UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Friday, 9 June, 2017 1:30pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS	***SPECIAL REQUIREMENTS***
Gold cover sheet	*None*
Green master cover sheet	

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Linear Algebra

Briefly explain the Gram–Schmidt orthogonalisation process in a real finite-dimensional inner product space V.

For a subspace U of V, define $U^{\perp}$, and show that $V = U \oplus U^{\perp}$.

For which positive integers n does

$$(f, g) = f(1)g(1) + f(2)g(2) + f(3)g(3)$$

define an inner product on the space of all real polynomials of degree at most n?

2E Groups, Rings and Modules

Let G be a non-trivial finite p-group and let $Z(G)$ be its centre. Show that $|Z(G)| > 1$. Show that if $|G| = p^3$ and if G is not abelian, then $|Z(G)| = p$.

3G Analysis II

State the chain rule for the composition of two differentiable functions $f \colon \mathbb{R}^m \to \mathbb{R}^n$ and $g \colon \mathbb{R}^n \to \mathbb{R}^p$.

Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable. For $c \in \mathbb{R}$, let $g(x) = f(x, c - x)$. Compute the derivative of g. Show that if $\partial f/\partial x = \partial f/\partial y$ throughout $\mathbb{R}^2$, then $f(x, y) = h(x + y)$ for some function $h \colon \mathbb{R} \to \mathbb{R}$.

4F Complex Analysis

Let D be a star-domain, and let f be a continuous complex-valued function on D. Suppose that for every triangle T contained in D we have

$$\int_{\partial T} f(z)\, dz \;=\; 0\;.$$

Show that f has an antiderivative on D.

If we assume instead that D is a domain (not necessarily a star-domain), does this conclusion still hold? Briefly justify your answer.

5A Methods

The Legendre polynomials, $P_n(x)$ for integers $n \geqslant 0$, satisfy the Sturm–Liouville equation

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}P_n(x)\right] + n(n+1)P_n(x) = 0$$

and the recursion formula

$$(n+1)\,P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x), \qquad P_0(x) = 1, \quad P_1(x) = x.$$

(i) For all $n \geqslant 0$, show that $P_n(x)$ is a polynomial of degree n with $P_n(1) = 1$.

(ii) For all $m, n \geqslant 0$, show that $P_n(x)$ and $P_m(x)$ are orthogonal over the range $x \in [-1, 1]$ when $m \neq n$.

(iii) For each $n \geqslant 0$ let

$$R_n(x) = \frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right].$$

Assume that for each n there is a constant α_n such that $P_n(x) = \alpha_n R_n(x)$ for all x. Determine α_n for each n.

6B Quantum Mechanics

(a) Give a physical interpretation of the wavefunction $\phi(x,t) = Ae^{ikx}e^{-iEt/\hbar}$ (where A, k and E are real constants).

(b) A particle of mass m and energy $E > 0$ is incident from the left on the potential step

$$V(x) = \begin{cases} 0 & \text{for } -\infty < x < a \\ V_0 & \text{for } a < x < \infty. \end{cases}$$

with $V_0 > 0$.

State the conditions satisfied by a stationary state at the point $x = a$.

Compute the probability that the particle is reflected as a function of E, and compare your result with the classical case.

7C Electromagnetism

A thin wire, in the form of a closed curve C, carries a constant current I. Using either the Biot–Savart law or the magnetic vector potential, show that the magnetic field far from the loop is of the approximate form

$$\mathbf{B}(\mathbf{r}) \approx \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r} - \mathbf{m}|\mathbf{r}|^2}{|\mathbf{r}|^5}\right],$$

where $\mathbf{m}$ is the magnetic dipole moment of the loop. Derive an expression for $\mathbf{m}$ in terms of I and the vector area spanned by the curve C.

8C Numerical Analysis

For the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 5 & 5 & 5 \\ 1 & 5 & 14 & 14 \\ 1 & 5 & 14 & \lambda \end{bmatrix}$$

find a factorization of the form

$$A = LDL^{\top},$$

where D is diagonal and L is lower triangular with ones on its diagonal.

For what values of λ is A positive definite?

In the case $\lambda = 30$ find the Cholesky factorization of A.

9H Markov Chains

Prove that the simple symmetric random walk on $\mathbb{Z}^3$ is transient.

[Any combinatorial inequality can be used without proof.]

SECTION II

10F Linear Algebra

What is the *dual* X^* of a finite-dimensional real vector space X? If X has a basis $e_1, \ldots, e_n$, define the dual basis, and prove that it is indeed a basis of X^*.

[No results on the dimension of duals may be assumed without proof.]

Write down (without making a choice of basis) an isomorphism from X to X^{**}. Prove that your map is indeed an isomorphism.

Does every basis of X^* arise as the dual basis of some basis of X? Justify your answer.

A subspace W of X^* is called *separating* if for every non-zero $x \in X$ there is a $T \in W$ with $T(x) \neq 0$. Show that the only separating subspace of X^* is X^* itself.

Now let X be the (infinite-dimensional) space of all real polynomials. Explain briefly how we may identify X^* with the space of all real sequences. Give an example of a proper subspace of X^* that is separating.

11E Groups, Rings and Modules

(a) State (without proof) the classification theorem for finitely generated modules over a Euclidean domain. Give the statement and the proof of the rational canonical form theorem.

(b) Let R be a principal ideal domain and let M be an R-submodule of R^n. Show that M is a free R-module.

12G Analysis II

Let $U \subset \mathbb{R}^m$ be a nonempty open set. What does it mean to say that a function $f \colon U \to \mathbb{R}^n$ is *differentiable*?

Let $f \colon U \to \mathbb{R}$ be a function, where $U \subset \mathbb{R}^2$ is open. Show that if the first partial derivatives of f exist and are continuous on U, then f is differentiable on U.

Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be the function

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \dfrac{x^3 + 2y^4}{x^2 + y^2} & (x,y) \neq (0,0). \end{cases}$$

Determine, with proof, where f is differentiable.

13E Metric and Topological Spaces

Let $f\colon X \to Y$ be a continuous map between topological spaces.

(a) Assume X is compact and that $Z \subseteq X$ is a closed subset. Prove that Z and $f(Z)$ are both compact.

(b) Suppose that

(i) $f^{-1}(\{y\})$ is compact for each $y \in Y$, and

(ii) if A is any closed subset of X, then $f(A)$ is a closed subset of Y.

Show that if $K \subseteq Y$ is compact, then $f^{-1}(K)$ is compact.

[*Hint: Given an open cover $f^{-1}(K) \subseteq \bigcup_{i \in I} U_i$, find a finite subcover, say $f^{-1}(\{y\}) \subseteq \bigcup_{i \in I_y} U_i$, for each $y \in K$; use closedness of $X \setminus \bigcup_{i \in I_y} U_i$ and property* (ii) *to produce an open cover of K.*]

14A Complex Methods

By using Fourier transforms and a conformal mapping

$$w = \sin\left(\frac{\pi z}{a}\right)$$

with $z = x + iy$ and $w = \xi + i\eta$, and a suitable real constant a, show that the solution to

$$\begin{aligned}
\nabla^2\phi &= 0 && -2\pi \leqslant x \leqslant 2\pi,\ y \geqslant 0,\\
\phi(x,0) &= f(x) && -2\pi \leqslant x \leqslant 2\pi,\\
\phi(\pm 2\pi, y) &= 0 && y > 0,\\
\phi(x,y) &\to 0 && y \to \infty,\ -2\pi \leqslant x \leqslant 2\pi,
\end{aligned}$$

is given by

$$\phi(\xi,\eta) = \frac{\eta}{\pi}\int_{-1}^{1} \frac{F(\xi')}{\eta^2 + (\xi - \xi')^2}\, d\xi',$$

where $F(\xi')$ is to be determined.

In the case of $f(x) = \sin\left(\dfrac{x}{4}\right)$, give $F(\xi')$ explicitly as a function of ξ'. [You need not evaluate the integral.]

15G Geometry

What is a *hyperbolic line* in (a) the disc model (b) the upper half-plane model of the hyperbolic plane? What is the *hyperbolic distance* $d(P,Q)$ between two points P, Q in the hyperbolic plane? Show that if γ is any continuously differentiable curve with endpoints P and Q then its length is at least $d(P,Q)$, with equality if and only if γ is a monotonic reparametrisation of the hyperbolic line segment joining P and Q.

What does it mean to say that two hyperbolic lines L, L' are (a) *parallel* (b) *ultraparallel*? Show that L and L' are ultraparallel if and only if they have a common perpendicular, and if so, then it is unique.

A *horocycle* is a curve in the hyperbolic plane which in the disc model is a Euclidean circle with exactly one point on the boundary of the disc. Describe the horocycles in the upper half-plane model. Show that for any pair of horocycles there exists a hyperbolic line which meets both orthogonally. For which pairs of horocycles is this line unique?

16D Variational Principles

Consider the functional

$$F[y] = \int_{\alpha}^{\beta} f(y, y', x)\, dx$$

of a function $y(x)$ defined for $x \in [\alpha, \beta]$, with y having fixed values at $x = \alpha$ and $x = \beta$.

By considering $F[y + \epsilon\xi]$, where $\xi(x)$ is an arbitrary function with $\xi(\alpha) = \xi(\beta) = 0$ and $\epsilon \ll 1$, determine that the second variation of F is

$$\delta^2 F[y, \xi] = \int_{\alpha}^{\beta} \left\{ \xi^2 \left[\frac{\partial^2 f}{\partial y^2} - \frac{d}{dx} \left(\frac{\partial^2 f}{\partial y \partial y'} \right) \right] + \xi'^2 \frac{\partial^2 f}{\partial y'^2} \right\} dx.$$

The surface area of an axisymmetric soap film joining two parallel, co-axial, circular rings of radius a distance $2L$ apart can be expressed by the functional

$$F[y] = \int_{-L}^{L} 2\pi y \sqrt{1 + y'^2}\, dx,$$

where x is distance in the axial direction and y is radial distance from the axis. Show that the surface area is stationary when

$$y = E \cosh \frac{x}{E},$$

where E is a constant that need not be determined, and that the stationary area is a local minimum if

$$\int_{-L/E}^{L/E} \left(\xi'^2 - \xi^2\right) \operatorname{sech}^2 z\, dz > 0$$

for all functions $\xi(z)$ that vanish at $z = \pm L/E$, where $z = x/E$.

17B Methods

(a)

(i) For the diffusion equation

$$\frac{\partial \phi}{\partial t} - K\frac{\partial^2 \phi}{\partial x^2} = 0 \qquad \text{on } -\infty < x < \infty \text{ and } t \geqslant 0,$$

with diffusion constant K, state the properties (in terms of the Dirac delta function) that define the *fundamental solution* $F(x,t)$ and the *Green's function* $G(x,t;y,\tau)$.
You are not required to give expressions for these functions.

(ii) Consider the initial value problem for the homogeneous equation:

$$\frac{\partial \phi}{\partial t} - K\frac{\partial^2 \phi}{\partial x^2} = 0, \quad \phi(x,t_0) = \alpha(x) \quad \text{on } -\infty < x < \infty \text{ and } t \geqslant t_0, \qquad \text{(A)}$$

and the forced equation with homogeneous initial condition (and given forcing term $h(x,t)$):

$$\frac{\partial \phi}{\partial t} - K\frac{\partial^2 \phi}{\partial x^2} = h(x,t), \quad \phi(x,0) = 0 \quad \text{on } -\infty < x < \infty \text{ and } t \geqslant 0. \qquad \text{(B)}$$

Given that F and G in part (i) are related by

$$G(x,t;y,\tau) = \Theta(t-\tau)F(x-y,t-\tau)$$

(where $\Theta(t)$ is the Heaviside step function having value 0 for $t < 0$ and 1 for $t > 0$), show how the solution of (B) can be expressed in terms of solutions of (A) with suitable initial conditions. Briefly interpret your expression.

(b) A semi-infinite conducting plate lies in the (x_1, x_2) plane in the region $x_1 \geqslant 0$. The boundary along the x_2 axis is perfectly insulated. Let (r,θ) denote standard polar co-ordinates on the plane. At time $t = 0$ the entire plate is at temperature zero except for the region defined by $-\pi/4 < \theta < \pi/4$ and $1 < r < 2$ which has constant initial temperature $T_0 > 0$. Subsequently the temperature of the plate obeys the two-dimensional heat equation with diffusion constant K. Given that the fundamental solution of the two-dimensional heat equation on $\mathbb{R}^2$ is

$$F(x_1,x_2,t) = \frac{1}{4\pi K t} e^{-(x_1^2+x_2^2)/(4Kt)},$$

show that the origin $(0,0)$ on the plate reaches its maximum temperature at time $t = 3/(8K\log 2)$.

18D Fluid Dynamics

The linearised equations governing the horizontal components of flow $\mathbf{u}(x, y, t)$ in a rapidly rotating shallow layer of depth $h = h_0 + \eta(x, y, t)$, where $\eta \ll h_0$, are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta,$$

$$\frac{\partial \eta}{\partial t} + h_0 \nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{f} = f\mathbf{e}_z$ is the constant Coriolis parameter, and $\mathbf{e}_z$ is the unit vector in the vertical direction.

Use these equations, either in vector form or using Cartesian components, to show that the potential vorticity

$$\mathbf{Q} = \boldsymbol{\zeta} - \frac{\eta}{h_0}\mathbf{f}$$

is independent of time, where $\boldsymbol{\zeta} = \nabla \times \mathbf{u}$ is the relative vorticity.

Derive the equation

$$\frac{\partial^2 \eta}{\partial t^2} - gh_0 \nabla^2 \eta + f^2 \eta = -h_0 \mathbf{f} \cdot \mathbf{Q}.$$

In the case that $\mathbf{Q} \equiv 0$, determine and sketch the dispersion relation $\omega(k)$ for plane waves with $\eta = Ae^{i(kx+\omega t)}$, where A is constant. Discuss the nature of the waves qualitatively: do long waves propagate faster or slower than short waves; how does the phase speed depend on wavelength; does rotation have more effect on long waves or short waves; how dispersive are the waves?

19H Statistics

(a) State and prove the Neyman–Pearson lemma.

(b) Let X be a real random variable with density $f(x) = (2\theta x + 1 - \theta)1_{[0,1]}(x)$ with $-1 \leqslant \theta \leqslant 1$.

Find a most powerful test of size α of $H_0 : \theta = 0$ versus $H_1 : \theta = 1$.

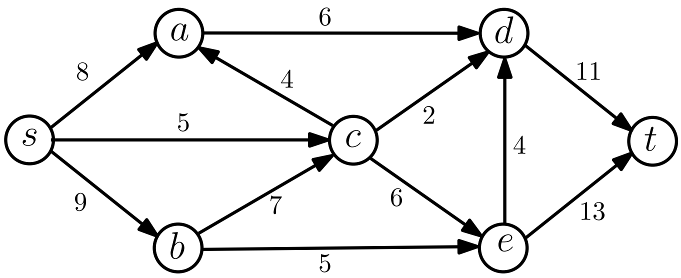
Find a uniformly most powerful test of size α of $H_0 : \theta = 0$ versus $H_1 : \theta > 0$.

20H Optimisation

(a) Let G be a flow network with capacities c_{ij} on the edges. Explain the maximum flow problem on this network defining all the notation you need.

(b) Describe the Ford–Fulkerson algorithm for finding a maximum flow and state the max-flow min-cut theorem.

(c) Apply the Ford–Fulkerson algorithm to find a maximum flow and a minimum cut of the following network:



(d) Suppose that we add $\varepsilon > 0$ to each capacity of a flow network. Is it true that the maximum flow will always increase by ε? Justify your answer.

END OF PAPER