UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part II

Friday, 9 June, 2017   9:00 am to 12:00 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most six questions*** *from Section I and from any number of questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, ..., ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1G Number Theory

Show that, for $x \geqslant 2$ a real number,

$$\prod_{\substack{p \leqslant x, \\ p \text{ is prime}}} \left(1 - \frac{1}{p}\right)^{-1} > \log x \, .$$

Hence prove that

$$\sum_{\substack{p \leqslant x, \\ p \text{ is prime}}} \frac{1}{p} > \log\log x + c \, ,$$

where c is a constant you should make explicit.

2F Topics In Analysis

If $x \in (0,1]$, set

$$x = \frac{1}{N(x) + T(x)} \, ,$$

where $N(x)$ is an integer and $1 > T(x) \geqslant 0$. Let $N(0) = T(0) = 0$.

If x is also irrational, write down the continued fraction expansion in terms of $NT^j(x)$ (where $NT^0(x) = N(x)$).

Let X be a random variable taking values in $[0,1]$ with probability density function

$$f(x) = \frac{1}{(\log 2)(1+x)} .$$

Show that $T(X)$ has the same distribution as X.

3G Coding & Cryptography

Describe the RSA system with public key (N, e) and private key d.

Give a simple example of how the system is vulnerable to a homomorphism attack.

Describe the El-Gamal signature scheme and explain how this can defeat a homomorphism attack.

4H Automata and Formal Languages

(a) Describe the process for converting a deterministic finite-state automaton D into a regular expression R defining the same language, $\mathcal{L}(D) = \mathcal{L}(R)$. [You need only outline the steps, without proof, but you should clearly define all terminology you introduce.]

(b) Consider the language L over the alphabet $\{0,1\}$ defined via

$$L := \{w01^n \mid w \in \{0,1\}^*, n \in \mathbb{K}\} \cup \{1\}^*.$$

Show that L satisfies the pumping lemma for regular languages but is not a regular language itself.

5J Statistical Modelling

A Cambridge scientist is testing approaches to slow the spread of a species of moth in certain trees. Two groups of 30 trees were treated with different organic pesticides, and a third group of 30 trees was kept under control conditions. At the end of the summer the trees are classified according to the level of leaf damage, obtaining the following contingency table.

```
> xtabs(count~group+damage.level,data=treeConditions)
             damage.level
group         Severe.Damage Moderate.Damage Some.Damage
  Control                22               5           3
  Treatment 1            18               4           8
  Treatment 2             14              3          13
```

Which of the following Generalised Linear Model fitting commands is appropriate for these data? Why? Describe the model being fit.

(a) `> fit <- glm(count~group+damage.level,data=treeConditions,family=poisson)`

(b) `> fit <- glm(count~group+damage.level,data=treeConditions,family=multinomial)`

(c) `> fit <- glm(damage.level~group,data=treeConditions,family=binomial)`

(d) `> fit <- glm(damage.level~group,data=treeConditions,family=binomial,`
`weights=count)`

6B Mathematical Biology

Consider an epidemic model with host demographics (natural births and deaths). The system is given by

$$\begin{aligned}\frac{dS}{dt} &= -\beta IS - \mu S + \mu N\,,\\ \frac{dI}{dt} &= +\beta IS - \nu I - \mu I\,,\end{aligned}$$

where $S(t)$ are the susceptibles, $I(t)$ are the infecteds, N is the total population size and the parameters β, μ and ν are positive. The basic reproduction ratio is defined as $R_0 = \beta N/(\mu + \nu)$.

Show that the system has an endemic equilibrium (where the disease is present) for $R_0 > 1$. Show that the endemic equilibrium is stable.

Interpret the meaning of the case $\nu \gg \mu$ and show that in this case the approximate period of (decaying) oscillation around the endemic equilibrium is given by

$$T = \frac{2\pi}{\sqrt{\mu\nu(R_0 - 1)}}\,.$$

Suppose now a vaccine is introduced which is given to some proportion of the population at birth, but not enough to eradicate the disease. What will be the effect on the period of (decaying) oscillations?

7E Further Complex Methods

Consider the differential equation

$$z\frac{d^2y}{dz^2} - 2\frac{dy}{dz} + zy = 0\,. \qquad (\star)$$

Laplace's method finds a solution of this differential equation by writing $y(z)$ in the form

$$y(z) = \int_C e^{zt} f(t)dt\,,$$

where C is a closed contour.

Determine $f(t)$. Hence find two linearly independent real solutions of $(\star)$ for z real.

8E Classical Dynamics

Consider the Poisson bracket structure on $\mathbb{R}^3$ given by

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y$$

and show that $\{f, \rho^2\} = 0$, where $\rho^2 = x^2 + y^2 + z^2$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is any polynomial function on $\mathbb{R}^3$.

Let $H = (Ax^2 + By^2 + Cz^2)/2$, where A, B, C are positive constants. Find the explicit form of Hamilton's equations

$$\dot{\mathbf{r}} = \{\mathbf{r}, H\}, \quad \text{where} \quad \mathbf{r} = (x, y, z).$$

Find a condition on A, B, C such that the oscillation described by

$$x = 1 + \alpha(t), \quad y = \beta(t), \quad z = \gamma(t)$$

is linearly unstable, where $\alpha(t), \beta(t), \gamma(t)$ are small.

9C Cosmology

(a) By considering a spherically symmetric star in hydrostatic equilibrium derive the pressure support equation

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2},$$

where r is the radial distance from the centre of the star, $M(r)$ is the stellar mass contained inside that radius, and $P(r)$ and $\rho(r)$ are the pressure and density at radius r respectively.

(b) Propose, and briefly justify, boundary conditions for this differential equation, both at the centre of the star $r = 0$, and at the stellar surface $r = R$.

Suppose that $P = K\rho^2$ for some $K > 0$. Show that the density satisfies the linear differential equation

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2 \frac{\partial \rho}{\partial x}\right) = -\rho$$

where $x = \alpha r$, for some constant α, is a rescaled radial coordinate. Find α.

SECTION II

10G Number Theory

(a) State Dirichlet's theorem on primes in arithmetic progression.

(b) Let d be the discriminant of a binary quadratic form, and let p be an odd prime. Show that p is represented by some binary quadratic form of discriminant d if and only if $x^2 \equiv d \pmod{p}$ is soluble.

(c) Let $f(x,y) = x^2 + 15y^2$ and $g(x,y) = 3x^2 + 5y^2$. Show that f and g each represent infinitely many primes. Are there any primes represented by both f and g?

11F Topics In Analysis

(a) Suppose that $\gamma : [0,1] \to \mathbb{C}$ is continuous with $\gamma(0) = \gamma(1)$ and $\gamma(t) \neq 0$ for all $t \in [0,1]$. Show that if $\gamma(0) = |\gamma(0)| \exp(i\theta_0)$ (with θ_0 real) we can define a continuous function $\theta : [0,1] \to \mathbb{R}$ such that $\theta(0) = \theta_0$ and $\gamma(t) = |\gamma(t)| \exp\big(i\theta(t)\big)$. Hence define the *winding number* $w(\gamma) = w(0,\gamma)$ of γ around 0.

(b) Show that $w(\gamma)$ can take any integer value.

(c) If γ_1 and γ_2 satisfy the requirements of the definition, and $(\gamma_1 \times \gamma_2)(t) = \gamma_1(t)\gamma_2(t)$, show that

$$w(\gamma_1 \times \gamma_2) = w(\gamma_1) + w(\gamma_2).$$

(d) If γ_1 and γ_2 satisfy the requirements of the definition and $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$ for all $t \in [0,1]$, show that

$$w(\gamma_1) = w(\gamma_2).$$

(e) State and prove a theorem that says that winding number is unchanged under an appropriate homotopy.

12J Statistical Modelling

The dataset `diesel` records the number of diesel cars which go through a block of Hills Road in 6 disjoint periods of 30 minutes, between 8AM and 11AM. The measurements are repeated each day for 10 days. Answer the following questions based on the code below, which is shown with partial output.

(a) Can we reject the model `fit.1` at a 1% level? Justify your answer.

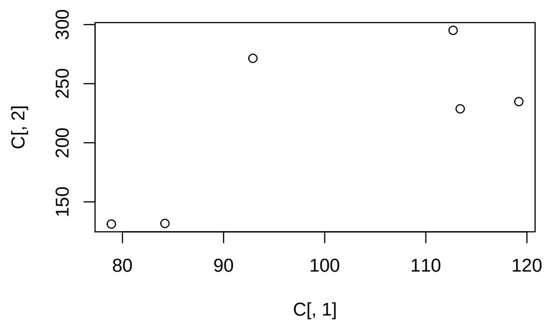
(b) What is the difference between the deviance of the models `fit.2` and `fit.3`?

(c) Which of `fit.2` and `fit.3` would you use to perform variable selection by backward stepwise selection? Why?

(d) How does the final plot differ from what you expect under the model in `fit.2`? Provide a possible explanation and suggest a better model.

```
> head(diesel)
  period num.cars day
1      1       69   1
2      2       97   1
3      3      103   1
4      4       99   1
5      5       67   1
6      6       91   1
> fit.1 = glm(num.cars~period,data=diesel,family=poisson)
> summary(fit.1)
Deviance Residuals:
    Min       1Q   Median       3Q      Max
-4.0188  -1.4837  -0.2117   1.6257   4.5965

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  4.628535   0.029288 158.035   <2e-16 ***
period      -0.006073   0.007551  -0.804    0.421
---
Signif. codes:  0 ?***? 0.001 ?**? 0.01 ?*? 0.05 ?.? 0.1 ? ? 1

(Dispersion parameter for poisson family taken to be 1)

    Null deviance: 262.36  on 59  degrees of freedom
Residual deviance: 261.72  on 58  degrees of freedom
AIC: 651.2

> diesel$period.factor = factor(diesel$period)
> fit.2 = glm(num.cars~period.factor,data=diesel,family=poisson)
> summary(fit.2)
Coefficients:
               Estimate Std. Error z value Pr(>|z|)
```

```
(Intercept)     4.36818    0.03560 122.698  < 2e-16 ***
period.factor2  0.35655    0.04642   7.681 1.58e-14 ***
period.factor3  0.41262    0.04590   8.991  < 2e-16 ***
period.factor4  0.36274    0.04636   7.824 5.10e-15 ***
period.factor5  0.06501    0.04955   1.312 0.189481
period.factor6  0.16334    0.04841   3.374 0.000741 ***
---
Signif. codes:  0 ?***? 0.001 ?**? 0.01 ?*? 0.05 ?.? 0.1 ? ? 1

> fit.3 = glm(num.cars~(period>1)+(period>2)+(period>3)+(period>4)+(period>5),
  data=diesel,family=poisson)
> summary(fit.3)

Coefficients:
               Estimate Std. Error z value Pr(>|z|)
(Intercept)     4.36818    0.03560 122.698  < 2e-16 ***
period > 1TRUE  0.35655    0.04642   7.681 1.58e-14 ***
period > 2TRUE  0.05607    0.04155   1.350   0.1771
period > 3TRUE -0.04988    0.04148  -1.202   0.2292
period > 4TRUE -0.29773    0.04549  -6.545 5.96e-11 ***
period > 5TRUE  0.09833    0.04758   2.066   0.0388 *
---
Signif. codes:  0 ?***? 0.001 ?**? 0.01 ?*? 0.05 ?.? 0.1 ? ? 1

> C = matrix(nrow=6,ncol=2)
> for (period in 1:6) {
  nums = diesel$num.cars[diesel$period == period]
  C[period,] = c(mean(nums),var(nums))
  }
plot(C[,1],C[,2])
```



13B Mathematical Biology

An activator-inhibitor system is described by the equations

$$\begin{aligned}\frac{\partial u}{\partial t} &= u(c+u-v)+\frac{\partial^2 u}{\partial x^2}\,,\\ \frac{\partial v}{\partial t} &= v(au-bv)+d\,\frac{\partial^2 v}{\partial x^2}\,,\end{aligned}$$

where $a, b, c, d > 0$.

Find and sketch the range of a, b for which the spatially homogeneous system has a stable stationary solution with $u > 0$ and $v > 0$.

Considering spatial perturbations of the form $\cos(kx)$ about the solution found above, find conditions for the system to be unstable. Sketch this region in the (a, b)-plane for fixed d (for a value of d such that the region is non-empty).

Show that k_c, the critical wavenumber at the onset of the instability, is given by

$$k_c = \sqrt{\frac{2ac}{d-a}}\,.$$

14E Classical Dynamics

Explain how geodesics of a Riemannian metric

$$g = g_{ab}(x^c)dx^a dx^b$$

arise from the kinetic Lagrangian

$$\mathcal{L} = \frac{1}{2}g_{ab}(x^c)\dot{x}^a\dot{x}^b,$$

where $a, b = 1, \ldots, n$.

Find geodesics of the metric on the upper half plane

$$\Sigma = \{(x, y) \in \mathbb{R}^2, y > 0\}$$

with the metric

$$g = \frac{dx^2 + dy^2}{y^2}$$

and sketch the geodesic containing the points $(2, 3)$ and $(10, 3)$.

[*Hint: Consider* dy/dx.]

15H Logic and Set Theory

Prove that every set has a transitive closure. [If you apply the Axiom of Replacement to a function-class F, you must explain clearly why F is indeed a function-class.]

State the Axiom of Foundation and the Principle of ϵ-Induction, and show that they are equivalent (in the presence of the other axioms of ZFC).

State the ϵ-Recursion Theorem.

Sets C_α are defined for each ordinal α by recursion, as follows: $C_0 = \emptyset$, $C_{\alpha+1}$ is the set of all countable subsets of C_α, and $C_\lambda = \cup_{\alpha<\lambda} C_\alpha$ for λ a non-zero limit. Does there exist an α with $C_{\alpha+1} = C_\alpha$? Justify your answer.

16H Graph Theory

Let G be a graph of maximum degree Δ. Show the following:

(i) Every eigenvalue λ of G satisfies $|\lambda| \leqslant \Delta$.

(ii) If G is regular then Δ is an eigenvalue.

(iii) If G is regular and connected then the multiplicity of Δ as an eigenvalue is 1.

(iv) If G is regular and not connected then the multiplicity of Δ as an eigenvalue is greater than 1.

Let A be the adjacency matrix of the Petersen graph. Explain why $A^2 + A - 2I = J$, where I is the identity matrix and J is the all-1 matrix. Find, with multiplicities, the eigenvalues of the Petersen graph.

17I Galois Theory

(a) State the Fundamental Theorem of Galois Theory.

(b) What does it mean for an extension L of $\mathbb{Q}$ to be *cyclotomic*? Show that a cyclotomic extension L of $\mathbb{Q}$ is a Galois extension and prove that its Galois group is Abelian.

(c) What is the Galois group G of $\mathbb{Q}(\eta)$ over $\mathbb{Q}$, where η is a primitive 7th root of unity? Identify the intermediate subfields M, with $\mathbb{Q} \leqslant M \leqslant \mathbb{Q}(\eta)$, in terms of η, and identify subgroups of G to which they correspond. Justify your answers.

18G Representation Theory

Let G =SU(2) and let V_n be the vector space of complex homogeneous polynomials of degree n in two variables.

(a) Prove that V_n has the structure of an irreducible representation for G.

(b) State and prove the Clebsch–Gordan theorem.

(c) Quoting without proof any properties of symmetric and exterior powers which you need, decompose $\mathrm{S}^2 V_n$ and $\Lambda^2 V_n$ ($n \geqslant 1$) into irreducible G-spaces.

19H Number Fields

(a) Write down $\mathcal{O}_K$, when $K = \mathbb{Q}(\sqrt{\delta})$, and $\delta \equiv 2$ or $3 \pmod 4$. [You need not prove your answer.]

Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{\delta})$, where $\delta \equiv 3 \pmod 4$ is a square-free integer. Find an integral basis of $\mathcal{O}_L$. [*Hint: Begin by considering the relative traces* $tr_{L/K}$, *for* K *a quadratic subfield of* L.]

(b) Compute the ideal class group of $\mathbb{Q}(\sqrt{-14})$.

20I Algebraic Topology

Recall that $\mathbb{R}P^n$ is real projective n-space, the quotient of S^n obtained by identifying antipodal points. Consider the standard embedding of S^n as the unit sphere in $\mathbb{R}^{n+1}$.

(a) For n odd, show that there exists a continuous map $f : S^n \to S^n$ such that $f(x)$ is orthogonal to x, for all $x \in S^n$.

(b) Exhibit a triangulation of $\mathbb{R}P^n$.

(c) Describe the map $H_n(S^n) \to H_n(S^n)$ induced by the antipodal map, justifying your answer.

(d) Show that, for n even, there is no continuous map $f : S^n \to S^n$ such that $f(x)$ is orthogonal to x for all $x \in S^n$.

21F Linear Analysis

Let H be a complex Hilbert space with inner product $(\cdot,\cdot)$ and let $T : H \to H$ be a bounded linear map.

(i) Define the *spectrum* $\sigma(T)$, the *point spectrum* $\sigma_p(T)$, the *continuous spectrum* $\sigma_c(T)$, and the *residual spectrum* $\sigma_r(T)$.

(ii) Show that T^*T is self-adjoint and that $\sigma(T^*T) \subset [0,\infty)$. Show that if T is compact then so is T^*T.

(iii) Assume that T is compact. Prove that T has a singular value decomposition: for $N < \infty$ or $N = \infty$, there exist orthonormal systems $(u_i)_{i=1}^N \subset H$ and $(v_i)_{i=1}^N \subset H$ and $(\lambda_i)_{i=1}^N \subset [0,\infty)$ such that, for any $x \in H$,

$$Tx = \sum_{i=1}^{N} \lambda_i (u_i, x) v_i.$$

[You may use the spectral theorem for compact self-adjoint linear operators.]

22F Analysis of Functions

Consider $\mathbb{R}^n$ with the Lebesgue measure. Denote by $\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} f(x)\,dx$ the Fourier transform of $f \in L^1(\mathbb{R}^n)$ and by $\hat{f}$ the Fourier–Plancherel transform of $f \in L^2(\mathbb{R}^n)$. Let $\chi_R(\xi) := \left(1 - \frac{|\xi|}{R}\right)\chi_{|\xi|\leqslant R}$ for $R > 0$ and define for $s \in \mathbb{R}_+$

$$H^s(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) \;\middle|\; (1+|\cdot|^2)^{s/2}\hat{f}(\cdot) \in L^2(\mathbb{R}^n) \right\}.$$

(i) Prove that $H^s(\mathbb{R}^n)$ is a vector subspace of $L^2(\mathbb{R}^n)$, and is a Hilbert space for the inner product $\langle f, g\rangle := \int_{\mathbb{R}^n} (1+|\xi|^2)^s \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

(ii) Construct a function $f \in H^s(\mathbb{R})$, $s \in (0, 1/2)$, that is not almost everywhere equal to a continuous function.

(iii) For $f \in L^1(\mathbb{R}^n)$, prove that $F_R : x \mapsto \int_{\mathbb{R}^n} \mathcal{F}f(\xi)\chi_R(\xi)e^{2i\pi x\cdot\xi}\,d\xi$ is a well-defined function and that $F_R \in L^1(\mathbb{R}^n)$ converges to f in $L^1(\mathbb{R}^n)$ as $R \to +\infty$.

[*Hint: Prove that* $F_R = K_R * f$ *where* K_R *is an approximation of the unit as* $R \to +\infty$.]

(iv) Deduce that if $f \in L^1(\mathbb{R}^n)$ and $(1+|\cdot|^2)^{s/2}\mathcal{F}f(\cdot) \in L^2(\mathbb{R}^n)$ then $f \in H^s(\mathbb{R}^n)$.

[*Hint: Prove that: (1) there is a sequence* $R_k \to +\infty$ *such that* $K_{R_k} * f$ *converges to* f *almost everywhere; (2)* $K_R * f$ *is uniformly bounded in* $L^2(\mathbb{R}^n)$ *as* $R \to +\infty$.]

23I Algebraic Geometry

(a) Let X and Y be non-singular projective curves over a field k and let $\varphi : X \to Y$ be a non-constant morphism. Define the *ramification degree* e_P of φ at a point $P \in X$.

(b) Suppose $\operatorname{char} k \neq 2$. Let $X = Z(f)$ be the plane cubic with $f = x_0x_2^2 - x_1^3 + x_0^2x_1$, and let $Y = \mathbb{P}^1$. Explain how the projection

$$(x_0 : x_1 : x_2) \mapsto (x_0 : x_1)$$

defines a morphism $\varphi : X \to Y$. Determine the degree of φ and the ramification degrees e_P for all $P \in X$.

(c) Let X be a non-singular projective curve and let $P \in X$. Show that there is a non-constant rational function on X which is regular on $X \setminus \{P\}$.

24I Differential Geometry

Let $S \subset \mathbb{R}^3$ be a surface and $p \in S$. Define the *exponential map* $\exp_p$ and compute its differential $d\exp_p|_0$. Deduce that $\exp_p$ is a local diffeomorphism.

Give an example of a surface S and a point $p \in S$ for which the exponential map $\exp_p$ fails to be defined globally on T_pS. Can this failure be remedied by extending the surface? In other words, for any such S, is there always a surface $S \subset \widehat{S} \subset \mathbb{R}^3$ such that the exponential map $\widehat{\exp}_p$ defined with respect to $\widehat{S}$ is globally defined on $T_pS = T_p\widehat{S}$?

State the version of the Gauss–Bonnet theorem with boundary term for a surface $S \subset \mathbb{R}^3$ and a closed disc $D \subset S$ whose boundary ∂D can be parametrized as a smooth closed curve in S.

Let $S \subset \mathbb{R}^3$ be a flat surface, i.e. $K = 0$. Can there exist a closed disc $D \subset S$, whose boundary ∂D can be parametrized as a smooth closed curve, and a surface $\tilde{S} \subset \mathbb{R}^3$ such that all of the following hold:

(i) $(S \setminus D) \cup \partial D \subset \tilde{S}$;

(ii) letting $\tilde{D}$ be $(\tilde{S} \setminus (S \setminus D)) \cup \partial D$, we have that $\tilde{D}$ is a closed disc in $\tilde{S}$ with boundary $\partial\tilde{D} = \partial D$;

(iii) the Gaussian curvature $\tilde{K}$ of $\tilde{S}$ satisfies $\tilde{K} \geqslant 0$, and there exists a $p \in \tilde{S}$ such that $\tilde{K}(p) > 0$?

Justify your answer.

25J Probability and Measure

(a) Suppose that $(E, \mathcal{E}, \mu)$ is a finite measure space and $\theta \colon E \to E$ is a measurable map. Prove that $\mu_\theta(A) = \mu(\theta^{-1}(A))$ defines a measure on $(E, \mathcal{E})$.

(b) Suppose that $\mathcal{A}$ is a π-system which generates $\mathcal{E}$. Using Dynkin's lemma, prove that θ is measure-preserving if and only if $\mu_\theta(A) = \mu(A)$ for all $A \in \mathcal{A}$.

(c) State Birkhoff's ergodic theorem and the maximal ergodic lemma.

(d) Consider the case $(E, \mathcal{E}, \mu) = ([0,1), \mathcal{B}([0,1)), \mu)$ where μ is Lebesgue measure on $[0,1)$. Let $\theta \colon [0,1) \to [0,1)$ be the following map. If $x = \sum_{n=1}^{\infty} 2^{-n}\omega_n$ is the binary expansion of x (where we disallow infinite sequences of 1s), then $\theta(x) = \sum_{n=1}^{\infty} 2^{-n}(\omega_{n-1}\mathbf{1}_{n\in E} + \omega_{n+1}\mathbf{1}_{n\in O})$ where E and O are respectively the even and odd elements of $\mathbb{N}$.

   (i) Prove that θ is measure-preserving. [You may assume that θ is measurable.]

   (ii) Prove or disprove: θ is ergodic.

26K Applied Probability

(a) Give the definition of an $M/M/1$ *queue*. Prove that if λ is the arrival rate and μ the service rate and $\lambda < \mu$, then the length of the queue is a positive recurrent Markov chain. What is the equilibrium distribution?

If the queue is in equilibrium and a customer arrives at some time t, what is the distribution of the waiting time (time spent waiting in the queue plus service time)?

(b) We now modify the above queue: on completion of service a customer leaves with probability δ, or goes to the back of the queue with probability $1-\delta$. Find the distribution of the total time a customer spends being served.

Hence show that equilibrium is possible if $\lambda < \delta\mu$ and find the stationary distribution.

Show that, in equilibrium, the departure process is Poisson.

[You may use relevant theorems provided you state them clearly.]

27K Principles of Statistics

For the statistical model $\{\mathcal{N}_d(\theta,\Sigma), \theta \in \mathbb{R}^d\}$, where Σ is a known, positive-definite $d \times d$ matrix, we want to estimate θ based on n i.i.d. observations $X_1,\ldots,X_n$ with distribution $\mathcal{N}_d(\theta,\Sigma)$.

(a) Derive the maximum likelihood estimator $\hat{\theta}_n$ of θ. What is the distribution of $\hat{\theta}_n$?

(b) For $\alpha \in (0,1)$, construct a confidence region C_n^α such that $\mathbf{P}_\theta(\theta \in C_n^\alpha) = 1-\alpha$.

(c) For $\Sigma = I_d$, compute the maximum likelihood estimator of θ for the following parameter spaces:

(i) $\Theta = \{\theta : \|\theta\|_2 = 1\}$.

(ii) $\Theta = \{\theta : v^\top\theta = 0\}$ for some unit vector $v \in \mathbb{R}^d$.

(d) For $\Sigma = I_d$, we want to test the null hypothesis $\Theta_0 = \{0\}$ (i.e. $\theta = 0$) against the composite alternative $\Theta_1 = \mathbb{R}^d \setminus \{0\}$. Compute the likelihood ratio statistic $\Lambda(\Theta_1,\Theta_0)$ and give its distribution under the null hypothesis. Compare this result with the statement of Wilks' theorem.

28J Stochastic Financial Models

(a) Describe the (Cox–Ross–Rubinstein) *binomial model.* When is the model arbitrage-free? How is the equivalent martingale measure characterised in this case?

(b) What is the price and the hedging strategy for any given contingent claim C in the binomial model?

(c) For any fixed $0 < t < T$ and $K > 0$, the payoff function of a forward-start-option is given by

$$\left(\frac{S_T^1}{S_t^1} - K\right)^+ .$$

Find a formula for the price of the forward-start-option in the binomial model.

29K Optimization and Control

A file of X gigabytes (GB) is to be transmitted over a communications link. At each time t the sender can choose a transmission rate $u(t)$ within the range $[0,1]$ GB per second. The charge for transmitting at rate $u(t)$ at time t is $u(t)p(t)$. The function p is fully known at time $t = 0$. If it takes a total time T to transmit the file then there is a delay cost of γT^2, $\gamma > 0$. Thus u and T are to be chosen to minimize

$$\int_0^T u(t)p(t)dt + \gamma T^2,$$

where $u(t) \in [0,1]$, $dx(t)/dt = -u(t)$, $x(0) = X$ and $x(T) = 0$. Using Pontryagin's maximum principle, or otherwise, show that a property of the optimal policy is that there exists p^* such that $u(t) = 1$ if $p(t) < p^*$ and $u(t) = 0$ if $p(t) > p^*$.

Show that the optimal p^* and T are related by $p^* = p(T) + 2\gamma T$.

Suppose $p(t) = t + 1/t$ and $X = 1$. Show that it is optimal to transmit at a constant rate $u(t) = 1$ between times $T - 1 \leqslant t \leqslant T$, where T is the unique positive solution to the equation

$$\frac{1}{(T-1)T} = 2\gamma T + 1 .$$

30E Asymptotic Methods

Consider solutions to the equation

$$\frac{d^2y}{dx^2} = \left(\frac{1}{4} + \frac{\mu^2 - \frac{1}{4}}{x^2}\right) y \qquad (\star)$$

of the form

$$y(x) \;=\; \exp\Big[S_0(x) + S_1(x) + S_2(x) + \ldots\Big]\,,$$

with the assumption that, for large positive x, the function $S_j(x)$ is small compared to $S_{j-1}(x)$ for all $j = 1, 2 \ldots$

Obtain equations for the $S_j(x),\ j = 0, 1, 2 \ldots$, which are formally equivalent to $(\star)$. Solve explicitly for S_0 and S_1. Show that it is consistent to assume that $S_j(x) = c_j x^{-(j-1)}$ for some constants c_j. Give a recursion relation for the c_j .

Deduce that there exist two linearly independent solutions to $(\star)$ with asymptotic expansions as $x \to +\infty$ of the form

$$y_{\pm}(x) \sim e^{\pm x/2}\Big(1 + \sum_{j=1}^{\infty} A_j^{\pm} x^{-j}\Big)\,.$$

Determine a recursion relation for the $A_j^{\pm}$. Compute $A_1^{\pm}$ and $A_2^{\pm}$.

31A Dynamical Systems

Consider the one-dimensional map $F : \mathbb{R} \to \mathbb{R}$ defined by

$$x_{i+1} = F(x_i; \mu) = x_i(ax_i^2 + bx_i + \mu),$$

where a and b are constants, μ is a parameter and $a \neq 0$.

(a) Find the fixed points of F and determine the linear stability of $x = 0$. Hence show that there are bifurcations at $\mu = 1$, at $\mu = -1$ and, if $b \neq 0$, at $\mu = 1 + b^2/(4a)$.

Sketch the bifurcation diagram for each of the cases:

$$\text{(i) } a > b = 0, \quad \text{(ii) } a, b > 0 \quad \text{and} \quad \text{(iii) } a, b < 0.$$

In each case show the locus and stability of the fixed points in the (μ, x)-plane, and state the type of each bifurcation. [Assume that there are no further bifurcations in the region sketched.]

(b) For the case $F(x) = x(\mu - x^2)$ (i.e. $a = -1$, $b = 0$), you may assume that

$$F^2(x) = x + x(\mu - 1 - x^2)(\mu + 1 - x^2)(1 - \mu x^2 + x^4).$$

Show that there are at most three 2-cycles and determine when they exist. By considering $F'(x_i)F'(x_{i+1})$, or otherwise, show further that one 2-cycle is always unstable when it exists and that the others are unstable when $\mu > \sqrt{5}$. Sketch the bifurcation diagram showing the locus and stability of the fixed points and 2-cycles. State briefly what you would expect to occur for $\mu > \sqrt{5}$.

32C Principles of Quantum Mechanics

The Hamiltonian for a quantum system in the Schrödinger picture is

$$H_0 + \lambda V(t)\,,$$

where H_0 is independent of time and the parameter λ is small. Define the interaction picture corresponding to this Hamiltonian and derive a time evolution equation for interaction picture states.

Let $|n\rangle$ and $|m\rangle$ be eigenstates of H_0 with distinct eigenvalues E_n and E_m respectively. Show that if the system was in the state $|n\rangle$ in the remote past, then the probability of measuring it to be in a different state $|m\rangle$ at a time t is

$$\frac{\lambda^2}{\hbar^2}\left|\int_{-\infty}^{t} dt'\langle m|V(t')|n\rangle e^{i(E_m-E_n)t'/\hbar}\right|^2 \;+\; O(\lambda^3)\;.$$

Let the system be a simple harmonic oscillator with $H_0 = \hbar\omega(a^\dagger a + \frac{1}{2})$, where $[a, a^\dagger] = 1$. Let $|0\rangle$ be the ground state which obeys $a|0\rangle = 0$. Suppose

$$V(t) = e^{-p|t|}(a + a^\dagger),$$

with $p > 0$. In the remote past the system was in the ground state. Find the probability, to lowest non-trivial order in λ, for the system to be in the first excited state in the far future.

33C Applications of Quantum Mechanics

(a) In one dimension, a particle of mass m is scattered by a potential $V(x)$ where $V(x) \to 0$ as $|x| \to \infty$. For wavenumber $k > 0$, the incoming ($\mathcal{I}$) and outgoing ($\mathcal{O}$) asymptotic plane wave states with positive (+) and negative (−) parity are given by

$$\mathcal{I}_+(x) = e^{-ik|x|}\,, \qquad \mathcal{I}_-(x) = \text{sign}(x)\, e^{-ik|x|}\,,$$
$$\mathcal{O}_+(x) = e^{+ik|x|}\,, \qquad \mathcal{O}_-(x) = -\text{sign}(x)\, e^{+ik|x|}\,.$$

(i) Explain how this basis may be used to define the *S-matrix*,

$$\mathcal{S}^P = \begin{pmatrix} S_{++} & S_{+-} \\ S_{-+} & S_{--} \end{pmatrix}.$$

(ii) For what choice of potential would you expect $S_{+-} = S_{-+} = 0$? Why?

(b) The potential $V(x)$ is given by

$$V(x) = V_0 \Big[\delta(x-a) + \delta(x+a) \Big]$$

with V_0 a constant.

(i) Show that

$$S_{--}(k) = e^{-2ika} \left[\frac{(2k - iU_0)e^{ika} + iU_0 e^{-ika}}{(2k + iU_0)e^{-ika} - iU_0 e^{ika}} \right],$$

where $U_0 = 2mV_0/\hbar^2$. Verify that $|S_{--}|^2 = 1$. Explain the physical meaning of this result.

(ii) For $V_0 < 0$, by considering the poles or zeros of $S_{--}(k)$, show that there exists one bound state of negative parity if $aU_0 < -1$.

(iii) For $V_0 > 0$ and $aU_0 \gg 1$, show that $S_{--}(k)$ has a pole at

$$ka = \pi + \alpha - i\gamma\,,$$

where α and γ are real and

$$\alpha = -\frac{\pi}{aU_0} + O\left(\frac{1}{(aU_0)^2}\right) \quad \text{and} \quad \gamma = \left(\frac{\pi}{aU_0}\right)^2 + O\left(\frac{1}{(aU_0)^3}\right).$$

Explain the physical significance of this result.

34D Statistical Physics

The van der Waals equation of state is

$$p = \frac{kT}{v-b} - \frac{a}{v^2},$$

where p is the pressure, $v = V/N$ is the volume divided by the number of particles, T is the temperature, k is Boltzmann's constant and a, b are positive constants.

(i) Prove that the Gibbs free energy $G = E + pV - TS$ satisfies $G = \mu N$. Hence obtain an expression for $(\partial\mu/\partial p)_{T,N}$ and use it to explain the Maxwell construction for determining the pressure at which the gas and liquid phases can coexist at a given temperature.

(ii) Explain what is meant by the critical point and determine the values p_c, v_c, T_c corresponding to this point.

(iii) By defining $\bar{p} = p/p_c$, $\bar{v} = v/v_c$ and $\bar{T} = T/T_c$, derive the law of corresponding states:

$$\bar{p} = \frac{8\bar{T}}{3\bar{v}-1} - \frac{3}{\bar{v}^2}.$$

(iv) To investigate the behaviour near the critical point, let $\bar{T} = 1 + t$ and $\bar{v} = 1 + \phi$, where t and ϕ are small. Expand $\bar{p}$ to cubic order in ϕ and hence show that

$$\left(\frac{\partial \bar{p}}{\partial \phi}\right)_t = -\frac{9}{2}\phi^2 + \mathcal{O}(\phi^3) + t\left[-6 + \mathcal{O}(\phi)\right].$$

At fixed small t, let $\phi_l(t)$ and $\phi_g(t)$ be the values of ϕ corresponding to the liquid and gas phases on the co-existence curve. By changing the integration variable from p to ϕ, use the Maxwell construction to show that $\phi_l(t) = -\phi_g(t)$. Deduce that, as the critical point is approached along the co-existence curve,

$$\bar{v}_{\text{gas}} - \bar{v}_{\text{liquid}} \sim (T_c - T)^{1/2}.$$

35D Electrodynamics

A dielectric material has a real, frequency-independent relative permittivity ϵ_r with $|\epsilon_r - 1| \ll 1$. In this case, the macroscopic polarization that develops when the dielectric is placed in an external electric field $\mathbf{E}_{\text{ext}}(t,\mathbf{x})$ is $\mathbf{P}(t,\mathbf{x}) \approx \epsilon_0(\epsilon_r - 1)\mathbf{E}_{\text{ext}}(t,\mathbf{x})$. Explain briefly why the associated bound current density is

$$\mathbf{J}_{\text{bound}}(t,\mathbf{x}) \approx \epsilon_0(\epsilon_r - 1)\frac{\partial \mathbf{E}_{\text{ext}}(t,\mathbf{x})}{\partial t}\,.$$

[You should ignore any magnetic response of the dielectric.]

A sphere of such a dielectric, with radius a, is centred on $\mathbf{x} = 0$. The sphere scatters an incident plane electromagnetic wave with electric field

$$\mathbf{E}(t,\mathbf{x}) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}\,,$$

where $\omega = c|\mathbf{k}|$ and $\mathbf{E}_0$ is a constant vector. Working in the Lorenz gauge, show that at large distances $r = |\mathbf{x}|$, for which both $r \gg a$ and $ka^2/r \ll 2\pi$, the magnetic vector potential $\mathbf{A}_{\text{scatt}}(t,\mathbf{x})$ of the scattered radiation is

$$\mathbf{A}_{\text{scatt}}(t,\mathbf{x}) \approx -i\omega\mathbf{E}_0 \frac{e^{i(kr-\omega t)}}{r}\frac{(\epsilon_r - 1)}{4\pi c^2}\int_{|\mathbf{x}'|\leqslant a} e^{i\mathbf{q}\cdot\mathbf{x}'}\, d^3\mathbf{x}'\,,$$

where $\mathbf{q} = \mathbf{k} - k\hat{\mathbf{x}}$ with $\hat{\mathbf{x}} = \mathbf{x}/r$.

In the far-field, where $kr \gg 1$, the electric and magnetic fields of the scattered radiation are given by

$$\mathbf{E}_{\text{scatt}}(t,\mathbf{x}) \approx -i\omega\hat{\mathbf{x}} \times [\hat{\mathbf{x}} \times \mathbf{A}_{\text{scatt}}(t,\mathbf{x})]\,,$$
$$\mathbf{B}_{\text{scatt}}(t,\mathbf{x}) \approx ik\hat{\mathbf{x}} \times \mathbf{A}_{\text{scatt}}(t,\mathbf{x})\,.$$

By calculating the Poynting vector of the scattered and incident radiation, show that the ratio of the time-averaged power scattered per unit solid angle to the time-averaged incident power per unit area (i.e. the differential cross-section) is

$$\frac{d\sigma}{d\Omega} = (\epsilon_r - 1)^2 k^4 \left(\frac{\sin(qa) - qa\cos(qa)}{q^3}\right)^2 |\hat{\mathbf{x}} \times \hat{\mathbf{E}}_0|^2\,,$$

where $\hat{\mathbf{E}}_0 = \mathbf{E}_0/|\mathbf{E}_0|$ and $q = |\mathbf{q}|$.

[*You may assume that, in the Lorenz gauge, the retarded potential due to a localised current distribution is*

$$\mathbf{A}(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(t_{\text{ret}},\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3\mathbf{x}'\,,$$

where the retarded time $t_{\text{ret}} = t - |\mathbf{x} - \mathbf{x}'|/c$.]

36D General Relativity

(a) In the transverse traceless gauge, a plane gravitational wave propagating in the z direction is described by a perturbation $h_{\alpha\beta}$ of the Minkowski metric $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ in Cartesian coordinates $x^{\alpha} = (t, x, y, z)$, where

$$h_{\alpha\beta} = H_{\alpha\beta} e^{ik_{\mu}x^{\mu}} , \qquad \text{where} \qquad k^{\mu} = \omega(1, 0, 0, 1) ,$$

and $H_{\alpha\beta}$ is a constant matrix. Spacetime indices in this question are raised or lowered with the Minkowski metric.

The energy-momentum tensor of a gravitational wave is defined to be

$$\tau_{\mu\nu} = \frac{1}{32\pi} (\partial_{\mu} h^{\alpha\beta})(\partial_{\nu} h_{\alpha\beta}) .$$

Show that $\partial^{\nu}\tau_{\mu\nu} = \frac{1}{2}\partial_{\mu}\tau^{\nu}{}_{\nu}$ and hence, or otherwise, show that energy and momentum are conserved.

(b) A point mass m undergoes harmonic motion along the z-axis with frequency ω and amplitude L. Compute the energy flux emitted in gravitational radiation.

[*Hint: The quadrupole formula for time-averaged energy flux radiated in gravitational waves is*

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{1}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle$$

where Q_{ij} is the reduced quadrupole tensor.]

37B Fluid Dynamics II

A horizontal layer of inviscid fluid of density ρ_1 occupying $0 < y < h$ flows with velocity $(U, 0)$ above a horizontal layer of inviscid fluid of density $\rho_2 > \rho_1$ occupying $-h < y < 0$ and flowing with velocity $(-U, 0)$, in Cartesian coordinates (x, y). There are rigid boundaries at $y = \pm h$. The interface between the two layers is perturbed to position $y = \mathrm{Re}(Ae^{ikx+\sigma t})$.

Write down the full set of equations and boundary conditions governing this flow. Derive the linearised boundary conditions appropriate in the limit $A \to 0$. Solve the linearised equations to show that the perturbation to the interface grows exponentially in time if

$$U^2 > \frac{\rho_2^2 - \rho_1^2}{\rho_1 \rho_2} \frac{g}{4k} \tanh kh.$$

Sketch the right-hand side of this inequality as a function of k. Thereby deduce the minimum value of U that makes the system unstable for all wavelengths.

38B Waves

Consider the Rossby-wave equation

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} - \ell^2\right)\varphi + \beta\frac{\partial \varphi}{\partial x} = 0\,,$$

where $\ell > 0$ and $\beta > 0$ are real constants. Find and sketch the dispersion relation for waves with wavenumber k and frequency $\omega(k)$. Find and sketch the phase velocity $c(k)$ and the group velocity $c_g(k)$, and identify in which direction(s) the wave crests travel, and the corresponding direction(s) of the group velocity.

Write down the solution with initial value

$$\varphi(x,0) = \int_{-\infty}^{\infty} A(k)e^{ikx}dk\,,$$

where $A(k)$ is real and $A(-k) = A(k)$. Use the method of stationary phase to obtain leading-order approximations to $\varphi(x,t)$ for large t, with x/t having the constant value V, for

(i) $0 < V < \beta/8\ell^2$,

(ii) $-\beta/\ell^2 < V \leqslant 0$,

where the solutions for the stationary points should be left in implicit form. [It is helpful to note that $\omega(-k) = -\omega(k)$.]

Briefly discuss the nature of the solution for $V > \beta/8\ell^2$ and $V < -\beta/\ell^2$. [Detailed calculations are not required.]

[*Hint: You may assume that*

$$\int_{-\infty}^{\infty} e^{\pm i\gamma u^2}du = \left(\frac{\pi}{\gamma}\right)^{\frac{1}{2}} e^{\pm i\pi/4}$$

for $\gamma > 0$.]

39A Numerical Analysis

(a) The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant T,$$

is approximated by the Crank–Nicolson scheme

$$u_m^{n+1} - \frac{1}{2}\mu\left(u_{m-1}^{n+1} - 2u_m^{n+1} + u_{m+1}^{n+1}\right) = u_m^n + \frac{1}{2}\mu\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right),$$

with $m = 1, \ldots, M$. Here $\mu = k/h^2$, $k = \Delta t$, $h = \Delta x = \frac{1}{M+1}$, and u_m^n is an approximation to $u(mh, nk)$. Assuming that $u(0,t) = u(1,t) = 0$, show that the above scheme can be written in the form

$$B\mathbf{u}^{n+1} = C\mathbf{u}^n, \quad 0 \leqslant n \leqslant T/k - 1,$$

where $\mathbf{u}^n = [u_1^n, \ldots, u_M^n]^T$ and the real matrices B and C should be found. Using matrix analysis, find the range of $\mu > 0$ for which the scheme is stable.

[*Hint: All Toeplitz symmetric tridiagonal (TST) matrices have the same set of orthogonal eigenvectors, and a TST matrix with the elements* $a_{i,i} = a$ *and* $a_{i,i\pm1} = b$ *has the eigenvalues* $\lambda_k = a + 2b\cos\frac{\pi k}{M+1}$.]

(b) The wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad x \in \mathbb{R}, \qquad t \geqslant 0,$$

with given initial conditions for u and $\partial u/\partial t$, is approximated by the scheme

$$u_m^{n+1} - 2u_m^n + u_m^{n-1} = \mu(u_{m+1}^n - 2u_m^n + u_{m-1}^n),$$

with the Courant number now $\mu = k^2/h^2$. Applying the Fourier technique, find the range of $\mu > 0$ for which the method is stable.

END OF PAPER