UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Wednesday, 6 June, 2018 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Numbers and Sets

State Fermat's theorem.

Let p be a prime such that $p \equiv 3 \pmod 4$. Prove that there is no solution to $x^2 \equiv -1 \pmod p$.

Show that there are infinitely many primes congruent to $1 \pmod 4$.

2E Numbers and Sets

Given $n \in \mathbb{N}$, show that $\sqrt{n}$ is either an integer or irrational.

Let α and β be irrational numbers and q be rational. Which of $\alpha + q$, $\alpha + \beta$, $\alpha\beta$, α^q and α^β *must* be irrational? Justify your answers. [*Hint: For the last part consider* $\sqrt{2}^{\sqrt{2}}$.]

3A Dynamics and Relativity

(a) Define an *inertial frame.*

(b) Specify three different types of Galilean transformation on inertial frames whose space coordinates are $\mathbf{x}$ and whose time coordinate is t.

(c) State the *Principle of Galilean Relativity.*

(d) Write down the equation of motion for a particle in one dimension x in a potential $V(x)$. Prove that energy is conserved. A particle is at position x_0 at time t_0. Find an expression for time t as a function of x in terms of an integral involving V.

(e) Write down the x values of any equilibria and state (without justification) whether they are stable or unstable for:
(i) $V(x) = (x^2 - 4)^2$
(ii) $V(x) = e^{-1/x^2}$ for $x \neq 0$ and $V(0) = 0$.

4A Dynamics and Relativity

Explain what is meant by a *central force* acting on a particle moving in three dimensions.

Show that the angular momentum of a particle about the origin for a central force is conserved, and hence that its path lies in a plane.

Show that, in the approximation in which the Sun is regarded as fixed and only its gravitational field is considered, a straight line joining the Sun and an orbiting planet sweeps out equal areas in equal time (Kepler's second law).

SECTION II

5E Numbers and Sets

Let n be a positive integer. Show that for any a coprime to n, there is a unique b (mod n) such that $ab \equiv 1 \pmod{n}$. Show also that if a and b are integers coprime to n, then ab is also coprime to n. [Any version of Bezout's theorem may be used without proof provided it is clearly stated.]

State and prove Wilson's theorem.

Let n be a positive integer and p be a prime. Show that the exponent of p in the prime factorisation of $n!$ is given by $\sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$ where $\lfloor x \rfloor$ denotes the integer part of x.

Evaluate $20! \pmod{23}$ and $1000! \pmod{10^{249}}$.

Let p be a prime and $0 < k < p^m$. Let ℓ be the exponent of p in the prime factorisation of k. Find the exponent of p in the prime factorisation of $\binom{p^m}{k}$, in terms of m and ℓ.

6E Numbers and Sets

For $n \in \mathbb{N}$ let $Q_n = \{0,1\}^n$ denote the set of all 0-1 sequences of length n. We define the *distance* $d(x,y)$ between two elements x and y of Q_n to be the number of coordinates in which they differ. Show that $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x, y, z \in Q_n$.

For $x \in Q_n$ and $1 \leqslant j \leqslant n$ let $B(x,j) = \{y \in Q_n : d(y,x) \leqslant j\}$. Show that $|B(x,j)| = \sum_{i=0}^{j} \binom{n}{i}$.

A subset C of Q_n is called a *k-code* if $d(x,y) \geqslant 2k+1$ for all $x, y \in C$ with $x \neq y$. Let $M(n,k)$ be the maximum possible value of $|C|$ for a k-code C in Q_n. Show that

$$2^n \left(\sum_{i=0}^{2k} \binom{n}{i} \right)^{-1} \leqslant M(n,k) \leqslant 2^n \left(\sum_{i=0}^{k} \binom{n}{i} \right)^{-1} .$$

Find $M(4,1)$, carefully justifying your answer.

7E Numbers and Sets

Let $n \in \mathbb{N}$ and $A_1, \ldots, A_n$ be subsets of a finite set X. Let $0 \leqslant t \leqslant n$. Show that if $x \in X$ belongs to A_i for exactly m values of i, then

$$\sum_{S \subset \{1,\ldots,n\}} \binom{|S|}{t} (-1)^{|S|-t} \mathbf{1}_{A_S}(x) = \begin{cases} 0 & \text{if } m \neq t \\ 1 & \text{if } m = t \end{cases}$$

where $A_S = \bigcap_{i \in S} A_i$ with the convention that $A_\emptyset = X$, and $\mathbf{1}_{A_S}$ denotes the indicator function of A_S. [*Hint: Set* $M = \{i : x \in A_i\}$ *and consider for which* $S \subset \{1, \ldots, n\}$ *one has* $\mathbf{1}_{A_S}(x) = 1$.]

Use this to show that the number of elements of X that belong to A_i for exactly t values of i is

$$\sum_{S \subset \{1,\ldots,n\}} \binom{|S|}{t} (-1)^{|S|-t} |A_S| \ .$$

Deduce the Inclusion-Exclusion Principle.

Using the Inclusion-Exclusion Principle, prove a formula for the Euler totient function $\varphi(N)$ in terms of the distinct prime factors of N.

A *Carmichael number* is a composite number n such that $x^{n-1} \equiv 1 \pmod{n}$ for every integer x coprime to n. Show that if $n = q_1 q_2 \ldots q_k$ is the product of $k \geqslant 2$ distinct primes $q_1, \ldots, q_k$ satisfying $q_j - 1 \mid n - 1$ for $j = 1, \ldots, k$, then n is a Carmichael number.

8E Numbers and Sets

Define what it means for a set to be *countable.*

Show that for any set X, there is no surjection from X onto the power set $\mathcal{P}(X)$. Deduce that the set $\{0,1\}^{\mathbb{N}}$ of all infinite 0-1 sequences is uncountable.

Let $\mathcal{L}$ be the set of sequences $(F_n)_{n=0}^{\infty}$ of subsets $F_0 \subset F_1 \subset F_2 \subset \ldots$ of $\mathbb{N}$ such that $|F_n| = n$ for all $n \in \mathbb{N}$ and $\bigcup_n F_n = \mathbb{N}$. Let $\mathcal{L}_0$ consist of all members $(F_n)_{n=0}^{\infty}$ of $\mathcal{L}$ for which $n \in F_n$ for all but finitely many $n \in \mathbb{N}$. Let $\mathcal{L}_1$ consist of all members $(F_n)_{n=0}^{\infty}$ of $\mathcal{L}$ for which $n \in F_{n+1}$ for all but finitely many $n \in \mathbb{N}$. For each of $\mathcal{L}_0$ and $\mathcal{L}_1$ determine whether it is countable or uncountable. Justify your answers.

9A Dynamics and Relativity

Consider a rigid body, whose shape and density distribution are rotationally symmetric about a horizontal axis. The body has mass M, radius a and moment of inertia I about its axis of rotational symmetry and is rolling down a non-slip slope inclined at an angle α to the horizontal. By considering its energy, calculate the acceleration of the disc down the slope in terms of the quantities introduced above and g, the acceleration due to gravity.

(a) A sphere with density proportional to r^c (where r is distance to the sphere's centre and c is a positive constant) is launched up a non-slip slope of constant incline at the same time, level and speed as a vertical disc of constant density. Find c such that the sphere and the disc return to their launch points at the same time.

(b) Two spherical glass marbles of equal radius are released from rest at time $t = 0$ on an inclined non-slip slope of constant incline from the same level. The glass in each marble is of constant and equal density, but the second marble has two spherical air bubbles in it whose radii are half the radius of the marbles, initially centred directly above and below the marble's centre, respectively. Each bubble is centred half-way between the centre of the marble and its surface. At a later time t, find the ratio of the distance travelled by the first marble and the second. [You may state without proof any theorems that you use and neglect the mass of air in the bubbles.]

10A Dynamics and Relativity

Define the *4-momentum* P of a particle of rest mass m and velocity $\mathbf{u}$. Calculate the power series expansion of the component P^0 for small $|\mathbf{u}|/c$ (where c is the speed of light in vacuo) up to and including terms of order $|\mathbf{u}|^4$, and interpret the first two terms.

(a) At CERN, anti-protons are made by colliding a moving proton with another proton at rest in a fixed target. The collision in question produces three protons and an anti-proton. Assume that the rest mass of a proton is identical to the rest mass of an anti-proton. What is the smallest possible speed of the incoming proton (measured in the laboratory frame)?

(b) A moving particle of rest mass M decays into N particles with 4-momenta Q_i, and rest masses m_i, where $i = 1, 2, \ldots, N$. Show that

$$M = \frac{1}{c}\sqrt{\left(\sum_{i=1}^{N} Q_i\right) \cdot \left(\sum_{j=1}^{N} Q_j\right)}.$$

Thus, show that

$$M \geqslant \sum_{i=1}^{N} m_i.$$

(c) A particle A decays into particle B and a massless particle 1. Particle B subsequently decays into particle C and a massless particle 2. Show that

$$0 \leqslant (Q_1 + Q_2) \cdot (Q_1 + Q_2) \leqslant \frac{(m_A^2 - m_B^2)(m_B^2 - m_C^2)c^2}{m_B^2},$$

where Q_1 and Q_2 are the 4-momenta of particles 1 and 2 respectively and m_A, m_B, m_C are the masses of particles A, B and C respectively.

11A Dynamics and Relativity

Write down the Lorentz force law for a charge q travelling at velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$.

In a space station which is in an inertial frame, an experiment is performed in vacuo where a ball is released from rest a distance h from a wall. The ball has charge $q > 0$ and at time t, it is a distance $z(t)$ from the wall. A constant electric field of magnitude E points toward the wall in a perpendicular direction, but there is no magnetic field. Find the speed of the ball on its first impact.

Every time the ball bounces, its speed is reduced by a factor $\gamma < 1$. Show that the total distance travelled by the ball before it comes to rest is

$$L = h\frac{q_1(\gamma)}{q_2(\gamma)}$$

where q_1 and q_2 are quadratic functions which you should find explicitly.

A gas leak fills the apparatus with an atmosphere and the experiment is repeated. The ball now experiences an additional drag force $\mathbf{D} = -\alpha|\mathbf{v}|\mathbf{v}$, where $\mathbf{v}$ is the instantaneous velocity of the ball and $\alpha > 0$. Solve the system before the first bounce, finding an explicit solution for the distance $z(t)$ between the ball and the wall as a function of time of the form

$$z(t) = h - Af(Bt)$$

where f is a function and A and B are dimensional constants, all of which you should find explicitly.

12A Dynamics and Relativity

The position $\mathbf{x} = (x, y, z)$ and velocity $\dot{\mathbf{x}}$ of a particle of mass m are measured in a frame which rotates at constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame. The particle is subject to a force $\mathbf{F} = -9m|\boldsymbol{\omega}|^2\mathbf{x}$. What is the equation of motion of the particle?

Find the trajectory of the particle in the coordinates (x, y, z) if $\boldsymbol{\omega} = (0, 0, \omega)$ and at $t = 0$, $\mathbf{x} = (1, 0, 0)$ and $\dot{\mathbf{x}} = (0, 0, 0)$.

Find the maximum value of the speed $|\dot{\mathbf{x}}|$ of the particle and the times at which it travels at this speed.

[*Hint: You may find using the variable $\xi = x + iy$ helpful.*]

END OF PAPER