UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 4 June, 2018 9:00 am to 12:00 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

Find the order and the sign of the permutation $(13)(2457)(815) \in S_8$.

How many elements of S_6 have order 6? And how many have order 3?

What is the greatest order of any element of A_9?

2D Groups

Prove that every member of $O(3)$ is a product of at most three reflections.

Is every member of $O(3)$ a product of at most two reflections? Justify your answer.

3C Vector Calculus

Derive a formula for the curvature of the two-dimensional curve $\mathbf{x}(u) = (u, f(u))$.

Verify your result for the semicircle with radius a given by $f(u) = \sqrt{a^2 - u^2}$.

4C Vector Calculus

In plane polar coordinates (r, θ), the orthonormal basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ satisfy

$$\frac{\partial \mathbf{e}_r}{\partial r} = \frac{\partial \mathbf{e}_\theta}{\partial r} = \mathbf{0}, \quad \frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \quad \text{and} \quad \boldsymbol{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} \,.$$

Hence derive the expression $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \phi = \dfrac{1}{r} \dfrac{\partial}{\partial r} \left(r \dfrac{\partial \phi}{\partial r} \right) + \dfrac{1}{r^2} \dfrac{\partial^2 \phi}{\partial \theta^2}$ for the Laplacian operator ∇^2.

Calculate the Laplacian of $\phi(r, \theta) = \alpha r^\beta \cos(\gamma \theta)$, where α, β and γ are constants. Hence find all solutions to the equation

$$\nabla^2 \phi = 0 \quad \text{in} \quad 0 \leqslant r \leqslant a, \quad \text{with} \quad \partial \phi / \partial r = \cos(2\theta) \text{ on } r = a \,.$$

Explain briefly how you know that there are no other solutions.

SECTION II

5D Groups

Define the *sign* of a permutation $\sigma \in S_n$. You should show that it is well-defined, and also that it is multiplicative (in other words, that it gives a homomorphism from S_n to $\{\pm 1\}$).

Show also that (for $n \geqslant 2$) this is the only surjective homomorphism from S_n to $\{\pm 1\}$.

6D Groups

Let g be an element of a group G. We define a map g^* from G to G by sending x to gxg^{-1}. Show that g^* is an *automorphism* of G (that is, an isomorphism from G to G).

Now let A denote the group of automorphisms of G (with the group operation being composition), and define a map θ from G to A by setting $\theta(g) = g^*$. Show that θ is a homomorphism. What is the kernel of θ?

Prove that the image of θ is a normal subgroup of A.

Show that if G is cyclic then A is abelian. If G is abelian, must A be abelian? Justify your answer.

7D Groups

Define the *quotient group* G/H, where H is a normal subgroup of a group G. You should check that your definition is well-defined. Explain why, for G finite, the greatest order of any element of G/H is at most the greatest order of any element of G.

Show that a subgroup H of a group G is normal if and only if there is a homomorphism from G to some group whose kernel is H.

A group is called *metacyclic* if it has a cyclic normal subgroup H such that G/H is cyclic. Show that every dihedral group is metacyclic.

Which groups of order 8 are metacyclic? Is A_4 metacyclic? For which $n \leqslant 5$ is S_n metacyclic?

8D Groups

State and prove the Direct Product Theorem.

Is the group $O(3)$ isomorphic to $SO(3) \times C_2$? Is $O(2)$ isomorphic to $SO(2) \times C_2$?

Let $U(2)$ denote the group of all invertible 2×2 complex matrices A with $A\overline{A}^{\mathrm{T}} = I$, and let $SU(2)$ be the subgroup of $U(2)$ consisting of those matrices with determinant 1.

Determine the centre of $U(2)$.

Write down a surjective homomorphism from $U(2)$ to the group T of all unit-length complex numbers whose kernel is $SU(2)$. Is $U(2)$ isomorphic to $SU(2) \times T$?

9C Vector Calculus

Given a one-to-one mapping $u = u(x, y)$ and $v = v(x, y)$ between the region D in the (x, y)-plane and the region D' in the (u, v)-plane, state the formula for transforming the integral $\iint_D f(x, y)\, dx\, dy$ into an integral over D', with the Jacobian expressed explicitly in terms of the partial derivatives of u and v.

Let D be the region $x^2 + y^2 \leqslant 1$, $y \geqslant 0$ and consider the change of variables $u = x + y$ and $v = x^2 + y^2$. Sketch D, the curves of constant u and the curves of constant v in the (x, y)-plane. Find and sketch the image D' of D in the (u, v)-plane.

Calculate $I = \iint_D (x + y)\, dx\, dy$ using this change of variables. Check your answer by calculating I directly.

10C Vector Calculus

State the formula of Stokes's theorem, specifying any orientation where needed.

Let $\mathbf{F} = (y^2 z, xz + 2xyz, 0)$. Calculate $\boldsymbol{\nabla} \times \mathbf{F}$ and verify that $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{F} = 0$.

Sketch the surface S defined as the union of the surface $z = -1$, $1 \leqslant x^2 + y^2 \leqslant 4$ and the surface $x^2 + y^2 + z = 3$, $1 \leqslant x^2 + y^2 \leqslant 4$.

Verify Stokes's theorem for $\mathbf{F}$ on S.

11C Vector Calculus

Use Maxwell's equations,

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t},$$

to derive expressions for $\dfrac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E}$ and $\dfrac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B}$ in terms of ρ and $\mathbf{J}$.

Now suppose that there exists a scalar potential ϕ such that $\mathbf{E} = -\nabla \phi$, and $\phi \to 0$ as $r \to \infty$. If $\rho = \rho(r)$ is spherically symmetric, calculate $\mathbf{E}$ using Gauss's flux method, i.e. by integrating a suitable equation inside a sphere centred at the origin. Use your result to find $\mathbf{E}$ and ϕ in the case when $\rho = 1$ for $r < a$ and $\rho = 0$ otherwise.

For each integer $n \geqslant 0$, let S_n be the sphere of radius 4^{-n} centred at the point $(1 - 4^{-n}, 0, 0)$. Suppose that ρ vanishes outside S_0, and has the constant value 2^n in the volume between S_n and S_{n+1} for $n \geqslant 0$. Calculate $\mathbf{E}$ and ϕ at the point $(1, 0, 0)$.

12C Vector Calculus

(a) Suppose that a tensor T_{ij} can be decomposed as

$$T_{ij} = S_{ij} + \epsilon_{ijk} V_k \,, \qquad (*)$$

where S_{ij} is symmetric. Obtain expressions for S_{ij} and V_k in terms of T_{ij}, and check that $(*)$ is satisfied.

(b) State the most general form of an isotropic tensor of rank k for $k = 0, 1, 2, 3$, and verify that your answers are isotropic.

(c) The general form of an isotropic tensor of rank 4 is

$$T_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk} \,.$$

Suppose that A_{ij} and B_{ij} satisfy the linear relationship $A_{ij} = T_{ijkl} B_{kl}$, where T_{ijkl} is isotropic. Express B_{ij} in terms of A_{ij}, assuming that $\beta^2 \neq \gamma^2$ and $3\alpha + \beta + \gamma \neq 0$. If instead $\beta = -\gamma \neq 0$ and $\alpha \neq 0$, find all B_{ij} such that $A_{ij} = 0$.

(d) Suppose that C_{ij} and D_{ij} satisfy the quadratic relationship $C_{ij} = T_{ijklmn} D_{kl} D_{mn}$, where T_{ijklmn} is an isotropic tensor of rank 6. If C_{ij} is symmetric and D_{ij} is antisymmetric, find the most general non-zero form of $T_{ijklmn} D_{kl} D_{mn}$ and prove that there are only two independent terms. [*Hint: You do not need to use the general form of an isotropic tensor of rank* 6.]

END OF PAPER