UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday, 1 June, 2018 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1B Differential Equations

Consider the following difference equation for real u_n:

$$u_{n+1} = a u_n (1 - u_n^2)$$

where a is a real constant.

For $-\infty < a < \infty$ find the steady-state solutions, i.e. those with $u_{n+1} = u_n$ for all n, and determine their stability, making it clear how the number of solutions and the stability properties vary with a. [You need not consider in detail particular values of a which separate intervals with different stability properties.]

2B Differential Equations

Show that for given $P(x, y)$, $Q(x, y)$ there is a function $F(x, y)$ such that, for any function $y(x)$,

$$P(x, y) + Q(x, y) \frac{dy}{dx} = \frac{d}{dx} F(x, y)$$

if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Now solve the equation

$$(2y + 3x) \frac{dy}{dx} + 4x^3 + 3y = 0 \ .$$

3F Probability

Let X and Y be independent Poisson random variables with parameters λ and μ respectively.

(i) Show that $X + Y$ is Poisson with parameter $\lambda + \mu$.

(ii) Show that the conditional distribution of X given $X + Y = n$ is binomial, and find its parameters.

4F Probability

(a) State the Cauchy–Schwarz inequality and Markov's inequality. State and prove Jensen's inequality.

(b) For a discrete random variable X, show that $\mathrm{Var}(X) = 0$ implies that X is constant, i.e. there is $x \in \mathbb{R}$ such that $\mathbb{P}(X = x) = 1$.

SECTION II

5B Differential Equations

By choosing a suitable basis, solve the equation

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} -2 & 5 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = e^{-4t} \begin{pmatrix} 3b \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} -3 \\ c-1 \end{pmatrix},$$

subject to the initial conditions $x(0) = 0$, $y(0) = 0$.

Explain briefly what happens in the cases $b = 2$ or $c = 2$.

6B Differential Equations

The function $u(x, y)$ satisfies the partial differential equation

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = 0,$$

where a, b and c are non-zero constants.

Defining the variables $\xi = \alpha x + y$ and $\eta = \beta x + y$, where α and β are constants, and writing $v(\xi, \eta) = u(x, y)$ show that

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} = A(\alpha, \beta)\frac{\partial^2 v}{\partial \xi^2} + B(\alpha, \beta)\frac{\partial^2 v}{\partial \xi \partial \eta} + C(\alpha, \beta)\frac{\partial^2 v}{\partial \eta^2},$$

where you should determine the functions $A(\alpha, \beta)$, $B(\alpha, \beta)$ and $C(\alpha, \beta)$.

If the quadratic $as^2 + bs + c = 0$ has distinct real roots then show that α and β can be chosen such that $A(\alpha, \beta) = C(\alpha, \beta) = 0$ and $B(\alpha, \beta) \neq 0$.

If the quadratic $as^2 + bs + c = 0$ has a repeated root then show that α and β can be chosen such that $A(\alpha, \beta) = B(\alpha, \beta) = 0$ and $C(\alpha, \beta) \neq 0$.

Hence find the general solutions of the equations

(i) $$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$

and

(ii) $$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

7B Differential Equations

Consider the differential equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - (x^2 + \alpha^2) y = 0.$$

What values of x are *ordinary points* of the differential equation? What values of x are *singular points* of the differential equation, and are they *regular singular points* or *irregular singular points*? Give clear definitions of these terms to support your answers.

For α not equal to an integer there are two linearly independent power series solutions about $x = 0$. Give the forms of the two power series and the recurrence relations that specify the relation between successive coefficients. Give explicitly the first three terms in each power series.

For α equal to an integer explain carefully why the forms you have specified do *not* give two linearly independent power series solutions. Show that for such values of α there is (up to multiplication by a constant) one power series solution, and give the recurrence relation between coefficients. Give explicitly the first three terms.

If $y_1(x)$ is a solution of the above second-order differential equation then

$$y_2(x) = y_1(x) \int_c^x \frac{1}{s[y_1(s)]^2}\, ds,$$

where c is an arbitrarily chosen constant, is a second solution that is linearly independent of $y_1(x)$. For the case $\alpha = 1$, taking $y_1(x)$ to be a power series, explain why the second solution $y_2(x)$ is *not* a power series.

[*You may assume that any power series you use are convergent.*]

8B Differential Equations

The temperature T in an oven is controlled by a heater which provides heat at rate $Q(t)$. The temperature of a pizza in the oven is U. Room temperature is the constant value T_r .

T and U satisfy the coupled differential equations

$$\begin{aligned}\frac{dT}{dt} &= -a(T - T_r) + Q(t)\\ \frac{dU}{dt} &= -b(U - T)\end{aligned}$$

where a and b are positive constants. Briefly explain the various terms appearing in the above equations.

Heating may be provided by a short-lived pulse at $t = 0$, with $Q(t) = Q_1(t) = \delta(t)$ or by constant heating over a finite period $0 < t < \tau$, with $Q(t) = Q_2(t) = \tau^{-1}(H(t) - H(t - \tau))$, where $\delta(t)$ and $H(t)$ are respectively the Dirac delta function and the Heaviside step function. Again briefly, explain how the given formulae for $Q_1(t)$ and $Q_2(t)$ are consistent with their description and why the total heat supplied by the two heating protocols is the same.

For $t < 0$, $T = U = T_r$. Find the solutions for $T(t)$ and $U(t)$ for $t > 0$, for each of $Q(t) = Q_1(t)$ and $Q(t) = Q_2(t)$, denoted respectively by $T_1(t)$ and $U_1(t)$, and $T_2(t)$ and $U_2(t)$. Explain clearly any assumptions that you make about continuity of the solutions in time.

Show that the solutions $T_2(t)$ and $U_2(t)$ tend respectively to $T_1(t)$ and $U_1(t)$ in the limit as $\tau \to 0$ and explain why.

9F Probability

(a) Let Y and Z be independent discrete random variables taking values in sets S_1 and S_2 respectively, and let $F : S_1 \times S_2 \to \mathbb{R}$ be a function.

Let $E(z) = \mathbb{E}F(Y, z)$. Show that

$$\mathbb{E}E(Z) = \mathbb{E}F(Y, Z) \, .$$

Let $V(z) = \mathbb{E}(F(Y, z)^2) - (\mathbb{E}F(Y, z))^2$. Show that

$$\mathrm{Var}F(Y, Z) = \mathbb{E}V(Z) + \mathrm{Var}E(Z) \, .$$

(b) Let $X_1, \ldots, X_n$ be independent Bernoulli(p) random variables. For any function $F : \{0, 1\} \to \mathbb{R}$, show that

$$\mathrm{Var}F(X_1) = p(1-p)(F(1) - F(0))^2 \, .$$

Let $\{0, 1\}^n$ denote the set of all 0-1 sequences of length n. By induction, or otherwise, show that for any function $F : \{0, 1\}^n \to \mathbb{R}$,

$$\mathrm{Var}F(X) \leqslant p(1-p) \sum_{i=1}^{n} \mathbb{E}((F(X) - F(X^i))^2)$$

where $X = (X_1, \ldots, X_n)$ and $X^i = (X_1, \ldots, X_{i-1}, 1 - X_i, X_{i+1}, \ldots, X_n)$.

10F Probability

(a) Let X and Y be independent random variables taking values ± 1, each with probability $\frac{1}{2}$, and let $Z = XY$. Show that X, Y and Z are pairwise independent. Are they independent?

(b) Let X and Y be discrete random variables with mean 0, variance 1, covariance ρ. Show that $\mathbb{E}\max\{X^2, Y^2\} \leqslant 1 + \sqrt{1 - \rho^2}$.

(c) Let X_1, X_2, X_3 be discrete random variables. Writing $a_{ij} = \mathbb{P}(X_i > X_j)$, show that $\min\{a_{12}, a_{23}, a_{31}\} \leqslant \frac{2}{3}$.

11F Probability

(a) Consider a Galton–Watson process (X_n). Prove that the extinction probability q is the smallest non-negative solution of the equation $q = F(q)$ where $F(t) = \mathbb{E}(t^{X_1})$. [You should prove any properties of Galton–Watson processes that you use.]

In the case of a Galton–Watson process with

$$\mathbb{P}(X_1 = 1) = 1/4, \quad \mathbb{P}(X_1 = 3) = 3/4,$$

find the mean population size and compute the extinction probability.

(b) For each $n \in \mathbb{N}$, let Y_n be a random variable with distribution Poisson(n). Show that

$$\frac{Y_n - n}{\sqrt{n}} \to Z$$

in distribution, where Z is a standard normal random variable.

Deduce that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

12F Probability

For a symmetric simple random walk (X_n) on $\mathbb{Z}$ starting at 0, let $M_n = \max_{i \leqslant n} X_i$.

(i) For $m \geqslant 0$ and $x \in \mathbb{Z}$, show that

$$\mathbb{P}(M_n \geqslant m, X_n = x) = \begin{cases} \mathbb{P}(X_n = x) & \text{if } x \geqslant m \\ \mathbb{P}(X_n = 2m - x) & \text{if } x < m. \end{cases}$$

(ii) For $m \geqslant 0$, show that $\mathbb{P}(M_n \geqslant m) = \mathbb{P}(X_n = m) + 2\sum_{x>m} \mathbb{P}(X_n = x)$ and that

$$\mathbb{P}(M_n = m) = \mathbb{P}(X_n = m) + \mathbb{P}(X_n = m + 1).$$

(iii) Prove that $\mathbb{E}(M_n^2) < \mathbb{E}(X_n^2)$.

END OF PAPER