UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 8 June, 2017 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups, Rings and Modules

Let R be a commutative ring and let M be an R-module. Show that M is a finitely generated R-module if and only if there exists a surjective R-module homomorphism $R^n \to M$ for some n.

Find an example of a $\mathbb{Z}$-module M such that there is no surjective $\mathbb{Z}$-module homomorphism $\mathbb{Z} \to M$ but there is a surjective $\mathbb{Z}$-module homomorphism $\mathbb{Z}^2 \to M$ which is not an isomorphism. Justify your answer.

2G Analysis II

What does it mean to say that a metric space is *complete*? Which of the following metric spaces are complete? Briefly justify your answers.

(i) $[0,1]$ with the Euclidean metric.

(ii) $\mathbb{Q}$ with the Euclidean metric.

(iii) The subset

$$\{\,(0,0)\,\} \cup \{\,(x, \sin(1/x)) \mid x > 0\,\} \subset \mathbb{R}^2$$

with the metric induced from the Euclidean metric on $\mathbb{R}^2$.

Write down a metric on $\mathbb{R}$ with respect to which $\mathbb{R}$ is not complete, justifying your answer.

[*You may assume throughout that* $\mathbb{R}$ *is complete with respect to the Euclidean metric.*]

3E Metric and Topological Spaces

Let X and Y be topological spaces.

(a) Define what is meant by the *product topology* on $X \times Y$. Define the *projection maps* $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ and show they are continuous.

(b) Consider $\Delta = \{(x,x) \mid x \in X\}$ in $X \times X$. Show that X is Hausdorff if and only if Δ is a closed subset of $X \times X$ in the product topology.

4A Complex Methods

By using the Laplace transform, show that the solution to

$$y'' - 4y' + 3y = t\,e^{-3t},$$

subject to the conditions $y(0) = 0$ and $y'(0) = 1$, is given by

$$y(t) = \frac{37}{72}e^{3t} - \frac{17}{32}e^{t} + \left(\frac{5}{288} + \frac{1}{24}t\right)e^{-3t}$$

when $t \geqslant 0$.

5G Geometry

Let

$$\pi(x, y, z) = \frac{x + iy}{1 - z}$$

be stereographic projection from the unit sphere S^2 in $\mathbb{R}^3$ to the Riemann sphere $\mathbb{C}_\infty$. Show that if r is a rotation of S^2, then $\pi r \pi^{-1}$ is a Möbius transformation of $\mathbb{C}_\infty$ which can be represented by an element of $SU(2)$. (You may assume without proof any result about generation of $SO(3)$ by a particular set of rotations, but should state it carefully.)

6D Variational Principles

(a) A *Pringle crisp* can be defined as the surface

$$z = xy \quad \text{with} \quad x^2 + y^2 \leqslant 1.$$

Use the method of Lagrange multipliers to find the minimum and maximum values of z on the boundary of the Pringle crisp and the (x, y) positions where these occur.

(b) A farmer wishes to construct a grain silo in the form of a hollow vertical cylinder of radius r and height h with a hollow hemispherical cap of radius r on top of the cylinder. The walls of the cylinder cost £x per unit area to construct and the surface of the cap costs £$2x$ per unit area to construct. Given that a total volume V is desired for the silo, what values of r and h should be chosen to minimise the cost?

7A Methods

Using the substitution $u(x, y) = v(x, y)e^{-x^2}$, find $u(x, y)$ that satisfies

$$u_x + x\,u_y + 2\,x\,u = e^{-x^2}$$

with boundary data $u(0, y) = y\,e^{-y^2}$.

8B Quantum Mechanics

A particle of mass m is confined to a one-dimensional box $0 \leqslant x \leqslant a$. The potential $V(x)$ is zero inside the box and infinite outside.

(a) Find the allowed energies of the particle and the normalised energy eigenstates.

(b) At time $t = 0$ the particle has wavefunction ψ_0 that is uniform in the left half of the box i.e. $\psi_0(x) = \sqrt{\frac{2}{a}}$ for $0 < x < a/2$ and $\psi_0(x) = 0$ for $a/2 < x < a$. Find the probability that a measurement of energy at time $t = 0$ will yield a value less than $5\hbar^2\pi^2/(2ma^2)$.

9H Markov Chains

(a) What does it mean to say that a Markov chain is *reversible*?

(b) Let G be a finite connected graph on n vertices. What does it mean to say that X is a *simple random walk* on G?

Find the unique invariant distribution π of X.

Show that X is reversible when $X_0 \sim \pi$.

[*You may use, without proof, results about detailed balance equations, but you should state them clearly.*]

SECTION II

10F Linear Algebra

Let f be a quadratic form on a finite-dimensional real vector space V. Prove that there exists a diagonal basis for f, meaning a basis with respect to which the matrix of f is diagonal.

Define the rank r and signature s of f in terms of this matrix. Prove that r and s are independent of the choice of diagonal basis.

In terms of r, s, and the dimension n of V, what is the greatest dimension of a subspace on which f is zero?

Now let f be the quadratic form on $\mathbb{R}^3$ given by $f(x,y,z) = x^2 - y^2$. For which points v in $\mathbb{R}^3$ is it the case that there is some diagonal basis for f containing v?

11E Groups, Rings and Modules

(a) Define what is meant by a *Euclidean domain*. Show that every Euclidean domain is a principal ideal domain.

(b) Let $p \in \mathbb{Z}$ be a prime number and let $f \in \mathbb{Z}[x]$ be a monic polynomial of positive degree. Show that the quotient ring $\mathbb{Z}[x]/(p,f)$ is finite.

(c) Let $\alpha \in \mathbb{Z}[\sqrt{-1}]$ and let P be a non-zero prime ideal of $\mathbb{Z}[\alpha]$. Show that the quotient $\mathbb{Z}[\alpha]/P$ is a finite ring.

12G Analysis II

What is a *contraction map* on a metric space X? State and prove the contraction mapping theorem.

Let (X,d) be a complete non-empty metric space. Show that if $f\colon X \to X$ is a map for which some iterate f^k $(k \geqslant 1)$ is a contraction map, then f has a unique fixed point. Show that f itself need not be a contraction map.

Let $f\colon [0,\infty) \to [0,\infty)$ be the function

$$f(x) = \frac{1}{3}\left(x + \sin x + \frac{1}{x+1}\right).$$

Show that f has a unique fixed point.

13F Complex Analysis

Let f be an entire function. Prove Taylor's theorem, that there exist complex numbers $c_0, c_1, \ldots$ such that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all z. [You may assume Cauchy's Integral Formula.]

For a positive real r, let $M_r = \sup\{|f(z)| : |z| = r\}$. Explain why we have

$$|c_n| \leqslant \frac{M_r}{r^n}$$

for all n.

Now let n and r be fixed. For which entire functions f do we have $|c_n| = \dfrac{M_r}{r^n}$?

14G Geometry

Let $\sigma \colon U \to \mathbb{R}^3$ be a parametrised surface, where $U \subset \mathbb{R}^2$ is an open set.

(a) Explain what are the *first and second fundamental forms* of the surface, and what is its *Gaussian curvature.* Compute the Gaussian curvature of the hyperboloid $\sigma(x, y) = (x, y, xy)$.

(b) Let $\mathbf{a}(x)$ and $\mathbf{b}(x)$ be parametrised curves in $\mathbb{R}^3$, and assume that

$$\sigma(x, y) = \mathbf{a}(x) + y\mathbf{b}(x).$$

Find a formula for the first fundamental form, and show that the Gaussian curvature vanishes if and only if

$$\mathbf{a}' \cdot (\mathbf{b} \times \mathbf{b}') = 0\,.$$

15A Methods

Let $\mathcal{L}$ be the linear differential operator

$$\mathcal{L}\, y \, = y''' - y'' - 2y'$$

where $'$ denotes differentiation with respect to x.

Find the Green's function, $G(x; \xi)$, for $\mathcal{L}$ satisfying the homogeneous boundary conditions $G(0; \xi) = 0$, $G'(0; \xi) = 0$, $G''(0; \xi) = 0$.

Using the Green's function, solve

$$\mathcal{L}y = e^x \Theta(x - 1)$$

with boundary conditions $y(0) = 1$, $y'(0) = -1$, $y''(0) = 0$. Here $\Theta(x)$ is the Heaviside step function having value 0 for $x < 0$ and 1 for $x > 0$.

16B Quantum Mechanics

(a) Given the position and momentum operators $\hat{x}_i = x_i$ and $\hat{p}_i = -i\hbar\,\partial/\partial x_i$ (for $i = 1,2,3$) in three dimensions, define the angular momentum operators $\hat{L}_i$ and the total angular momentum $\hat{L}^2$.
Show that $\hat{L}_3$ is Hermitian.

(b) Derive the generalised uncertainty relation for the observables $\hat{L}_3$ and $\hat{x}_1$ in the form

$$\Delta_\psi \hat{L}_3 \; \Delta_\psi \hat{x}_1 \geqslant M$$

for any state ψ and a suitable expression M that you should determine. [*Hint: It may be useful to consider the operator* $\hat{L}_3 + i\lambda\hat{x}_1$.]

(c) Consider a particle with wavefunction

$$\psi = K(x_1 + x_2 + 2x_3)e^{-\alpha r}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and K and α are real positive constants.
Show that ψ is an eigenstate of total angular momentum $\hat{L}^2$ and find the corresponding angular momentum quantum number l. Find also the expectation value of a measurement of $\hat{L}_3$ on the state ψ.

17C Electromagnetism

(i) Two point charges, of opposite sign and unequal magnitude, are placed at two different locations. Show that the combined electrostatic potential vanishes on a sphere that encloses only the charge of smaller magnitude.

(ii) A grounded, conducting sphere of radius a is centred at the origin. A point charge q is located outside the sphere at position vector $\mathbf{p}$. Formulate the differential equation and boundary conditions for the electrostatic potential outside the sphere. Using the result of part (i) or otherwise, show that the electric field outside the sphere is identical to that generated (in the absence of any conductors) by the point charge q and an image charge q' located inside the sphere at position vector $\mathbf{p}'$, provided that $\mathbf{p}'$ and q' are chosen correctly.

Calculate the magnitude and direction of the force experienced by the charge q.

18D Fluid Dynamics

Use Euler's equations to derive the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u},$$

where $\mathbf{u}$ is the fluid velocity and $\boldsymbol{\omega}$ is the vorticity.

Consider axisymmetric, incompressible, inviscid flow between two rigid plates at $z = h(t)$ and $z = -h(t)$ in cylindrical polar coordinates (r, θ, z), where t is time. Using mass conservation, or otherwise, find the complete flow field whose radial component is independent of z.

Now suppose that the flow has angular velocity $\boldsymbol{\Omega} = \Omega(t)\mathbf{e}_z$ and that $\Omega = \Omega_0$ when $h = h_0$. Use the vorticity equation to determine the angular velocity for subsequent times as a function of h. What physical principle does your result illustrate?

19C Numerical Analysis

Let $p_n \in \mathbb{P}_n$ be the nth monic orthogonal polynomial with respect to the inner product

$$\langle f, g \rangle = \int_a^b w(x) f(x) g(x)\, dx$$

on $C[a, b]$, where w is a positive weight function.

Prove that, for $n \geqslant 1$, p_n has n distinct zeros in the interval (a, b).

Let $c_1, c_2, \ldots, c_n \in [a, b]$ be n distinct points. Show that the quadrature formula

$$\int_a^b w(x) f(x)\, dx \approx \sum_{i=1}^{n} b_i f(c_i)$$

is exact for all $f \in \mathbb{P}_{n-1}$ if the weights b_i are chosen to be

$$b_i = \int_a^b w(x) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - c_j}{c_i - c_j}\, dx\,.$$

Show further that the quadrature formula is exact for all $f \in \mathbb{P}_{2n-1}$ if the nodes c_i are chosen to be the zeros of p_n (Gaussian quadrature). [*Hint: Write f as $qp_n + r$, where $q, r \in \mathbb{P}_{n-1}$.*]

Use the Peano kernel theorem to write an integral expression for the approximation error of Gaussian quadrature for sufficiently differentiable functions. (You should give a formal expression for the Peano kernel but are *not* required to evaluate it.)

20H Statistics

Consider the general linear model

$$\boldsymbol{Y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon},$$

where X is a known $n \times p$ matrix of full rank $p < n$, $\boldsymbol{\varepsilon} \sim \mathcal{N}_n(0, \sigma^2 I)$ with σ^2 known and $\boldsymbol{\beta} \in \mathbb{R}^p$ is an unknown vector.

(a) State without proof the Gauss–Markov theorem.

Find the maximum likelihood estimator $\widehat{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}$. Is it unbiased?

Let $\boldsymbol{\beta}^*$ be any unbiased estimator for $\boldsymbol{\beta}$ which is linear in (Y_i). Show that

$$\operatorname{var}(\boldsymbol{t}^T\widehat{\boldsymbol{\beta}}) \leqslant \operatorname{var}(\boldsymbol{t}^T\boldsymbol{\beta}^*)$$

for all $\boldsymbol{t} \in \mathbb{R}^p$.

(b) Suppose now that $p = 1$ and that $\boldsymbol{\beta}$ and σ^2 are both unknown. Find the maximum likelihood estimator for σ^2. What is the joint distribution of $\widehat{\boldsymbol{\beta}}$ and $\widehat{\sigma}^2$ in this case? Justify your answer.

21H Optimisation

(a) Explain what is meant by a *two-person zero-sum game* with payoff matrix $A = (a_{ij} : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$ and define what is an *optimal strategy* (also known as a maximin strategy) for each player.

(b) Suppose the payoff matrix A is antisymmetric, i.e. $m = n$ and $a_{ij} = -a_{ji}$ for all i, j. What is the value of the game? Justify your answer.

(c) Consider the following two-person zero-sum game. Let $n \geqslant 3$. Both players simultaneously call out one of the numbers $\{1, \ldots, n\}$. If the numbers differ by one, the player with the higher number **wins** £1 from the other player. If the players' choices differ by 2 or more, the player with the higher number **pays** £2 to the other player. In the event of a tie, no money changes hands.

Write down the payoff matrix.

For the case when $n = 3$ find the value of the game and an optimal strategy for each player.

Find the value of the game and an optimal strategy for each player for all n.

[*You may use results from the course provided you state them clearly.*]

END OF PAPER