UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Tuesday, 5 June, 2018 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Linear Algebra

State the Rank-Nullity Theorem.

If $\alpha : V \to W$ and $\beta : W \to X$ are linear maps and W is finite dimensional, show that

$$\dim \operatorname{Im}(\alpha) = \dim \operatorname{Im}(\beta\alpha) + \dim(\operatorname{Im}(\alpha) \cap \operatorname{Ker}(\beta)).$$

If $\gamma : U \to V$ is another linear map, show that

$$\dim \operatorname{Im}(\beta\alpha) + \dim \operatorname{Im}(\alpha\gamma) \leqslant \dim \operatorname{Im}(\alpha) + \dim \operatorname{Im}(\beta\alpha\gamma).$$

2A Complex Analysis or Complex Methods

(a) Show that

$$w = \log(z)$$

is a conformal mapping from the right half z-plane, $\operatorname{Re}(z) > 0$, to the strip

$$S = \left\{ w \,:\, -\frac{\pi}{2} < \operatorname{Im}(w) < \frac{\pi}{2} \right\},$$

for a suitably chosen branch of $\log(z)$ that you should specify.

(b) Show that

$$w = \frac{z-1}{z+1}$$

is a conformal mapping from the right half z-plane, $\operatorname{Re}(z) > 0$, to the unit disc

$$D = \{ w \,:\, |w| < 1 \}.$$

(c) Deduce a conformal mapping from the strip S to the disc D.

3G Geometry

(a) State the Gauss–Bonnet theorem for spherical triangles.

(b) Prove that any geodesic triangulation of the sphere has Euler number equal to 2.

(c) Prove that there is no geodesic triangulation of the sphere in which every vertex is adjacent to exactly 6 triangles.

4B Variational Principles

Find, using a Lagrange multiplier, the four stationary points in $\mathbb{R}^3$ of the function $x^2 + y^2 + z^2$ subject to the constraint $x^2 + 2y^2 - z^2 = 1$. By sketching sections of the constraint surface in each of the coordinate planes, or otherwise, identify the nature of the constrained stationary points.

How would the location of the stationary points differ if, instead, the function $x^2 + 2y^2 - z^2$ were subject to the constraint $x^2 + y^2 + z^2 = 1$?

5D Fluid Dynamics

Show that the flow with velocity potential

$$\phi = \frac{q}{2\pi} \ln r$$

in two-dimensional, plane-polar coordinates (r, θ) is incompressible in $r > 0$. Determine the flux of fluid across a closed contour C that encloses the origin. What does this flow represent?

Show that the flow with velocity potential

$$\phi = \frac{q}{4\pi} \ln\left(x^2 + (y-a)^2\right) + \frac{q}{4\pi} \ln\left(x^2 + (y+a)^2\right)$$

has no normal flow across the line $y = 0$. What fluid flow does this represent in the unbounded plane? What flow does it represent for fluid occupying the domain $y > 0$?

6D Numerical Analysis

The Trapezoidal Rule for solving the differential equation

$$y'(t) = f(t, y), \qquad t \in [0, T], \qquad y(0) = y_0$$

is defined by

$$y_{n+1} = y_n + \tfrac{1}{2}h\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right],$$

where $h = t_{n+1} - t_n$.

Determine the minimum order of convergence k of this rule for general functions f that are sufficiently differentiable. Show with an explicit example that there is a function f for which the local truncation error is Ah^{k+1} for some constant A.

7H Statistics

$X_1, X_2, \ldots, X_n$ form a random sample from a distribution whose probability density function is

$$f(x; \theta) = \begin{cases} \dfrac{2x}{\theta^2} & 0 \leqslant x \leqslant \theta \\ 0 & \text{otherwise,} \end{cases}$$

where the value of the positive parameter θ is unknown. Determine the maximum likelihood estimator of the median of this distribution.

8H Optimisation

What is meant by a *transportation problem*? Illustrate the transportation algorithm by solving the problem with three sources and three destinations described by the table

	Destinations			
Sources	4	3	1	10
	6	10	3	8
	3	5	7	8
	3	9	14	

where the figures in the boxes denote transportation costs, the right-hand column denotes supplies, and the bottom row denotes requirements.

SECTION II

9E Linear Algebra

Define a *Jordan block* $J_m(\lambda)$. What does it mean for a complex $n \times n$ matrix to be in *Jordan normal form*?

If A is a matrix in Jordan normal form for an endomorphism $\alpha : V \to V$, prove that

$$\dim \operatorname{Ker}((\alpha - \lambda I)^r) - \dim \operatorname{Ker}((\alpha - \lambda I)^{r-1})$$

is the number of Jordan blocks $J_m(\lambda)$ of A with $m \geqslant r$.

Find a matrix in Jordan normal form for $J_m(\lambda)^2$. *[Consider all possible values of* λ*.]*

Find a matrix in Jordan normal form for the complex matrix

$$\begin{bmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & 0 \\ a_4 & 0 & 0 & 0 \end{bmatrix}$$

assuming it is invertible.

10G Groups, Rings and Modules

(a) State Sylow's theorems.

(b) Prove Sylow's first theorem.

(c) Let G be a group of order 12. Prove that either G has a unique Sylow 3-subgroup or $G \cong A_4$.

11F Analysis II

Let $U \subset \mathbb{R}^n$ be a non-empty open set and let $f : U \to \mathbb{R}^n$.

(a) What does it mean to say that f is *differentiable*? What does it mean to say that f is a C^1 function?

If f is differentiable, show that f is continuous.

State the inverse function theorem.

(b) Suppose that U is convex, f is C^1 and that its derivative $Df(a)$ at a satisfies $\|Df(a) - I\| < 1$ for all $a \in U$, where $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map and $\|\cdot\|$ denotes the operator norm. Show that f is injective.

Explain why $f(U)$ is an open subset of $\mathbb{R}^n$.

Must it be true that $f(U) = \mathbb{R}^n$? What if $U = \mathbb{R}^n$? Give proofs or counter-examples as appropriate.

(c) Find the largest set $U \subset \mathbb{R}^2$ such that the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x^2 - y^2, 2xy)$ satisfies $\|Df(a) - I\| < 1$ for every $a \in U$.

12E Metric and Topological Spaces

What does it mean to say that a topological space is *compact*? Prove directly from the definition that $[0, 1]$ is compact. Hence show that the unit circle $S^1 \subset \mathbb{R}^2$ is compact, proving any results that you use. *[You may use without proof the continuity of standard functions.]*

The set $\mathbb{R}^2$ has a topology $\mathcal{T}$ for which the closed sets are the empty set and the finite unions of vector subspaces. Let X denote the set $\mathbb{R}^2 \setminus \{0\}$ with the subspace topology induced by $\mathcal{T}$. By considering the subspace topology on $S^1 \subset \mathbb{R}^2$, or otherwise, show that X is compact.

13A Complex Analysis or Complex Methods

(a) Let C be a rectangular contour with vertices at $\pm R + \pi\, i$ and $\pm R - \pi\, i$ for some $R > 0$ taken in the anticlockwise direction. By considering

$$\lim_{R\to\infty} \oint_C \frac{e^{iz^2/4\pi}}{e^{z/2} - e^{-z/2}} dz,$$

show that

$$\lim_{R\to\infty} \int_{-R}^{R} e^{ix^2/4\pi} dx = 2\pi e^{\pi i/4}.$$

(b) By using a semi-circular contour in the upper half plane, calculate

$$\int_0^\infty \frac{x\sin(\pi x)}{x^2 + a^2}\, dx$$

for $a > 0$.

[You may use Jordan's Lemma without proof.]

14C Methods

Define the *convolution* $f * g$ of two functions f and g. Defining the *Fourier transform* $\tilde{f}$ of f by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\, \mathrm{d}x\,,$$

show that

$$\widetilde{f * g}\,(k) = \tilde{f}(k)\, \tilde{g}(k)\,.$$

Given that the Fourier transform of $f(x) = 1/x$ is

$$\tilde{f}(k) = -\mathrm{i}\pi\, \mathrm{sgn}(k)\,,$$

find the Fourier transform of $\sin(x)/x^2$.

15B Quantum Mechanics

The relative motion of a neutron and proton is described by the Schrödinger equation for a single particle of mass m under the influence of the central potential

$$V(r)=\begin{cases}-U & r<a\\ 0 & r>a\end{cases}$$

where U and a are positive constants. Solve this equation for a spherically symmetric state of the deuteron, which is a bound state of a proton and a neutron, giving the condition on U for this state to exist.

[*If* ψ *is spherically symmetric then* $\nabla^2\psi=\frac{1}{r}\frac{d^2}{dr^2}(r\psi)$.]

16C Electromagnetism

Starting from the Lorentz force law acting on a current distribution $\mathbf{J}$ obeying $\boldsymbol{\nabla}\cdot\mathbf{J}=0$, show that the energy of a magnetic dipole $\mathbf{m}$ in the presence of a time–independent magnetic field $\mathbf{B}$ is

$$U=-\mathbf{m}\cdot\mathbf{B}\,.$$

State clearly any approximations you make.

[*You may use without proof the fact that*

$$\int(\mathbf{a}\cdot\mathbf{r})\ \mathbf{J}(\mathbf{r})\ \mathrm{d}V=-\frac{1}{2}\mathbf{a}\times\int(\mathbf{r}\times\mathbf{J}(\mathbf{r}))\ \mathrm{d}V$$

for any constant vector $\mathbf{a}$, *and the identity*

$$(\mathbf{b}\times\boldsymbol{\nabla})\times\mathbf{c}=\boldsymbol{\nabla}(\mathbf{b}\cdot\mathbf{c})-\mathbf{b}(\boldsymbol{\nabla}\cdot\mathbf{c}),$$

which holds when $\mathbf{b}$ *is constant.*]

A beam of slowly moving, randomly oriented magnetic dipoles enters a region where the magnetic field is

$$\mathbf{B}=\hat{\mathbf{z}}B_0+(y\hat{\mathbf{x}}+x\hat{\mathbf{y}})B_1\,,$$

with B_0 and B_1 constants. By considering their energy, briefly describe what happens to those dipoles that are parallel to, and those that are anti-parallel to the direction of $\mathbf{B}$.

17D Fluid Dynamics

A layer of fluid of dynamic viscosity μ, density ρ and uniform thickness h flows down a rigid vertical plane. The adjacent air has uniform pressure p_0 and exerts a tangential stress on the fluid that is proportional to the surface velocity and opposes the flow, with constant of proportionality k. The acceleration due to gravity is g.

(a) Draw a diagram of this situation, including indications of the applied stresses and body forces, a suitable coordinate system and a representation of the expected velocity profile.

(b) Write down the equations and boundary conditions governing the flow, with a brief description of each, paying careful attention to signs. Solve these equations to determine the pressure and velocity fields in terms of the parameters given above.

(c) Show that the surface velocity of the fluid layer is $\dfrac{\rho g h^2}{2\mu}\left(1+\dfrac{kh}{\mu}\right)^{-1}$.

(d) Determine the volume flux per unit width of the plane for general values of k and its limiting values when $k \to 0$ and $k \to \infty$.

18D Numerical Analysis

Show that if $\mathbf{u} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ then the $m \times m$ matrix transformation

$$H_{\mathbf{u}} = I - 2\frac{\mathbf{u}\mathbf{u}^\top}{\|\mathbf{u}\|^2}$$

is orthogonal. Show further that, for any two vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^m$ of equal length,

$$H_{\mathbf{a}-\mathbf{b}}\mathbf{a} = \mathbf{b}.$$

Explain how to use such transformations to convert an $m \times n$ matrix A with $m \geqslant n$ into the form $A = QR$, where Q is an orthogonal matrix and R is an upper-triangular matrix, and illustrate the method using the matrix

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

19H Statistics

(a) Consider the general linear model $Y = X\theta + \varepsilon$ where X is a known $n \times p$ matrix, θ is an unknown $p \times 1$ vector of parameters, and ε is an $n \times 1$ vector of independent $N(0, \sigma^2)$ random variables with unknown variances σ^2. Show that, provided the matrix X is of rank p, the least squares estimate of θ is

$$\hat{\theta} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}Y.$$

Let

$$\hat{\varepsilon} = Y - X\hat{\theta}.$$

What is the distribution of $\hat{\varepsilon}^{\mathrm{T}}\hat{\varepsilon}$? Write down, in terms of $\hat{\varepsilon}^{\mathrm{T}}\hat{\varepsilon}$, an unbiased estimator of σ^2.

(b) Four points on the ground form the vertices of a plane quadrilateral with interior angles $\theta_1, \theta_2, \theta_3, \theta_4$, so that $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 2\pi$. Aerial observations Z_1, Z_2, Z_3, Z_4 are made of these angles, where the observations are subject to independent errors distributed as $N(0, \sigma^2)$ random variables.

(i) Represent the preceding model as a general linear model with observations $(Z_1, Z_2, Z_3, Z_4 - 2\pi)$ and unknown parameters $(\theta_1, \theta_2, \theta_3)$.

(ii) Find the least squares estimates $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$.

(iii) Determine an unbiased estimator of σ^2. What is its distribution?

20H Markov Chains

A coin-tossing game is played by two players, A_1 and A_2. Each player has a coin and the probability that the coin tossed by player A_i comes up heads is p_i, where $0 < p_i < 1$, $i = 1, 2$. The players toss their coins according to the following scheme: A_1 tosses first and then after each head, A_2 pays A_1 one pound and A_1 has the next toss, while after each tail, A_1 pays A_2 one pound and A_2 has the next toss.

Define a Markov chain to describe the state of the game. Find the probability that the game ever returns to a state where neither player has lost money.

END OF PAPER