UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 31 May, 2018 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Vectors and Matrices

For $z, w \in \mathbb{C}$ define the *principal value* of z^w. State de Moivre's theorem.

Hence solve the equations

$$\text{(i) } z^6 = \sqrt{3} + i, \quad \text{(ii) } z^{1/6} = \sqrt{3} + i, \quad \text{(iii) } i^z = \sqrt{3} + i, \quad \text{(iv) } \left(e^{5i\pi/2}\right)^z = \sqrt{3} + i\,.$$

[In each expression, the principal value is to be taken.]

2A Vectors and Matrices

The map $\mathbf{\Phi}(\mathbf{x}) = \alpha(\mathbf{n} \cdot \mathbf{x})\mathbf{n} - \mathbf{n} \times (\mathbf{n} \times \mathbf{x})$ is defined for $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$ and α is a real constant.

(i) Find the values of α for which the inverse map $\mathbf{\Phi}^{-1}$ exists, as well as the inverse map itself in these cases.

(ii) When $\mathbf{\Phi}$ is not invertible, find its image and kernel. What is the value of the rank and the value of the nullity of $\mathbf{\Phi}$?

(iii) Let $\mathbf{y} = \mathbf{\Phi}(\mathbf{x})$. Find the components A_{ij} of the matrix A such that $y_i = A_{ij}x_j$. When $\mathbf{\Phi}$ is invertible, find the components of the matrix B such that $x_i = B_{ij}y_j$.

3E Analysis I

Prove that an increasing sequence in $\mathbb{R}$ that is bounded above converges.

Let $f \colon \mathbb{R} \to (0, \infty)$ be a decreasing function. Let $x_1 = 1$ and $x_{n+1} = x_n + f(x_n)$. Prove that $x_n \to \infty$ as $n \to \infty$.

4D Analysis I

Define the *radius of convergence* R of a complex power series $\sum a_n z^n$. Prove that $\sum a_n z^n$ converges whenever $|z| < R$ and diverges whenever $|z| > R$.

If $|a_n| \leqslant |b_n|$ for all n does it follow that the radius of convergence of $\sum a_n z^n$ is at least that of $\sum b_n z^n$? Justify your answer.

SECTION II

5C Vectors and Matrices

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be non-zero real vectors. Define the *inner product* $\mathbf{x} \cdot \mathbf{y}$ in terms of the components x_i and y_i, and define the *norm* $|\mathbf{x}|$. Prove that $\mathbf{x} \cdot \mathbf{y} \leqslant |\mathbf{x}|\,|\mathbf{y}|$. When does equality hold? Express the angle between $\mathbf{x}$ and $\mathbf{y}$ in terms of their inner product.

Use suffix notation to expand $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c})$.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be given unit vectors in $\mathbb{R}^3$, and let $\mathbf{m} = (\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a})$. Obtain expressions for the angle between $\mathbf{m}$ and each of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $|\mathbf{m}|$. Calculate $|\mathbf{m}|$ for the particular case when the angles between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are all equal to θ, and check your result for an example with $\theta = 0$ and an example with $\theta = \pi/2$.

Consider three planes in $\mathbb{R}^3$ passing through the points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively, with unit normals $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. State a condition that must be satisfied for the three planes to intersect at a single point, and find the intersection point.

6B Vectors and Matrices

(a) Consider the matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

representing a rotation about the z-axis through an angle θ.

Show that R has three eigenvalues in $\mathbb{C}$ each with modulus 1, of which one is real and two are complex (in general), and give the relation of the real eigenvector and the two complex eigenvalues to the properties of the rotation.

Now consider the rotation composed of a rotation by angle $\pi/2$ about the z-axis followed by a rotation by angle $\pi/2$ about the x-axis. Determine the rotation axis $\mathbf{n}$ and the magnitude of the angle of rotation ϕ.

(b) A surface in $\mathbb{R}^3$ is given by

$$7x^2 + 4xy + 3y^2 + 2xz + 3z^2 = 1.$$

By considering a suitable eigenvalue problem, show that the surface is an ellipsoid, find the lengths of its semi-axes and find the position of the two points on the surface that are closest to the origin.

7B Vectors and Matrices
Let A be a real symmetric $n \times n$ matrix.

(a) Prove the following:

(i) Each eigenvalue of A is real and there is a corresponding real eigenvector.
(ii) Eigenvectors corresponding to different eigenvalues are orthogonal.
(iii) If there are n distinct eigenvalues then the matrix is diagonalisable.

Assuming that A has n distinct eigenvalues, explain briefly how to choose (up to an arbitrary scalar factor) the vector v such that $\dfrac{v^T A v}{v^T v}$ is maximised.

(b) A scalar λ and a non-zero vector v such that

$$Av = \lambda B v$$

are called, for a specified $n \times n$ matrix B, respectively a *generalised eigenvalue* and a *generalised eigenvector* of A.

Assume the matrix B is real, symmetric and positive definite (i.e. $(u^*)^T B u > 0$ for all non-zero complex vectors u).

Prove the following:

(i) If λ is a generalised eigenvalue of A then it is a root of $\det(A - \lambda B) = 0$.
(ii) Each generalised eigenvalue of A is real and there is a corresponding real generalised eigenvector.
(iii) Two generalised eigenvectors u, v, corresponding to different generalised eigenvalues, are orthogonal in the sense that $u^T B v = 0$.

(c) Find, up to an arbitrary scalar factor, the vector v such that the value of $F(v) = \dfrac{v^T A v}{v^T B v}$ is maximised, and the corresponding value of $F(v)$, where

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 10 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

8A Vectors and Matrices

What is the definition of an *orthogonal matrix* M?

Write down a $2{\times}2$ matrix R representing the rotation of a 2-dimensional vector (x, y) by an angle θ around the origin. Show that R is indeed orthogonal.

Take a matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

where a, b, c are real. Suppose that the $2{\times}2$ matrix $B = RAR^T$ is diagonal. Determine all possible values of θ.

Show that the diagonal entries of B are the eigenvalues of A and express them in terms of the determinant and trace of A.

Using the above results, or otherwise, find the elements of the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{2N}$$

as a function of N, where N is a natural number.

9F Analysis I

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $x \in \mathbb{R}$. Define what it means for f to be *continuous* at x. Show that f is continuous at x if and only if $f(x_n) \to f(x)$ for every sequence (x_n) with $x_n \to x$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant polynomial. Show that its image $\{f(x) : x \in \mathbb{R}\}$ is either the real line $\mathbb{R}$, the interval $[a, \infty)$ for some $a \in \mathbb{R}$, or the interval $(-\infty, a]$ for some $a \in \mathbb{R}$.

(c) Let $\alpha > 1$, let $f : (0, \infty) \to \mathbb{R}$ be continuous, and assume that $f(x) = f(x^\alpha)$ holds for all $x > 0$. Show that f must be constant.

Is this also true when the condition that f be continuous is dropped?

10F Analysis

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$. Show that f is continuous at x_0.

(b) State the Mean Value Theorem. Prove the following inequalities:

$$|\cos(e^{-x}) - \cos(e^{-y})| \leqslant |x - y| \quad \text{for } x, y \geqslant 0$$

and

$$\log(1 + x) \leqslant \frac{x}{\sqrt{1+x}} \quad \text{for } x \geqslant 0.$$

(c) Determine at which points the following functions from $\mathbb{R}$ to $\mathbb{R}$ are differentiable, and find their derivatives at the points at which they are differentiable:

$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases} \qquad g(x) = \cos(|x|), \qquad h(x) = x|x|.$$

(d) Determine the points at which the following function from $\mathbb{R}$ to $\mathbb{R}$ is continuous:

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \text{ or } x = 0 \\ 1/q & \text{if } x = p/q \text{ where } p \in \mathbb{Z} \setminus \{0\} \text{ and } q \in \mathbb{N} \text{ are relatively prime.} \end{cases}$$

11E Analysis I

State and prove the Comparison Test for real series.

Assume $0 \leqslant x_n < 1$ for all $n \in \mathbb{N}$. Show that if $\sum x_n$ converges, then so do $\sum x_n^2$ and $\sum \frac{x_n}{1-x_n}$. In each case, does the converse hold? Justify your answers.

Let (x_n) be a decreasing sequence of positive reals. Show that if $\sum x_n$ converges, then $nx_n \to 0$ as $n \to \infty$. Does the converse hold? If $\sum x_n$ converges, must it be the case that $(n \log n)x_n \to 0$ as $n \to \infty$? Justify your answers.

12D Analysis I

(a) Let $q_1, q_2, \ldots$ be a fixed enumeration of the rationals in $[0, 1]$. For positive reals $a_1, a_2, \ldots$, define a function f from $[0, 1]$ to $\mathbb{R}$ by setting $f(q_n) = a_n$ for each n and $f(x) = 0$ for x irrational. Prove that if $a_n \to 0$ then f is Riemann integrable. If $a_n \not\to 0$, can f be Riemann integrable? Justify your answer.

(b) State and prove the Fundamental Theorem of Calculus.

Let f be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$, and set $g(x) = f'(x)$ for $0 \leqslant x \leqslant 1$. Must g be Riemann integrable on $[0, 1]$?

END OF PAPER