UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part IB

Tuesday, 6 June, 2017 9:00 am to 12:00 pm

## PAPER 1

### Before you begin read these instructions carefully.

*The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.*

*Candidates may obtain credit from attempts on* ***at most four*** *questions from Section I and* ***at most six*** *questions from Section II.*

*Write on* ***one side*** *of the paper only and begin each answer on a separate sheet.*

*Write legibly; otherwise, you place yourself at a grave disadvantage.*

***At the end of the examination:***

*Tie up your answers in separate bundles labelled* ***A, B, …, H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

*Attach a completed gold cover sheet to each bundle.*

*You must also complete a green master cover sheet listing all the questions you have attempted.*

***Every cover sheet must bear your examination number and desk number.***

***STATIONERY REQUIREMENTS***
*Gold cover sheet*
*Green master cover sheet*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

## SECTION I

### 1F Linear Algebra

State and prove the Steinitz Exchange Lemma.

Deduce that, for a subset $S$ of $\mathbb{R}^n$, any two of the following imply the third:

(i) $S$ is linearly independent

(ii) $S$ is spanning

(iii) $S$ has exactly $n$ elements

Let $e_1, e_2$ be a basis of $\mathbb{R}^2$. For which values of $\lambda$ do $\lambda e_1 + e_2,\ e_1 + \lambda e_2$ form a basis of $\mathbb{R}^2$?

### 2A Complex Analysis or Complex Methods

Let $F(z) = u(x,y) + i\,v(x,y)$ where $z = x + i\,y$. Suppose $F(z)$ is an analytic function of $z$ in a domain $\mathcal{D}$ of the complex plane.

Derive the Cauchy-Riemann equations satisfied by $u$ and $v$.

For $u = \dfrac{x}{x^2+y^2}$ find a suitable function $v$ and domain $\mathcal{D}$ such that $F = u + i\,v$ is analytic in $\mathcal{D}$.

### 3G Geometry

Give the definition for the *area* of a hyperbolic triangle with interior angles $\alpha$, $\beta$, $\gamma$.

Let $n \geqslant 3$. Show that the area of a convex hyperbolic $n$-gon with interior angles $\alpha_1, \ldots, \alpha_n$ is $(n-2)\pi - \sum \alpha_i$.

Show that for every $n \geqslant 3$ and for every $A$ with $0 < A < (n-2)\pi$ there exists a regular hyperbolic $n$-gon with area $A$.

**4D Variational Principles**

Derive the Euler-Lagrange equation for the function $u(x, y)$ that gives a stationary value of

$$I[u] = \int_{\mathcal{D}} L\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \, dx \; dy,$$

where $\mathcal{D}$ is a bounded domain in the $(x, y)$-plane and $u$ is fixed on the boundary $\partial\mathcal{D}$.

Find the equation satisfied by the function $u$ that gives a stationary value of

$$I = \int_{\mathcal{D}} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + k^2 u^2\right] dx \; dy,$$

where $k$ is a constant and $u$ is prescribed on $\partial\mathcal{D}$.

**5D Fluid Dynamics**

For each of the flows

(i) $\mathbf{u} = (2xy, x^2 + y^2)$

(ii) $\mathbf{u} = (-2y, -2x)$

determine whether or not the flow is incompressible and/or irrotational. Find the associated velocity potential and/or stream function when appropriate. For either **one** of the flows, sketch the streamlines of the flow, indicating the direction of the flow.

**6C Numerical Analysis**

Given $n+1$ real points $x_0 < x_1 < \cdots < x_n$, define the *Lagrange cardinal polynomials* $\ell_i(x)$, $i = 0, 1, \ldots, n$. Let $p(x)$ be the polynomial of degree $n$ that interpolates the function $f \in C^n[x_0, x_n]$ at these points. Express $p(x)$ in terms of the values $f_i = f(x_i)$, $i = 0, 1, \ldots, n$ and the Lagrange cardinal polynomials.

Define the *divided difference* $f[x_0, x_1, \ldots, x_n]$ and give an expression for it in terms of $f_0, f_1, \ldots, f_n$ and $x_0, x_1, \ldots, x_n$. Prove that

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(\xi)$$

for some number $\xi \in [x_0, x_n]$.

**7H Statistics**

(a) State and prove the Rao–Blackwell theorem.

(b) Let $X_1, \ldots, X_n$ be an independent sample from $Poisson(\lambda)$ with $\theta = e^{-\lambda}$ to be estimated. Show that $Y = 1_{\{0\}}(X_1)$ is an unbiased estimator of $\theta$ and that $T = \sum_i X_i$ is a sufficient statistic.

What is $\mathbb{E}[Y \mid T]$?

**8H Optimisation**

Solve the following linear programming problem using the simplex method:

$$\begin{aligned} \max(x_1 + 2x_2 + x_3) \\ \text{subject to} \quad x_1, x_2, x_3 \geqslant 0 \\ x_1 + x_2 + 2x_3 \leqslant 10 \\ 2x_1 + x_2 + 3x_3 \leqslant 15. \end{aligned}$$

Suppose we now subtract $\Delta \in [0, 10]$ from the right hand side of the last two constraints. Find the new optimal value.

## SECTION II

**9F Linear Algebra**

Let $U$ and $V$ be finite-dimensional real vector spaces, and let $\alpha : U \to V$ be a surjective linear map. Which of the following are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) There is a linear map $\beta : V \to U$ such that $\beta\alpha$ is the identity map on $U$.

(ii) There is a linear map $\beta : V \to U$ such that $\alpha\beta$ is the identity map on $V$.

(iii) There is a subspace $W$ of $U$ such that the restriction of $\alpha$ to $W$ is an isomorphism from $W$ to $V$.

(iv) If $X$ and $Y$ are subspaces of $U$ with $U = X \oplus Y$ then $V = \alpha(X) \oplus \alpha(Y)$.

(v) If $X$ and $Y$ are subspaces of $U$ with $V = \alpha(X) \oplus \alpha(Y)$ then $U = X \oplus Y$.

**10E Groups, Rings and Modules**

(a) State Sylow's theorem.

(b) Let $G$ be a finite simple non-abelian group. Let $p$ be a prime number. Show that if $p$ divides $|G|$, then $|G|$ divides $n_p!/2$ where $n_p$ is the number of Sylow $p$-subgroups of $G$.

(c) Let $G$ be a group of order 48. Show that $G$ is not simple. Find an example of $G$ which has no normal Sylow 2-subgroup.

**11G Analysis II**

What does it mean to say that a real-valued function on a metric space is *uniformly continuous*? Show that a continuous function on a closed interval in $\mathbb{R}$ is uniformly continuous.

What does it mean to say that a real-valued function on a metric space is *Lipschitz*? Show that if a function is Lipschitz then it is uniformly continuous.

Which of the following statements concerning continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ are true and which are false? Justify your answers.

(i) If $f$ is bounded then $f$ is uniformly continuous.

(ii) If $f$ is differentiable and $f'$ is bounded, then $f$ is uniformly continuous.

(iii) There exists a sequence of uniformly continuous functions converging pointwise to $f$.

**12E Metric and Topological Spaces**

Consider $\mathbb{R}$ and $\mathbb{R}^2$ with their usual Euclidean topologies.

(a) Show that a non-empty subset of $\mathbb{R}$ is connected if and only if it is an interval. Find a compact subset $K \subset \mathbb{R}$ for which $\mathbb{R} \setminus K$ has infinitely many connected components.

(b) Let $T$ be a countable subset of $\mathbb{R}^2$. Show that $\mathbb{R}^2 \setminus T$ is path-connected. Deduce that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$.

**13A Complex Analysis or Complex Methods**

(a) Let $f(z)$ be defined on the complex plane such that $zf(z) \to 0$ as $|z| \to \infty$ and $f(z)$ is analytic on an open set containing $\mathrm{Im}(z) \geqslant -c$, where $c$ is a positive real constant.

Let $C_1$ be the horizontal contour running from $-\infty - ic$ to $+\infty - ic$ and let

$$F(\lambda) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z - \lambda} dz.$$

By evaluating the integral, show that $F(\lambda)$ is analytic for $\mathrm{Im}(\lambda) > -c$.

(b) Let $g(z)$ be defined on the complex plane such that $z\, g(z) \to 0$ as $|z| \to \infty$ with $\mathrm{Im}(z) \geqslant -c$. Suppose $g(z)$ is analytic at all points except $z = \alpha_+$ and $z = \alpha_-$ which are simple poles with $\mathrm{Im}(\alpha_+) > c$ and $\mathrm{Im}(\alpha_-) < -c$.

Let $C_2$ be the horizontal contour running from $-\infty + ic$ to $+\infty + ic$, and let

$$H(\lambda) = \frac{1}{2\pi i} \int_{C_1} \frac{g(z)}{z - \lambda} dz,$$
$$J(\lambda) = -\frac{1}{2\pi i} \int_{C_2} \frac{g(z)}{z - \lambda} dz.$$

(i) Show that $H(\lambda)$ is analytic for $\mathrm{Im}(\lambda) > -c$.

(ii) Show that $J(\lambda)$ is analytic for $\mathrm{Im}(\lambda) < c$.

(iii) Show that if $-c < \mathrm{Im}(\lambda) < c$ then $H(\lambda) + J(\lambda) = g(\lambda)$.

[You should be careful to make sure you consider all points in the required regions.]

**14B Methods**

(a)

(i) Compute the Fourier transform $\tilde{h}(k)$ of $h(x) = e^{-a|x|}$, where $a$ is a real positive constant.

(ii) Consider the boundary value problem

$$-\frac{d^2u}{dx^2} + \omega^2 u = e^{-|x|} \quad \text{on } -\infty < x < \infty$$

with real constant $\omega \neq \pm 1$ and boundary condition $u(x) \to 0$ as $|x| \to \infty$.
Find the Fourier transform $\tilde{u}(k)$ of $u(x)$ and hence solve the boundary value problem. You should clearly state any properties of the Fourier transform that you use.

(b) Consider the wave equation

$$v_{tt} = v_{xx} \quad \text{on} \quad -\infty < x < \infty \text{ and } t > 0$$

with initial conditions

$$v(x,0) = f(x) \qquad v_t(x,0) = g(x).$$

Show that the Fourier transform $\tilde{v}(k,t)$ of the solution $v(x,t)$ with respect to the variable $x$ is given by

$$\tilde{v}(k,t) = \tilde{f}(k)\cos kt + \frac{\tilde{g}(k)}{k}\sin kt$$

where $\tilde{f}(k)$ and $\tilde{g}(k)$ are the Fourier transforms of the initial conditions.
Starting from $\tilde{v}(k,t)$ derive d'Alembert's solution for the wave equation:

$$v(x,t) = \frac{1}{2}\Big(f(x-t) + f(x+t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)d\xi\,.$$

You should state clearly any properties of the Fourier transform that you use.

**15B Quantum Mechanics**

Consider the time-independent Schrödinger equation in one dimension for a particle of mass $m$ with potential $V(x)$.

(a) Show that if the potential is an even function then any non-degenerate stationary state has definite parity.

(b) A particle of mass $m$ is subject to the potential $V(x)$ given by

$$V(x) = -\lambda\Big(\delta(x-a) + \delta(x+a)\Big)$$

where $\lambda$ and $a$ are real positive constants and $\delta(x)$ is the Dirac delta function.

Derive the conditions satisfied by the wavefunction $\psi(x)$ around the points $x = \pm a$.

Show (using a graphical method or otherwise) that there is a bound state of even parity for any $\lambda > 0$, and that there is an odd parity bound state only if $\lambda > \hbar^2/(2ma)$. [*Hint: You may assume without proof that the functions* $x \tanh x$ *and* $x \coth x$ *are monotonically increasing for* $x > 0$.]

**16C Electromagnetism**

Write down Maxwell's equations for the electric field $\mathbf{E}(\mathbf{x}, t)$ and the magnetic field $\mathbf{B}(\mathbf{x}, t)$ in a vacuum. Deduce that both $\mathbf{E}$ and $\mathbf{B}$ satisfy a wave equation, and relate the wave speed $c$ to the physical constants $\epsilon_0$ and $\mu_0$.

Verify that there exist plane-wave solutions of the form

$$\mathbf{E}(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{e}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right],$$
$$\mathbf{B}(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{b}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right],$$

where $\mathbf{e}$ and $\mathbf{b}$ are constant complex vectors, $\mathbf{k}$ is a constant real vector and $\omega$ is a real constant. Derive the dispersion relation that relates the angular frequency $\omega$ of the wave to the wavevector $\mathbf{k}$, and give the algebraic relations between the vectors $\mathbf{e}$, $\mathbf{b}$ and $\mathbf{k}$ implied by Maxwell's equations.

Let $\mathbf{n}$ be a constant real unit vector. Suppose that a perfect conductor occupies the region $\mathbf{n}\cdot\mathbf{x} < 0$ with a plane boundary $\mathbf{n}\cdot\mathbf{x} = 0$. In the vacuum region $\mathbf{n}\cdot\mathbf{x} > 0$, a plane electromagnetic wave of the above form, with $\mathbf{k}\cdot\mathbf{n} < 0$, is incident on the plane boundary. Write down the boundary conditions on $\mathbf{E}$ and $\mathbf{B}$ at the surface of the conductor. Show that Maxwell's equations and the boundary conditions are satisfied if the solution in the vacuum region is the sum of the incident wave given above and a reflected wave of the form

$$\mathbf{E}'(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{e}'\, e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega t)}\right],$$
$$\mathbf{B}'(\mathbf{x}, t) = \mathrm{Re}\left[\mathbf{b}'\, e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega t)}\right],$$

where

$$\mathbf{e}' = -\mathbf{e} + 2(\mathbf{n}\cdot\mathbf{e})\mathbf{n},$$
$$\mathbf{b}' = \mathbf{b} - 2(\mathbf{n}\cdot\mathbf{b})\mathbf{n},$$
$$\mathbf{k}' = \mathbf{k} - 2(\mathbf{n}\cdot\mathbf{k})\mathbf{n}.$$

**17D Fluid Dynamics**

A layer of thickness $h$ of fluid of density $\rho$ and dynamic viscosity $\mu$ flows steadily down and parallel to a rigid plane inclined at angle $\alpha$ to the horizontal. Wind blows over the surface of the fluid and exerts a stress $S$ on the surface of the fluid in the upslope direction.

(a) Draw a diagram of this situation, including indications of the applied stresses and body forces, a suitable coordinate system and a representation of the expected velocity profile.

(b) Write down the equations and boundary conditions governing the flow, with a brief description of each, paying careful attention to signs. Solve these equations to determine the pressure and velocity fields.

(c) Determine the volume flux and show that there is no net flux if

$$S = \frac{2}{3}\rho g h \sin\alpha.$$

Draw a sketch of the corresponding velocity profile.

(d) Determine the value of $S$ for which the shear stress on the rigid plane is zero and draw a sketch of the corresponding velocity profile.

**18C Numerical Analysis**

A three-stage explicit Runge–Kutta method for solving the autonomous ordinary differential equation

$$\frac{dy}{dt} = f(y)$$

is given by

$$y_{n+1} = y_n + h(b_1k_1 + b_2k_2 + b_3k_3)\,,$$

where

$$\begin{aligned} k_1 &= f(y_n)\,, \\ k_2 &= f(y_n + ha_1k_1)\,, \\ k_3 &= f(y_n + h(a_2k_1 + a_3k_2)) \end{aligned}$$

and $h > 0$ is the time-step. Derive sufficient conditions on the coefficients $b_1$, $b_2$, $b_3$, $a_1$, $a_2$ and $a_3$ for the method to be of third order.

Assuming that these conditions hold, verify that $-\frac{5}{2}$ belongs to the linear stability domain of the method.

**19H Statistics**

(a) Give the definitions of a *sufficient* and a *minimal sufficient* statistic $T$ for an unknown parameter $\theta$.

Let $X_1, X_2, \ldots, X_n$ be an independent sample from the geometric distribution with success probability $1/\theta$ and mean $\theta > 1$, i.e. with probability mass function

$$p(m) = \frac{1}{\theta}\left(1 - \frac{1}{\theta}\right)^{m-1} \quad \text{for } m = 1, 2, \ldots .$$

Find a minimal sufficient statistic for $\theta$. Is your statistic a biased estimator of $\theta$?

[You may use results from the course provided you state them clearly.]

(b) Define the *bias* of an estimator. What does it mean for an estimator to be *unbiased*?

Suppose that $Y$ has the truncated Poisson distribution with probability mass function

$$p(y) = (e^{\theta} - 1)^{-1} \cdot \frac{\theta^y}{y!} \quad \text{for } y = 1, 2, \ldots .$$

Show that the only unbiased estimator $T$ of $1 - e^{-\theta}$ based on $Y$ is obtained by taking $T = 0$ if $Y$ is odd and $T = 2$ if $Y$ is even.

Is this a useful estimator? Justify your answer.

**20H Markov Chains**

A rich and generous man possesses $n$ pounds. Some poor cousins arrive at his mansion. Being generous he decides to give them money. On day 1, he chooses uniformly at random an integer between $n-1$ and 1 inclusive and gives it to the first cousin. Then he is left with $x$ pounds. On day 2, he chooses uniformly at random an integer between $x-1$ and 1 inclusive and gives it to the second cousin and so on. If $x = 1$ then he does not give the next cousin any money. His choices of the uniform numbers are independent. Let $X_i$ be his fortune at the end of day $i$.

Show that $X$ is a Markov chain and find its transition probabilities.

Let $\tau$ be the first time he has 1 pound left, i.e. $\tau = \min\{i \geqslant 1 \ : \ X_i = 1\}$. Show that

$$\mathbb{E}[\tau] = \sum_{i=1}^{n-1} \frac{1}{i}.$$

## END OF PAPER