. (Two vertices are *adjacent* if the Euclidean distance between them is one.) The initial vertex occupied by the particle is $(0, 0, \ldots, 0)$. Calculate the expected number of steps until the particle

(i) first returns to $(0, 0, \ldots, 0)$,

(ii) first visits $(0, 0, \ldots, 0, 1)$,

(iii) first visits $(0, 0, \ldots, 0, 1, 1)$.

END OF PAPER