UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part IB

Wednesday, 7 June, 2017 9:00 am to 12:00 pm

## PAPER 2

### Before you begin read these instructions carefully.

*The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.*

*Candidates may obtain credit from attempts on* ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

*Write on* ***one side*** *of the paper only and begin each answer on a separate sheet.*

*Write legibly; otherwise, you place yourself at a grave disadvantage.*

***At the end of the examination:***

*Tie up your answers in separate bundles labelled* ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

*Attach a completed gold cover sheet to each bundle.*

*You must also complete a green master cover sheet listing all the questions you have attempted.*

***Every cover sheet must bear your examination number and desk number.***

***STATIONERY REQUIREMENTS***
*Gold cover sheet*
*Green master cover sheet*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

## SECTION I

**1F Linear Algebra**

State and prove the Rank–Nullity theorem.

Let $\alpha$ be a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$ of rank 2. Give an example to show that $\mathbb{R}^3$ may be the direct sum of the kernel of $\alpha$ and the image of $\alpha$, and also an example where this is not the case.

**2E Groups, Rings and Modules**

(a) Define what is meant by a *unique factorisation domain* and by a *principal ideal domain*. State Gauss's lemma and Eisenstein's criterion, without proof.

(b) Find an example, with justification, of a ring $R$ and a subring $S$ such that

(i) $R$ is a principal ideal domain, and

(ii) $S$ is a unique factorisation domain but not a principal ideal domain.

**3G Analysis II**

Let $X \subset \mathbb{R}$. What does it mean to say that a sequence of real-valued functions on $X$ is *uniformly convergent*?

Let $f, f_n\ (n \geqslant 1)\colon \mathbb{R} \to \mathbb{R}$ be functions.

(a) Show that if each $f_n$ is continuous, and $(f_n)$ converges uniformly on $\mathbb{R}$ to $f$, then $f$ is also continuous.

(b) Suppose that, for every $M > 0$, $(f_n)$ converges uniformly on $[-M, M]$. Need $(f_n)$ converge uniformly on $\mathbb{R}$? Justify your answer.

**4E Metric and Topological Spaces**

Let $f\colon (X,d) \to (Y,e)$ be a function between metric spaces.

(a) Give the $\epsilon$-$\delta$ definition for $f$ to be *continuous*. Show that $f$ is continuous if and only if $f^{-1}(U)$ is an open subset of $X$ for each open subset $U$ of $Y$.

(b) Give an example of $f$ such that $f$ is not continuous but $f(V)$ is an open subset of $Y$ for every open subset $V$ of $X$.

**5B Methods**

Expand $f(x) = x$ as a Fourier series on $-\pi < x < \pi$.

By integrating the series show that $x^2$ on $-\pi < x < \pi$ can be written as

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \ ,$$

where $a_n$, $n = 1, 2, \ldots$, should be determined and

$$a_0 = 8 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

By evaluating $a_0$ another way show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

**6C Electromagnetism**

State Gauss's Law in the context of electrostatics.

A spherically symmetric capacitor consists of two conductors in the form of concentric spherical shells of radii $a$ and $b$, with $b > a$. The inner sphere carries a charge $Q$ and the outer sphere carries a charge $-Q$. Determine the electric field $\mathbf{E}$ and the electrostatic potential $\phi$ in the regions $r < a$, $a < r < b$ and $r > b$. Show that the capacitance is

$$C = \frac{4\pi\epsilon_0 ab}{b-a}$$

and calculate the electrostatic energy of the system in terms of $Q$ and $C$.

**7D Fluid Dynamics**

From Euler's equations describing steady inviscid fluid flow under the action of a conservative force, derive Bernoulli's equation for the pressure along a streamline of the flow, defining all variables that you introduce.

Water fills an inverted, open, circular cone (radius increasing upwards) of half angle $\pi/4$ to a height $h_0$ above its apex. At time $t = 0$, the tip of the cone is removed to leave a small hole of radius $\epsilon \ll h_0$. Assuming that the flow is approximately steady while the depth of water $h(t)$ is much larger than $\epsilon$, show that the time taken for the water to drain is approximately

$$\left(\frac{2}{25}\frac{h_0^5}{\epsilon^4 g}\right)^{1/2}.$$

**8H Statistics**

(a) Define a $100\gamma\%$ *confidence interval* for an unknown parameter $\theta$.

(b) Let $X_1, \ldots, X_n$ be i.i.d. random variables with distribution $N(\mu, 1)$ with $\mu$ unknown. Find a 95% confidence interval for $\mu$.

[You may use the fact that $\Phi(1.96) \simeq 0.975$.]

(c) Let $U_1, U_2$ be independent $U[\theta - 1, \theta + 1]$ with $\theta$ to be estimated. Find a 50% confidence interval for $\theta$.

Suppose that we have two observations $u_1 = 10$ and $u_2 = 11.5$. What might be a better interval to report in this case?

**9H Optimisation**

Consider the following optimisation problem

$$\mathrm{P}: \quad \min f(x) \quad \text{subject to} \quad g(x) = b, \; x \in X.$$

(a) Write down the Lagrangian for this problem. State the Lagrange sufficiency theorem.

(b) Formulate the dual problem. State and prove the weak duality property.

## SECTION II

**10F Linear Algebra**

Let $\alpha : U \to V$ and $\beta : V \to W$ be linear maps between finite-dimensional real vector spaces.

Show that the rank $r(\beta\alpha)$ satisfies $r(\beta\alpha) \leqslant \min(r(\beta), r(\alpha))$. Show also that $r(\beta\alpha) \geqslant r(\alpha) + r(\beta) - \dim V$. For each of these two inequalities, give examples to show that we may or may not have equality.

Now let $V$ have dimension $2n$ and let $\alpha : V \to V$ be a linear map of rank $2n-2$ such that $\alpha^n = 0$. Find the rank of $\alpha^k$ for each $1 \leqslant k \leqslant n-1$.

**11E Groups, Rings and Modules**

Let $R$ be a commutative ring.

(a) Let $N$ be the set of nilpotent elements of $R$, that is,

$$N = \{r \in R \mid r^n = 0 \;\text{ for some } n \in \mathbb{N}\}.$$

Show that $N$ is an ideal of $R$.

(b) Assume $R$ is Noetherian and assume $S \subset R$ is a non-empty subset such that if $s, t \in S$, then $st \in S$. Let $I$ be an ideal of $R$ disjoint from $S$. Show that there is a prime ideal $P$ of $R$ containing $I$ and disjoint from $S$.

(c) Again assume $R$ is Noetherian and let $N$ be as in part (a). Let $\mathcal{P}$ be the set of all prime ideals of $R$. Show that

$$N = \bigcap_{P \in \mathcal{P}} P.$$

**12G Analysis II**

Let $V$ be a real vector space. What is a *norm* on $V$? Show that if $\|-\|$ is a norm on $V$, then the maps $T_v \colon x \mapsto x+v$ (for $v \in V$) and $m_a \colon x \mapsto ax$ (for $a \in \mathbb{R}$) are continuous with respect to the norm.

Let $B \subset V$ be a subset containing 0. Show that there exists at most one norm on $V$ for which $B$ is the open unit ball.

Suppose that $B$ satisfies the following two properties:

- if $v \in V$ is a nonzero vector, then the line $\mathbb{R}v \subset V$ meets $B$ in a set of the form $\{tv \ : \ -\lambda < t < \lambda\}$ for some $\lambda > 0$;
- if $x,\, y \in B$ and $s,\, t > 0$ then $(s+t)^{-1}(sx+ty) \in B$.

Show that there exists a norm $\|-\|_B$ for which $B$ is the open unit ball.

Identify $\|-\|_B$ in the following two cases:

(i) $V = \mathbb{R}^n$, $B = \{\, (x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ -1 < x_i < 1 \ \text{ for all } i \,\}$.

(ii) $V = \mathbb{R}^2$, $B$ the interior of the square with vertices $(\pm 1, 0)$, $(0, \pm 1)$.

Let $C \subset \mathbb{R}^2$ be the set

$$C = \{\, (x_1, x_2) \in \mathbb{R}^2 \ : \ |x_1| < 1,\ |x_2| < 1,\ \text{and } (|x_1|-1)^2 + (|x_2|-1)^2 > 1 \,\}\,.$$

Is there a norm on $\mathbb{R}^2$ for which $C$ is the open unit ball? Justify your answer.

**13A Complex Analysis or Complex Methods**

State the residue theorem.

By considering

$$\oint_C \frac{z^{1/2} \log z}{1+z^2} dz$$

with $C$ a suitably chosen contour in the upper half plane or otherwise, evaluate the real integrals

$$\int_0^\infty \frac{x^{1/2} \log x}{1+x^2} dx$$

and

$$\int_0^\infty \frac{x^{1/2}}{1+x^2} dx$$

where $x^{1/2}$ is taken to be the positive square root.

**14G Geometry**

Let $H = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u} \cdot \mathbf{x} = c\}$ be a hyperplane in $\mathbb{R}^n$, where $\mathbf{u}$ is a unit vector and $c$ is a constant. Show that the reflection map

$$\mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{u} \cdot \mathbf{x} - c)\mathbf{u}$$

is an isometry of $\mathbb{R}^n$ which fixes $H$ pointwise.

Let $\mathbf{p}$, $\mathbf{q}$ be distinct points in $\mathbb{R}^n$. Show that there is a unique reflection $R$ mapping $\mathbf{p}$ to $\mathbf{q}$, and that $R \in O(n)$ if and only if $\mathbf{p}$ and $\mathbf{q}$ are equidistant from the origin.

Show that every isometry of $\mathbb{R}^n$ can be written as a product of at most $n+1$ reflections. Give an example of an isometry of $\mathbb{R}^2$ which cannot be written as a product of fewer than 3 reflections.

**15D Variational Principles**

A proto-planet of mass $m$ in a uniform galactic dust cloud has kinetic and potential energies

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2, \qquad V = kmr^2$$

where $k$ is constant. State Hamilton's principle and use it to determine the equations of motion for the proto-planet.

Write down two conserved quantities of the motion and state why their existence illustrates Noether's theorem.

Determine the Hamiltonian $H(\mathbf{p}, \mathbf{x})$ of this system, where $\mathbf{p} = (p_r, p_\phi)$, $\mathbf{x} = (r, \phi)$ and $(p_r, p_\phi)$ are the conjugate momenta corresponding to $(r, \phi)$.

Write down Hamilton's equations for this system and use them to show that

$$m\ddot{r} = -V'_{\text{eff}}(r), \quad \text{where} \quad V_{\text{eff}}(r) = m\left(\frac{h^2}{2m^2r^2} + kr^2\right)$$

and $h$ is a constant. With the aid of a diagram, explain why there is a stable circular orbit.

**16A Methods**

Laplace's equation for $\phi$ in cylindrical coordinates $(r, \theta, z)$, is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

Use separation of variables to find an expression for the general solution to Laplace's equation in cylindrical coordinates that is $2\pi$-periodic in $\theta$.

Find the bounded solution $\phi(r, \theta, z)$ that satisfies

$$\nabla^2 \phi = 0 \qquad z \geqslant 0, \quad 0 \leqslant r \leqslant 1,$$
$$\phi(1, \theta, z) = e^{-4z}(\cos\theta + \sin 2\theta) + 2\,e^{-z}\,\sin 2\theta.$$

**17B Quantum Mechanics**

(a) The potential for the one-dimensional harmonic oscillator is $V(x) = \frac{1}{2}m\omega^2 x^2$. By considering the associated time-independent Schrödinger equation for the wavefunction $\psi(x)$ with substitutions

$$\xi = \left(\frac{m\omega}{\hbar}\right)^{1/2} x \quad \text{and} \quad \psi(x) = f(\xi)e^{-\xi^2/2},$$

show that the allowed energy levels are given by $E_n = (n + \frac{1}{2})\hbar\omega$ for $n = 0, 1, 2, \ldots$. [You may assume without proof that $f$ must be a polynomial for $\psi$ to be normalisable.]

(b) Consider a particle with charge $q$ and mass $m = 1$ subject to the one-dimensional harmonic oscillator potential $U_0(x) = x^2/2$. You may assume that the normalised ground state of this potential is

$$\psi_0(x) = \left(\frac{1}{\pi\hbar}\right)^{1/4} e^{-x^2/(2\hbar)}.$$

The particle is in the stationary state corresponding to $\psi_0(x)$ when at time $t = t_0$, an electric field of constant strength $E$ is turned on, adding an extra term $U_1(x) = -qEx$ to the harmonic potential.

(i) Using the result of part (a) or otherwise, find the energy levels of the new potential.

(ii) Show that the probability of finding the particle in the ground state immediately after $t_0$ is given by $e^{-q^2E^2/(2\hbar)}$. [You may assume that $\int_{-\infty}^{\infty} e^{-x^2+2Ax}\,dx = \sqrt{\pi}e^{A^2}$.]

**18C Electromagnetism**

In special relativity, the electromagnetic fields can be derived from a 4-vector potential $A^\mu = (\phi/c, \mathbf{A})$. Using the Minkowski metric tensor $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$, state how the electromagnetic tensor $F_{\mu\nu}$ is related to the 4-potential, and write out explicitly the components of both $F_{\mu\nu}$ and $F^{\mu\nu}$ in terms of those of $\mathbf{E}$ and $\mathbf{B}$.

If $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is a Lorentz transformation of the spacetime coordinates from one inertial frame $\mathcal{S}$ to another inertial frame $\mathcal{S}'$, state how $F'^{\mu\nu}$ is related to $F^{\mu\nu}$.

Write down the Lorentz transformation matrix for a boost in standard configuration, such that frame $\mathcal{S}'$ moves relative to frame $\mathcal{S}$ with speed $v$ in the $+x$ direction. Deduce the transformation laws

$$\begin{aligned}
E'_x &= E_x \, , \\
E'_y &= \gamma(E_y - vB_z) \, , \\
E'_z &= \gamma(E_z + vB_y) \, , \\
B'_x &= B_x \, , \\
B'_y &= \gamma\left(B_y + \frac{v}{c^2}E_z\right) \, , \\
B'_z &= \gamma\left(B_z - \frac{v}{c^2}E_y\right) \, ,
\end{aligned}$$

where $\gamma = \left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$.

In frame $\mathcal{S}$, an infinitely long wire of negligible thickness lies along the $x$ axis. The wire carries $n$ positive charges $+q$ per unit length, which travel at speed $u$ in the $+x$ direction, and $n$ negative charges $-q$ per unit length, which travel at speed $u$ in the $-x$ direction. There are no other sources of the electromagnetic field. Write down the electric and magnetic fields in $\mathcal{S}$ in terms of Cartesian coordinates. Calculate the electric field in frame $\mathcal{S}'$, which is related to $\mathcal{S}$ by a boost by speed $v$ as described above. Give an explanation of the physical origin of your expression.

**19C Numerical Analysis**

Define the *linear least-squares problem* for the equation $A\mathbf{x} = \mathbf{b}$, where $A$ is an $m \times n$ matrix with $m > n$, $\mathbf{b} \in \mathbb{R}^m$ is a given vector and $\mathbf{x} \in \mathbb{R}^n$ is an unknown vector.

If $A = QR$, where $Q$ is an orthogonal matrix and $R$ is an upper triangular matrix in standard form, explain why the least-squares problem is solved by minimizing the Euclidean norm $\|R\mathbf{x} - Q^\top \mathbf{b}\|$.

Using the method of Householder reflections, find a QR factorization of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Hence find the solution of the least-squares problem in the case

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix}.$$

**20H Markov Chains**

Let $Y_1, Y_2, \ldots$ be i.i.d. random variables with values in $\{1, 2, \ldots\}$ and $\mathbb{E}[Y_1] = \mu < \infty$. Moreover, suppose that the greatest common divisor of $\{n : \mathbb{P}(Y_1 = n) > 0\}$ is 1. Consider the following process

$$X_n = \inf\{m \geqslant n : Y_1 + \ldots + Y_k = m, \quad \text{for some } k \geqslant 0\} - n.$$

(a) Show that $X$ is a Markov chain and find its transition probabilities.

(b) Let $T_0 = \inf\{n \geqslant 1 : X_n = 0\}$. Find $\mathbb{E}_0[T_0]$.

(c) Find the limit as $n \to \infty$ of $\mathbb{P}(X_n = 0)$. State carefully any theorems from the course that you are using.

**END OF PAPER**