UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA 2018

List of Courses

Paper 1, Section I

3E Analysis I

Prove that an increasing sequence in $\mathbb{R}$ that is bounded above converges.

Let $f\colon \mathbb{R} \to (0,\infty)$ be a decreasing function. Let $x_1 = 1$ and $x_{n+1} = x_n + f(x_n)$. Prove that $x_n \to \infty$ as $n \to \infty$.

Paper 1, Section I

4D Analysis I

Define the *radius of convergence* R of a complex power series $\sum a_n z^n$. Prove that $\sum a_n z^n$ converges whenever $|z| < R$ and diverges whenever $|z| > R$.

If $|a_n| \leqslant |b_n|$ for all n does it follow that the radius of convergence of $\sum a_n z^n$ is at least that of $\sum b_n z^n$? Justify your answer.

Paper 1, Section II

9F Analysis I

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $x \in \mathbb{R}$. Define what it means for f to be *continuous* at x. Show that f is continuous at x if and only if $f(x_n) \to f(x)$ for every sequence (x_n) with $x_n \to x$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant polynomial. Show that its image $\{f(x) : x \in \mathbb{R}\}$ is either the real line $\mathbb{R}$, the interval $[a,\infty)$ for some $a \in \mathbb{R}$, or the interval $(-\infty, a]$ for some $a \in \mathbb{R}$.

(c) Let $\alpha > 1$, let $f : (0,\infty) \to \mathbb{R}$ be continuous, and assume that $f(x) = f(x^\alpha)$ holds for all $x > 0$. Show that f must be constant.

Is this also true when the condition that f be continuous is dropped?

Paper 1, Section II
10F Analysis I

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$. Show that f is continuous at x_0.

(b) State the Mean Value Theorem. Prove the following inequalities:

$$|\cos(e^{-x}) - \cos(e^{-y})| \leqslant |x - y| \quad \text{for } x, y \geqslant 0$$

and

$$\log(1 + x) \leqslant \frac{x}{\sqrt{1 + x}} \quad \text{for } x \geqslant 0.$$

(c) Determine at which points the following functions from $\mathbb{R}$ to $\mathbb{R}$ are differentiable, and find their derivatives at the points at which they are differentiable:

$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases} \qquad g(x) = \cos(|x|), \qquad h(x) = x|x|.$$

(d) Determine the points at which the following function from $\mathbb{R}$ to $\mathbb{R}$ is continuous:

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \text{ or } x = 0 \\ 1/q & \text{if } x = p/q \text{ where } p \in \mathbb{Z} \setminus \{0\} \text{ and } q \in \mathbb{N} \text{ are relatively prime.} \end{cases}$$

Paper 1, Section II
11E Analysis I

State and prove the Comparison Test for real series.

Assume $0 \leqslant x_n < 1$ for all $n \in \mathbb{N}$. Show that if $\sum x_n$ converges, then so do $\sum x_n^2$ and $\sum \frac{x_n}{1-x_n}$. In each case, does the converse hold? Justify your answers.

Let (x_n) be a decreasing sequence of positive reals. Show that if $\sum x_n$ converges, then $nx_n \to 0$ as $n \to \infty$. Does the converse hold? If $\sum x_n$ converges, must it be the case that $(n \log n)x_n \to 0$ as $n \to \infty$? Justify your answers.

Paper 1, Section II

12D Analysis I

(a) Let $q_1, q_2, \ldots$ be a fixed enumeration of the rationals in $[0,1]$. For positive reals $a_1, a_2, \ldots$, define a function f from $[0,1]$ to $\mathbb{R}$ by setting $f(q_n) = a_n$ for each n and $f(x) = 0$ for x irrational. Prove that if $a_n \to 0$ then f is Riemann integrable. If $a_n \not\to 0$, can f be Riemann integrable? Justify your answer.

(b) State and prove the Fundamental Theorem of Calculus.

Let f be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$, and set $g(x) = f'(x)$ for $0 \leqslant x \leqslant 1$. Must g be Riemann integrable on $[0,1]$?

Paper 2, Section I

1B Differential Equations

Consider the following difference equation for real u_n:

$$u_{n+1} = au_n(1 - u_n^2)$$

where a is a real constant.

For $-\infty < a < \infty$ find the steady-state solutions, i.e. those with $u_{n+1} = u_n$ for all n, and determine their stability, making it clear how the number of solutions and the stability properties vary with a. [You need not consider in detail particular values of a which separate intervals with different stability properties.]

Paper 2, Section I

2B Differential Equations

Show that for given $P(x, y)$, $Q(x, y)$ there is a function $F(x, y)$ such that, for any function $y(x)$,

$$P(x, y) + Q(x, y)\frac{dy}{dx} = \frac{d}{dx}F(x, y)$$

if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Now solve the equation

$$(2y + 3x)\frac{dy}{dx} + 4x^3 + 3y = 0\ .$$

Paper 2, Section II

5B Differential Equations

By choosing a suitable basis, solve the equation

$$\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{pmatrix} -2 & 5 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = e^{-4t}\begin{pmatrix} 3b \\ 2 \end{pmatrix} + e^{-t}\begin{pmatrix} -3 \\ c-1 \end{pmatrix},$$

subject to the initial conditions $x(0) = 0$, $y(0) = 0$.

Explain briefly what happens in the cases $b = 2$ or $c = 2$.

Paper 2, Section II

6B Differential Equations

The function $u(x,y)$ satisfies the partial differential equation

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}=0,$$

where a, b and c are non-zero constants.

Defining the variables $\xi=\alpha x+y$ and $\eta=\beta x+y$, where α and β are constants, and writing $v(\xi,\eta)=u(x,y)$ show that

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}=A(\alpha,\beta)\frac{\partial^2 v}{\partial \xi^2}+B(\alpha,\beta)\frac{\partial^2 v}{\partial \xi\partial\eta}+C(\alpha,\beta)\frac{\partial^2 v}{\partial \eta^2},$$

where you should determine the functions $A(\alpha,\beta)$, $B(\alpha,\beta)$ and $C(\alpha,\beta)$.

If the quadratic $as^2+bs+c=0$ has distinct real roots then show that α and β can be chosen such that $A(\alpha,\beta)=C(\alpha,\beta)=0$ and $B(\alpha,\beta)\neq 0$.

If the quadratic $as^2+bs+c=0$ has a repeated root then show that α and β can be chosen such that $A(\alpha,\beta)=B(\alpha,\beta)=0$ and $C(\alpha,\beta)\neq 0$.

Hence find the general solutions of the equations

(i) $$\frac{\partial^2 u}{\partial x^2}+3\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial^2 u}{\partial y^2}=0$$

and

(ii) $$\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0.$$

Paper 2, Section II

7B Differential Equations

Consider the differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x^2+\alpha^2)y=0.$$

What values of x are *ordinary points* of the differential equation? What values of x are *singular points* of the differential equation, and are they *regular singular points* or *irregular singular points*? Give clear definitions of these terms to support your answers.

For α not equal to an integer there are two linearly independent power series solutions about $x=0$. Give the forms of the two power series and the recurrence relations that specify the relation between successive coefficients. Give explicitly the first three terms in each power series.

For α equal to an integer explain carefully why the forms you have specified do *not* give two linearly independent power series solutions. Show that for such values of α there is (up to multiplication by a constant) one power series solution, and give the recurrence relation between coefficients. Give explicitly the first three terms.

If $y_1(x)$ is a solution of the above second-order differential equation then

$$y_2(x)=y_1(x)\int_c^x\frac{1}{s[y_1(s)]^2}\,ds,$$

where c is an arbitrarily chosen constant, is a second solution that is linearly independent of $y_1(x)$. For the case $\alpha=1$, taking $y_1(x)$ to be a power series, explain why the second solution $y_2(x)$ is *not* a power series.

[*You may assume that any power series you use are convergent.*]

Paper 2, Section II

8B Differential Equations

The temperature T in an oven is controlled by a heater which provides heat at rate $Q(t)$. The temperature of a pizza in the oven is U. Room temperature is the constant value T_r .

T and U satisfy the coupled differential equations

$$\begin{aligned} \frac{dT}{dt} &= -a(T-T_r)+Q(t) \\ \frac{dU}{dt} &= -b(U-T) \end{aligned}$$

where a and b are positive constants. Briefly explain the various terms appearing in the above equations.

Heating may be provided by a short-lived pulse at $t=0$, with $Q(t)=Q_1(t)=\delta(t)$ or by constant heating over a finite period $0<t<\tau$, with $Q(t)=Q_2(t)=\tau^{-1}(H(t)-H(t-\tau))$, where $\delta(t)$ and $H(t)$ are respectively the Dirac delta function and the Heaviside step function. Again briefly, explain how the given formulae for $Q_1(t)$ and $Q_2(t)$ are consistent with their description and why the total heat supplied by the two heating protocols is the same.

For $t<0$, $T=U=T_r$. Find the solutions for $T(t)$ and $U(t)$ for $t>0$, for each of $Q(t)=Q_1(t)$ and $Q(t)=Q_2(t)$, denoted respectively by $T_1(t)$ and $U_1(t)$, and $T_2(t)$ and $U_2(t)$. Explain clearly any assumptions that you make about continuity of the solutions in time.

Show that the solutions $T_2(t)$ and $U_2(t)$ tend respectively to $T_1(t)$ and $U_1(t)$ in the limit as $\tau\to 0$ and explain why.

Paper 4, Section I

3A Dynamics and Relativity

(a) Define an *inertial frame.*

(b) Specify three different types of Galilean transformation on inertial frames whose space coordinates are $\mathbf{x}$ and whose time coordinate is t.

(c) State the *Principle of Galilean Relativity.*

(d) Write down the equation of motion for a particle in one dimension x in a potential $V(x)$. Prove that energy is conserved. A particle is at position x_0 at time t_0. Find an expression for time t as a function of x in terms of an integral involving V.

(e) Write down the x values of any equilibria and state (without justification) whether they are stable or unstable for:

(i) $V(x) = (x^2 - 4)^2$

(ii) $V(x) = e^{-1/x^2}$ for $x \neq 0$ and $V(0) = 0$.

Paper 4, Section I

4A Dynamics and Relativity

Explain what is meant by a *central force* acting on a particle moving in three dimensions.

Show that the angular momentum of a particle about the origin for a central force is conserved, and hence that its path lies in a plane.

Show that, in the approximation in which the Sun is regarded as fixed and only its gravitational field is considered, a straight line joining the Sun and an orbiting planet sweeps out equal areas in equal time (Kepler's second law).

Paper 4, Section II

9A Dynamics and Relativity

Consider a rigid body, whose shape and density distribution are rotationally symmetric about a horizontal axis. The body has mass M, radius a and moment of inertia I about its axis of rotational symmetry and is rolling down a non-slip slope inclined at an angle α to the horizontal. By considering its energy, calculate the acceleration of the disc down the slope in terms of the quantities introduced above and g, the acceleration due to gravity.

(a) A sphere with density proportional to r^c (where r is distance to the sphere's centre and c is a positive constant) is launched up a non-slip slope of constant incline at the same time, level and speed as a vertical disc of constant density. Find c such that the sphere and the disc return to their launch points at the same time.

(b) Two spherical glass marbles of equal radius are released from rest at time $t = 0$ on an inclined non-slip slope of constant incline from the same level. The glass in each marble is of constant and equal density, but the second marble has two spherical air bubbles in it whose radii are half the radius of the marbles, initially centred directly above and below the marble's centre, respectively. Each bubble is centred half-way between the centre of the marble and its surface. At a later time t, find the ratio of the distance travelled by the first marble and the second. [You may state without proof any theorems that you use and neglect the mass of air in the bubbles.]

Paper 4, Section II

10A Dynamics and Relativity

Define the *4-momentum* P of a particle of rest mass m and velocity $\mathbf{u}$. Calculate the power series expansion of the component P^0 for small $|\mathbf{u}|/c$ (where c is the speed of light in vacuo) up to and including terms of order $|\mathbf{u}|^4$, and interpret the first two terms.

(a) At CERN, anti-protons are made by colliding a moving proton with another proton at rest in a fixed target. The collision in question produces three protons and an anti-proton. Assume that the rest mass of a proton is identical to the rest mass of an anti-proton. What is the smallest possible speed of the incoming proton (measured in the laboratory frame)?

(b) A moving particle of rest mass M decays into N particles with 4-momenta Q_i, and rest masses m_i, where $i = 1, 2, \ldots, N$. Show that

$$M = \frac{1}{c}\sqrt{\left(\sum_{i=1}^{N} Q_i\right) \cdot \left(\sum_{j=1}^{N} Q_j\right)}.$$

Thus, show that

$$M \geqslant \sum_{i=1}^{N} m_i.$$

(c) A particle A decays into particle B and a massless particle 1. Particle B subsequently decays into particle C and a massless particle 2. Show that

$$0 \leqslant (Q_1 + Q_2) \cdot (Q_1 + Q_2) \leqslant \frac{(m_A^2 - m_B^2)(m_B^2 - m_C^2)c^2}{m_B^2},$$

where Q_1 and Q_2 are the 4-momenta of particles 1 and 2 respectively and m_A, m_B, m_C are the masses of particles A, B and C respectively.

Paper 4, Section II

11A Dynamics and Relativity

Write down the Lorentz force law for a charge q travelling at velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$.

In a space station which is in an inertial frame, an experiment is performed in vacuo where a ball is released from rest a distance h from a wall. The ball has charge $q > 0$ and at time t, it is a distance $z(t)$ from the wall. A constant electric field of magnitude E points toward the wall in a perpendicular direction, but there is no magnetic field. Find the speed of the ball on its first impact.

Every time the ball bounces, its speed is reduced by a factor $\gamma < 1$. Show that the total distance travelled by the ball before it comes to rest is

$$L = h\frac{q_1(\gamma)}{q_2(\gamma)}$$

where q_1 and q_2 are quadratic functions which you should find explicitly.

A gas leak fills the apparatus with an atmosphere and the experiment is repeated. The ball now experiences an additional drag force $\mathbf{D} = -\alpha|\mathbf{v}|\mathbf{v}$, where $\mathbf{v}$ is the instantaneous velocity of the ball and $\alpha > 0$. Solve the system before the first bounce, finding an explicit solution for the distance $z(t)$ between the ball and the wall as a function of time of the form

$$z(t) = h - Af(Bt)$$

where f is a function and A and B are dimensional constants, all of which you should find explicitly.

Paper 4, Section II

12A Dynamics and Relativity

The position $\mathbf{x} = (x, y, z)$ and velocity $\dot{\mathbf{x}}$ of a particle of mass m are measured in a frame which rotates at constant angular velocity $\boldsymbol{\omega}$ with respect to an inertial frame. The particle is subject to a force $\mathbf{F} = -9m|\boldsymbol{\omega}|^2\mathbf{x}$. What is the equation of motion of the particle?

Find the trajectory of the particle in the coordinates (x, y, z) if $\boldsymbol{\omega} = (0, 0, \omega)$ and at $t = 0$, $\mathbf{x} = (1, 0, 0)$ and $\dot{\mathbf{x}} = (0, 0, 0)$.

Find the maximum value of the speed $|\dot{\mathbf{x}}|$ of the particle and the times at which it travels at this speed.

[*Hint: You may find using the variable* $\xi = x + iy$ *helpful.*]

Paper 3, Section I

1D Groups

Find the order and the sign of the permutation $(13)(2457)(815) \in S_8$.

How many elements of S_6 have order 6? And how many have order 3?

What is the greatest order of any element of A_9?

Paper 3, Section I

2D Groups

Prove that every member of $O(3)$ is a product of at most three reflections.

Is every member of $O(3)$ a product of at most two reflections? Justify your answer.

Paper 3, Section II

5D Groups

Define the *sign* of a permutation $\sigma \in S_n$. You should show that it is well-defined, and also that it is multiplicative (in other words, that it gives a homomorphism from S_n to $\{\pm 1\}$).

Show also that (for $n \geqslant 2$) this is the only surjective homomorphism from S_n to $\{\pm 1\}$.

Paper 3, Section II

6D Groups

Let g be an element of a group G. We define a map g^* from G to G by sending x to gxg^{-1}. Show that g^* is an *automorphism* of G (that is, an isomorphism from G to G).

Now let A denote the group of automorphisms of G (with the group operation being composition), and define a map θ from G to A by setting $\theta(g) = g^*$. Show that θ is a homomorphism. What is the kernel of θ?

Prove that the image of θ is a normal subgroup of A.

Show that if G is cyclic then A is abelian. If G is abelian, must A be abelian? Justify your answer.

Paper 3, Section II

7D Groups

Define the *quotient group* G/H, where H is a normal subgroup of a group G. You should check that your definition is well-defined. Explain why, for G finite, the greatest order of any element of G/H is at most the greatest order of any element of G.

Show that a subgroup H of a group G is normal if and only if there is a homomorphism from G to some group whose kernel is H.

A group is called *metacyclic* if it has a cyclic normal subgroup H such that G/H is cyclic. Show that every dihedral group is metacyclic.

Which groups of order 8 are metacyclic? Is A_4 metacyclic? For which $n \leqslant 5$ is S_n metacyclic?

Paper 3, Section II

8D Groups

State and prove the Direct Product Theorem.

Is the group $O(3)$ isomorphic to $SO(3) \times C_2$? Is $O(2)$ isomorphic to $SO(2) \times C_2$?

Let $U(2)$ denote the group of all invertible 2×2 complex matrices A with $A\overline{A}^{\mathrm{T}} = I$, and let $SU(2)$ be the subgroup of $U(2)$ consisting of those matrices with determinant 1.

Determine the centre of $U(2)$.

Write down a surjective homomorphism from $U(2)$ to the group T of all unit-length complex numbers whose kernel is $SU(2)$. Is $U(2)$ isomorphic to $SU(2) \times T$?

Paper 4, Section I

1E Numbers and Sets

State Fermat's theorem.

Let p be a prime such that $p \equiv 3 \pmod 4$. Prove that there is no solution to $x^2 \equiv -1 \pmod p$.

Show that there are infinitely many primes congruent to $1 \pmod 4$.

Paper 4, Section I

2E Numbers and Sets

Given $n \in \mathbb{N}$, show that $\sqrt{n}$ is either an integer or irrational.

Let α and β be irrational numbers and q be rational. Which of $\alpha + q$, $\alpha + \beta$, $\alpha\beta$, α^q and α^β *must* be irrational? Justify your answers. [*Hint: For the last part consider* $\sqrt{2}^{\sqrt{2}}$.]

Paper 4, Section II

5E Numbers and Sets

Let n be a positive integer. Show that for any a coprime to n, there is a unique $b \pmod n$ such that $ab \equiv 1 \pmod n$. Show also that if a and b are integers coprime to n, then ab is also coprime to n. [Any version of Bezout's theorem may be used without proof provided it is clearly stated.]

State and prove Wilson's theorem.

Let n be a positive integer and p be a prime. Show that the exponent of p in the prime factorisation of $n!$ is given by $\sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor$ where $\lfloor x \rfloor$ denotes the integer part of x.

Evaluate $20! \pmod{23}$ and $1000! \pmod{10^{249}}$.

Let p be a prime and $0 < k < p^m$. Let ℓ be the exponent of p in the prime factorisation of k. Find the exponent of p in the prime factorisation of $\binom{p^m}{k}$, in terms of m and ℓ.

Paper 4, Section II
6E Numbers and Sets

For $n \in \mathbb{N}$ let $Q_n = \{0,1\}^n$ denote the set of all 0-1 sequences of length n. We define the *distance* $d(x, y)$ between two elements x and y of Q_n to be the number of coordinates in which they differ. Show that $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in Q_n$.

For $x \in Q_n$ and $1 \leqslant j \leqslant n$ let $B(x, j) = \{y \in Q_n : d(y, x) \leqslant j\}$. Show that $|B(x, j)| = \sum_{i=0}^{j} \binom{n}{i}$.

A subset C of Q_n is called a *k-code* if $d(x, y) \geqslant 2k + 1$ for all $x, y \in C$ with $x \neq y$. Let $M(n, k)$ be the maximum possible value of $|C|$ for a k-code C in Q_n. Show that

$$2^n \left(\sum_{i=0}^{2k} \binom{n}{i} \right)^{-1} \leqslant M(n, k) \leqslant 2^n \left(\sum_{i=0}^{k} \binom{n}{i} \right)^{-1} .$$

Find $M(4, 1)$, carefully justifying your answer.

Paper 4, Section II
7E Numbers and Sets

Let $n \in \mathbb{N}$ and $A_1, \ldots, A_n$ be subsets of a finite set X. Let $0 \leqslant t \leqslant n$. Show that if $x \in X$ belongs to A_i for exactly m values of i, then

$$\sum_{S \subset \{1,\ldots,n\}} \binom{|S|}{t} (-1)^{|S|-t} \mathbf{1}_{A_S}(x) = \begin{cases} 0 & \text{if } m \neq t \\ 1 & \text{if } m = t \end{cases}$$

where $A_S = \bigcap_{i \in S} A_i$ with the convention that $A_\emptyset = X$, and $\mathbf{1}_{A_S}$ denotes the indicator function of A_S. [*Hint: Set* $M = \{i : x \in A_i\}$ *and consider for which* $S \subset \{1, \ldots, n\}$ *one has* $\mathbf{1}_{A_S}(x) = 1$.]

Use this to show that the number of elements of X that belong to A_i for exactly t values of i is

$$\sum_{S \subset \{1,\ldots,n\}} \binom{|S|}{t} (-1)^{|S|-t} |A_S| .$$

Deduce the Inclusion-Exclusion Principle.

Using the Inclusion-Exclusion Principle, prove a formula for the Euler totient function $\varphi(N)$ in terms of the distinct prime factors of N.

A *Carmichael number* is a composite number n such that $x^{n-1} \equiv 1 \pmod{n}$ for every integer x coprime to n. Show that if $n = q_1 q_2 \ldots q_k$ is the product of $k \geqslant 2$ distinct primes $q_1, \ldots, q_k$ satisfying $q_j - 1 \mid n - 1$ for $j = 1, \ldots, k$, then n is a Carmichael number.

Paper 4, Section II

8E Numbers and Sets

Define what it means for a set to be *countable*.

Show that for any set X, there is no surjection from X onto the power set $\mathcal{P}(X)$. Deduce that the set $\{0,1\}^{\mathbb{N}}$ of all infinite 0-1 sequences is uncountable.

Let $\mathcal{L}$ be the set of sequences $(F_n)_{n=0}^{\infty}$ of subsets $F_0 \subset F_1 \subset F_2 \subset \ldots$ of $\mathbb{N}$ such that $|F_n| = n$ for all $n \in \mathbb{N}$ and $\bigcup_n F_n = \mathbb{N}$. Let $\mathcal{L}_0$ consist of all members $(F_n)_{n=0}^{\infty}$ of $\mathcal{L}$ for which $n \in F_n$ for all but finitely many $n \in \mathbb{N}$. Let $\mathcal{L}_1$ consist of all members $(F_n)_{n=0}^{\infty}$ of $\mathcal{L}$ for which $n \in F_{n+1}$ for all but finitely many $n \in \mathbb{N}$. For each of $\mathcal{L}_0$ and $\mathcal{L}_1$ determine whether it is countable or uncountable. Justify your answers.

Paper 2, Section I

3F Probability

Let X and Y be independent Poisson random variables with parameters λ and μ respectively.

(i) Show that $X + Y$ is Poisson with parameter $\lambda + \mu$.

(ii) Show that the conditional distribution of X given $X + Y = n$ is binomial, and find its parameters.

Paper 2, Section I

4F Probability

(a) State the Cauchy–Schwarz inequality and Markov's inequality. State and prove Jensen's inequality.

(b) For a discrete random variable X, show that $\mathrm{Var}(X) = 0$ implies that X is constant, i.e. there is $x \in \mathbb{R}$ such that $\mathbb{P}(X = x) = 1$.

Paper 2, Section II
9F Probability

(a) Let Y and Z be independent discrete random variables taking values in sets S_1 and S_2 respectively, and let $F : S_1 \times S_2 \to \mathbb{R}$ be a function.

Let $E(z) = \mathbb{E}F(Y, z)$. Show that

$$\mathbb{E}E(Z) = \mathbb{E}F(Y, Z)\ .$$

Let $V(z) = \mathbb{E}(F(Y, z)^2) - (\mathbb{E}F(Y, z))^2$. Show that

$$\mathrm{Var}F(Y, Z) = \mathbb{E}V(Z) + \mathrm{Var}E(Z)\ .$$

(b) Let $X_1, \ldots, X_n$ be independent Bernoulli(p) random variables. For any function $F : \{0, 1\} \to \mathbb{R}$, show that

$$\mathrm{Var}F(X_1) = p(1-p)(F(1) - F(0))^2\ .$$

Let $\{0, 1\}^n$ denote the set of all 0-1 sequences of length n. By induction, or otherwise, show that for any function $F : \{0, 1\}^n \to \mathbb{R}$,

$$\mathrm{Var}F(X) \leqslant p(1-p)\sum_{i=1}^{n} \mathbb{E}((F(X) - F(X^i))^2)$$

where $X = (X_1, \ldots, X_n)$ and $X^i = (X_1, \ldots, X_{i-1}, 1 - X_i, X_{i+1}, \ldots, X_n)$.

Paper 2, Section II
10F Probability

(a) Let X and Y be independent random variables taking values ± 1, each with probability $\frac{1}{2}$, and let $Z = XY$. Show that X, Y and Z are pairwise independent. Are they independent?

(b) Let X and Y be discrete random variables with mean 0, variance 1, covariance ρ. Show that $\mathbb{E}\max\{X^2, Y^2\} \leqslant 1 + \sqrt{1-\rho^2}$.

(c) Let X_1, X_2, X_3 be discrete random variables. Writing $a_{ij} = \mathbb{P}(X_i > X_j)$, show that $\min\{a_{12}, a_{23}, a_{31}\} \leqslant \frac{2}{3}$.

Paper 2, Section II
11F Probability

(a) Consider a Galton–Watson process (X_n). Prove that the extinction probability q is the smallest non-negative solution of the equation $q = F(q)$ where $F(t) = \mathbb{E}(t^{X_1})$. [You should prove any properties of Galton–Watson processes that you use.]

In the case of a Galton–Watson process with

$$\mathbb{P}(X_1 = 1) = 1/4, \quad \mathbb{P}(X_1 = 3) = 3/4,$$

find the mean population size and compute the extinction probability.

(b) For each $n \in \mathbb{N}$, let Y_n be a random variable with distribution Poisson(n). Show that

$$\frac{Y_n - n}{\sqrt{n}} \to Z$$

in distribution, where Z is a standard normal random variable.

Deduce that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

Paper 2, Section II
12F Probability

For a symmetric simple random walk (X_n) on $\mathbb{Z}$ starting at 0, let $M_n = \max_{i\leqslant n} X_i$.

(i) For $m \geqslant 0$ and $x \in \mathbb{Z}$, show that

$$\mathbb{P}(M_n \geqslant m, X_n = x) = \begin{cases} \mathbb{P}(X_n = x) & \text{if } x \geqslant m \\ \mathbb{P}(X_n = 2m - x) & \text{if } x < m. \end{cases}$$

(ii) For $m \geqslant 0$, show that $\mathbb{P}(M_n \geqslant m) = \mathbb{P}(X_n = m) + 2\sum_{x>m} \mathbb{P}(X_n = x)$ and that

$$\mathbb{P}(M_n = m) = \mathbb{P}(X_n = m) + \mathbb{P}(X_n = m + 1).$$

(iii) Prove that $\mathbb{E}(M_n^2) < \mathbb{E}(X_n^2)$.

Paper 3, Section I

3C Vector Calculus

Derive a formula for the curvature of the two-dimensional curve $\mathbf{x}(u) = (u, f(u))$.

Verify your result for the semicircle with radius a given by $f(u) = \sqrt{a^2 - u^2}$.

Paper 3, Section I

4C Vector Calculus

In plane polar coordinates (r, θ), the orthonormal basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ satisfy

$$\frac{\partial \mathbf{e}_r}{\partial r} = \frac{\partial \mathbf{e}_\theta}{\partial r} = \mathbf{0}, \quad \frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r, \quad \text{and} \quad \boldsymbol{\nabla} = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\,.$$

Hence derive the expression $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\phi = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial \phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 \phi}{\partial \theta^2}$ for the Laplacian operator ∇^2.

Calculate the Laplacian of $\phi(r, \theta) = \alpha r^\beta \cos(\gamma\theta)$, where α, β and γ are constants. Hence find all solutions to the equation

$$\nabla^2 \phi = 0 \quad \text{in} \quad 0 \leqslant r \leqslant a, \quad \text{with} \quad \partial\phi/\partial r = \cos(2\theta) \text{ on } r = a\,.$$

Explain briefly how you know that there are no other solutions.

Paper 3, Section II

9C Vector Calculus

Given a one-to-one mapping $u = u(x, y)$ and $v = v(x, y)$ between the region D in the (x, y)-plane and the region D' in the (u, v)-plane, state the formula for transforming the integral $\iint_D f(x, y)\, dx\, dy$ into an integral over D', with the Jacobian expressed explicitly in terms of the partial derivatives of u and v.

Let D be the region $x^2 + y^2 \leqslant 1$, $y \geqslant 0$ and consider the change of variables $u = x + y$ and $v = x^2 + y^2$. Sketch D, the curves of constant u and the curves of constant v in the (x, y)-plane. Find and sketch the image D' of D in the (u, v)-plane.

Calculate $I = \iint_D (x + y)\, dx\, dy$ using this change of variables. Check your answer by calculating I directly.

Paper 3, Section II
10C Vector Calculus

State the formula of Stokes's theorem, specifying any orientation where needed.

Let $\mathbf{F} = (y^2z, xz + 2xyz, 0)$. Calculate $\nabla \times \mathbf{F}$ and verify that $\nabla \cdot \nabla \times \mathbf{F} = 0$.

Sketch the surface S defined as the union of the surface $z = -1$, $1 \leqslant x^2 + y^2 \leqslant 4$ and the surface $x^2 + y^2 + z = 3$, $1 \leqslant x^2 + y^2 \leqslant 4$.

Verify Stokes's theorem for $\mathbf{F}$ on S.

Paper 3, Section II
11C Vector Calculus

Use Maxwell's equations,

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t},$$

to derive expressions for $\dfrac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E}$ and $\dfrac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B}$ in terms of ρ and $\mathbf{J}$.

Now suppose that there exists a scalar potential ϕ such that $\mathbf{E} = -\nabla \phi$, and $\phi \to 0$ as $r \to \infty$. If $\rho = \rho(r)$ is spherically symmetric, calculate $\mathbf{E}$ using Gauss's flux method, i.e. by integrating a suitable equation inside a sphere centred at the origin. Use your result to find $\mathbf{E}$ and ϕ in the case when $\rho = 1$ for $r < a$ and $\rho = 0$ otherwise.

For each integer $n \geqslant 0$, let S_n be the sphere of radius 4^{-n} centred at the point $(1 - 4^{-n}, 0, 0)$. Suppose that ρ vanishes outside S_0, and has the constant value 2^n in the volume between S_n and S_{n+1} for $n \geqslant 0$. Calculate $\mathbf{E}$ and ϕ at the point $(1, 0, 0)$.

Paper 3, Section II

12C Vector Calculus

(a) Suppose that a tensor T_{ij} can be decomposed as

$$T_{ij} = S_{ij} + \epsilon_{ijk} V_k \,, \qquad (*)$$

where S_{ij} is symmetric. Obtain expressions for S_{ij} and V_k in terms of T_{ij}, and check that $(*)$ is satisfied.

(b) State the most general form of an isotropic tensor of rank k for $k = 0, 1, 2, 3$, and verify that your answers are isotropic.

(c) The general form of an isotropic tensor of rank 4 is

$$T_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk} \,.$$

Suppose that A_{ij} and B_{ij} satisfy the linear relationship $A_{ij} = T_{ijkl} B_{kl}$, where T_{ijkl} is isotropic. Express B_{ij} in terms of A_{ij}, assuming that $\beta^2 \neq \gamma^2$ and $3\alpha + \beta + \gamma \neq 0$. If instead $\beta = -\gamma \neq 0$ and $\alpha \neq 0$, find all B_{ij} such that $A_{ij} = 0$.

(d) Suppose that C_{ij} and D_{ij} satisfy the quadratic relationship $C_{ij} = T_{ijklmn} D_{kl} D_{mn}$, where T_{ijklmn} is an isotropic tensor of rank 6. If C_{ij} is symmetric and D_{ij} is antisymmetric, find the most general non-zero form of $T_{ijklmn} D_{kl} D_{mn}$ and prove that there are only two independent terms. [*Hint: You do not need to use the general form of an isotropic tensor of rank* 6.]

Paper 1, Section I

1C Vectors and Matrices

For $z, w \in \mathbb{C}$ define the *principal value* of z^w. State de Moivre's theorem.

Hence solve the equations

$$\text{(i) } z^6 = \sqrt{3} + i, \quad \text{(ii) } z^{1/6} = \sqrt{3} + i, \quad \text{(iii) } i^z = \sqrt{3} + i, \quad \text{(iv) } \left(e^{5i\pi/2}\right)^z = \sqrt{3} + i\,.$$

[In each expression, the principal value is to be taken.]

Paper 1, Section I

2A Vectors and Matrices

The map $\mathbf{\Phi}(\mathbf{x}) = \alpha(\mathbf{n}\cdot\mathbf{x})\mathbf{n} - \mathbf{n}\times(\mathbf{n}\times\mathbf{x})$ is defined for $\mathbf{x} \in \mathbb{R}^3$, where $\mathbf{n}$ is a unit vector in $\mathbb{R}^3$ and α is a real constant.

(i) Find the values of α for which the inverse map $\mathbf{\Phi}^{-1}$ exists, as well as the inverse map itself in these cases.

(ii) When $\mathbf{\Phi}$ is not invertible, find its image and kernel. What is the value of the rank and the value of the nullity of $\mathbf{\Phi}$?

(iii) Let $\mathbf{y} = \mathbf{\Phi}(\mathbf{x})$. Find the components A_{ij} of the matrix A such that $y_i = A_{ij}x_j$. When $\mathbf{\Phi}$ is invertible, find the components of the matrix B such that $x_i = B_{ij}y_j$.

Paper 1, Section II

5C Vectors and Matrices

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be non-zero real vectors. Define the *inner product* $\mathbf{x}\cdot\mathbf{y}$ in terms of the components x_i and y_i, and define the *norm* $|\mathbf{x}|$. Prove that $\mathbf{x}\cdot\mathbf{y} \leqslant |\mathbf{x}|\,|\mathbf{y}|$. When does equality hold? Express the angle between $\mathbf{x}$ and $\mathbf{y}$ in terms of their inner product.

Use suffix notation to expand $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})$.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be given unit vectors in $\mathbb{R}^3$, and let $\mathbf{m} = (\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a})$. Obtain expressions for the angle between $\mathbf{m}$ and each of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $|\mathbf{m}|$. Calculate $|\mathbf{m}|$ for the particular case when the angles between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are all equal to θ, and check your result for an example with $\theta = 0$ and an example with $\theta = \pi/2$.

Consider three planes in $\mathbb{R}^3$ passing through the points $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively, with unit normals $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. State a condition that must be satisfied for the three planes to intersect at a single point, and find the intersection point.

Paper 1, Section II
6B Vectors and Matrices

(a) Consider the matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

representing a rotation about the z-axis through an angle θ.

Show that R has three eigenvalues in $\mathbb{C}$ each with modulus 1, of which one is real and two are complex (in general), and give the relation of the real eigenvector and the two complex eigenvalues to the properties of the rotation.

Now consider the rotation composed of a rotation by angle $\pi/2$ about the z-axis followed by a rotation by angle $\pi/2$ about the x-axis. Determine the rotation axis $\mathbf{n}$ and the magnitude of the angle of rotation ϕ.

(b) A surface in $\mathbb{R}^3$ is given by

$$7x^2 + 4xy + 3y^2 + 2xz + 3z^2 = 1.$$

By considering a suitable eigenvalue problem, show that the surface is an ellipsoid, find the lengths of its semi-axes and find the position of the two points on the surface that are closest to the origin.

Paper 1, Section II

7B Vectors and Matrices

Let A be a real symmetric $n \times n$ matrix.

(a) Prove the following:

(i) Each eigenvalue of A is real and there is a corresponding real eigenvector.

(ii) Eigenvectors corresponding to different eigenvalues are orthogonal.

(iii) If there are n distinct eigenvalues then the matrix is diagonalisable.

Assuming that A has n distinct eigenvalues, explain briefly how to choose (up to an arbitrary scalar factor) the vector v such that $\dfrac{v^T A v}{v^T v}$ is maximised.

(b) A scalar λ and a non-zero vector v such that

$$Av = \lambda B v$$

are called, for a specified $n \times n$ matrix B, respectively a *generalised eigenvalue* and a *generalised eigenvector* of A.

Assume the matrix B is real, symmetric and positive definite (i.e. $(u^*)^T B u > 0$ for all non-zero complex vectors u).

Prove the following:

(i) If λ is a generalised eigenvalue of A then it is a root of $\det(A - \lambda B) = 0$.

(ii) Each generalised eigenvalue of A is real and there is a corresponding real generalised eigenvector.

(iii) Two generalised eigenvectors u, v, corresponding to different generalised eigenvalues, are orthogonal in the sense that $u^T B v = 0$.

(c) Find, up to an arbitrary scalar factor, the vector v such that the value of $F(v) = \dfrac{v^T A v}{v^T B v}$ is maximised, and the corresponding value of $F(v)$, where

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 10 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Paper 1, Section II

8A Vectors and Matrices

What is the definition of an *orthogonal matrix* M?

Write down a 2×2 matrix R representing the rotation of a 2-dimensional vector (x, y) by an angle θ around the origin. Show that R is indeed orthogonal.

Take a matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

where a, b, c are real. Suppose that the 2×2 matrix $B = RAR^T$ is diagonal. Determine all possible values of θ.

Show that the diagonal entries of B are the eigenvalues of A and express them in terms of the determinant and trace of A.

Using the above results, or otherwise, find the elements of the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{2N}$$

as a function of N, where N is a natural number.