UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Wednesday, 7 June, 2017 1:30 pm to 4:30 pm

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold Cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Numbers and Sets

(a) Show that for all positive integers z and n, either $z^{2n} \equiv 0 \pmod 3$ or $z^{2n} \equiv 1 \pmod 3$.

(b) If the positive integers x, y, z satisfy $x^2 + y^2 = z^2$, show that at least one of x and y must be divisible by 3. Can both x and y be odd?

2D Numbers and Sets

(a) Give the definitions of *relation* and *equivalence relation* on a set S.

(b) Let Σ be the set of ordered pairs (A, f) where A is a non-empty subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$. Let $\mathcal{R}$ be the relation on Σ defined by requiring $(A, f)\,\mathcal{R}\,(B, g)$ if the following two conditions hold:

(i) $(A \setminus B) \cup (B \setminus A)$ is finite and

(ii) there is a finite set $F \subset A \cap B$ such that $f(x) = g(x)$ for all $x \in A \cap B \setminus F$.

Show that $\mathcal{R}$ is an equivalence relation on Σ.

3A Dynamics and Relativity

Consider a system of particles with masses m_i and position vectors $\mathbf{x}_i$. Write down the definition of the position of the *centre of mass* $\mathbf{R}$ of the system. Let T be the total kinetic energy of the system. Show that

$$T = \frac{1}{2} M \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \frac{1}{2} \sum_i m_i \dot{\mathbf{y}}_i \cdot \dot{\mathbf{y}}_i \,,$$

where M is the total mass and $\mathbf{y}_i$ is the position vector of particle i with respect to $\mathbf{R}$.

The particles are connected to form a rigid body which rotates with angular speed ω about an axis $\mathbf{n}$ through $\mathbf{R}$, where $\mathbf{n} \cdot \mathbf{n} = 1$. Show that

$$T = \frac{1}{2} M \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \frac{1}{2} I \omega^2 \,,$$

where $I = \sum_i I_i$ and I_i is the moment of inertia of particle i about $\mathbf{n}$.

4A Dynamics and Relativity

A tennis ball of mass m is projected vertically upwards with initial speed u_0 and reaches its highest point at time T. In addition to uniform gravity, the ball experiences air resistance, which produces a frictional force of magnitude αv, where v is the ball's speed and α is a positive constant. Show by dimensional analysis that T can be written in the form

$$T = \frac{m}{\alpha} f(\lambda)$$

for some function f of a dimensionless quantity λ.

Use the equation of motion of the ball to find $f(\lambda)$.

SECTION II

5D Numbers and Sets

(a) State and prove the Fermat–Euler Theorem. Deduce Fermat's Little Theorem. State Wilson's Theorem.

(b) Let p be an odd prime. Prove that $X^2 \equiv -1 \pmod{p}$ is solvable if and only if $p \equiv 1 \pmod{4}$.

(c) Let p be prime. If h and k are non-negative integers with $h + k = p - 1$, prove that $h!k! + (-1)^h \equiv 0 \pmod{p}$.

6D Numbers and Sets

(a) Define what it means for a set to be *countable*.

(b) Let A be an infinite subset of the set of natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$. Prove that there is a bijection $f : \mathbb{N} \to A$.

(c) Let A_n be the set of natural numbers whose decimal representation ends with exactly $n - 1$ zeros. For example, $71 \in A_1$, $70 \in A_2$ and $15000 \in A_4$. By applying the result of part (b) with $A = A_n$, construct a bijection $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Deduce that the set of rationals is countable.

(d) Let A be an infinite set of positive real numbers. If every sequence $(a_j)_{j=1}^{\infty}$ of distinct elements with $a_j \in A$ for each j has the property that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} a_j = 0,$$

prove that A is countable.

[*You may assume without proof that a countable union of countable sets is countable.*]

7D Numbers and Sets

(a) For positive integers n, m, k with $k \leqslant n$, show that

$$\binom{n}{k}\left(\frac{k}{n}\right)^m = \binom{n-1}{k-1}\sum_{\ell=0}^{m-1} a_{n,m,\ell}\left(\frac{k-1}{n-1}\right)^{m-1-\ell}$$

giving an explicit formula for $a_{n,m,\ell}$. [You may wish to consider the expansion of $\left(\frac{k-1}{n-1}+\frac{1}{n-1}\right)^{m-1}$.]

(b) For a function $f : [0,1] \to \mathbb{R}$ and each integer $n \geqslant 1$, the function $B_n(f) : [0,1] \to \mathbb{R}$ is defined by

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

For any integer $m \geqslant 0$ let $f_m(x) = x^m$. Show that $B_n(f_0)(x) = 1$ and $B_n(f_1)(x) = x$ for all $n \geqslant 1$ and $x \in [0,1]$.

Show that for each integer $m \geqslant 0$ and each $x \in [0,1]$,

$$B_n(f_m)(x) \to f_m(x) \quad \text{as} \quad n \to \infty\,.$$

Deduce that for each integer $m \geqslant 0$,

$$\lim_{n\to\infty}\frac{1}{4^n}\sum_{k=0}^{2n}\left(\frac{k}{n}\right)^m\binom{2n}{k} = 1.$$

8D Numbers and Sets

Let $(a_k)_{k=1}^{\infty}$ be a sequence of real numbers.

(a) Define what it means for $(a_k)_{k=1}^{\infty}$ to converge. Define what it means for the series $\sum_{k=1}^{\infty} a_k$ to converge.

Show that if $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k)_{k=1}^{\infty}$ converges to 0.

If $(a_k)_{k=1}^{\infty}$ converges to $a \in \mathbb{R}$, show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = a\,.$$

(b) Suppose $a_k > 0$ for every k. Let $u_n = \sum_{k=1}^{n}\left(a_k + \frac{1}{a_k}\right)$ and $v_n = \sum_{k=1}^{n}\left(a_k - \frac{1}{a_k}\right)$.

Show that $(u_n)_{n=1}^{\infty}$ does not converge.

Give an example of a sequence $(a_k)_{k=1}^{\infty}$ with $a_k > 0$ and $a_k \neq 1$ for every k such that $(v_n)_{n=1}^{\infty}$ converges.

If $(v_n)_{n=1}^{\infty}$ converges, show that $\frac{u_n}{n} \to 2$.

9A Dynamics and Relativity

(a) A photon with energy E_1 in the laboratory frame collides with an electron of rest mass m that is initially at rest in the laboratory frame. As a result of the collision the photon is deflected through an angle θ as measured in the laboratory frame and its energy changes to E_2.

Derive an expression for $\frac{1}{E_2} - \frac{1}{E_1}$ in terms of θ, m and c.

(b) A deuterium atom with rest mass m_1 and energy E_1 in the laboratory frame collides with another deuterium atom that is initially at rest in the laboratory frame. The result of this collision is a proton of rest mass m_2 and energy E_2, and a tritium atom of rest mass m_3. Show that, if the proton is emitted perpendicular to the incoming trajectory of the deuterium atom as measured in the laboratory frame, then

$$m_3^2 = m_2^2 + 2\left(m_1 + \frac{E_1}{c^2}\right)\left(m_1 - \frac{E_2}{c^2}\right).$$

10A Dynamics and Relativity

A particle of unit mass moves under the influence of a central force. By considering the components of the acceleration in polar coordinates (r,θ) prove that the magnitude of the angular momentum is conserved. [You may use $\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}$.]

Now suppose that the central force is derived from the potential k/r, where k is a constant.

(a) Show that the total energy of the particle can be written in the form

$$E = \tfrac{1}{2}\dot{r}^2 + V_{\text{eff}}(r).$$

Sketch $V_{\text{eff}}(r)$ in the cases $k > 0$ and $k < 0$.

(b) The particle is projected from a very large distance from the origin with speed v and impact parameter b. [The *impact parameter* is the distance of closest approach to the origin in absence of any force.]

(i) In the case $k < 0$, sketch the particle's trajectory and find the shortest distance p between the particle and the origin, and the speed u of the particle when $r = p$.

(ii) In the case $k > 0$, sketch the particle's trajectory and find the corresponding shortest distance $\widetilde{p}$ between the particle and the origin, and the speed $\widetilde{u}$ of the particle when $r = \widetilde{p}$.

(iii) Find $p\widetilde{p}$ and $u\widetilde{u}$ in terms of b and v. [In answering part (iii) you should assume that $|k|$ is the same in parts (i) and (ii).]

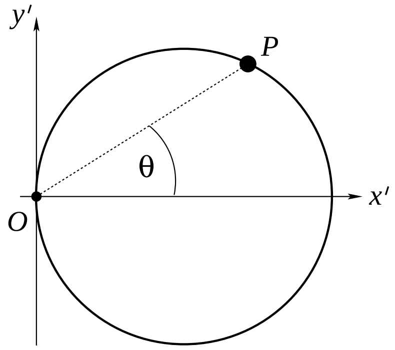
11A Dynamics and Relativity

(a) Consider an inertial frame S, and a frame S' which rotates with constant angular velocity $\boldsymbol{\omega}$ relative to S. The two frames share a common origin. Identify each term in the equation

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{S'} = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_S - 2\,\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{S'} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

(b) A small bead P of unit mass can slide without friction on a circular hoop of radius a. The hoop is horizontal and rotating with constant angular speed ω about a fixed vertical axis through a point O on its circumference.

(i) Using Cartesian axes in the rotating frame S', with origin at O and x'-axis along the diameter of the hoop through O, write down the position vector of P in terms of a and the angle θ shown in the diagram $(-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi)$.



(ii) Working again in the rotating frame, find, in terms of a, θ, $\dot{\theta}$ and ω, an expression for the horizontal component of the force exerted by the hoop on the bead.

(iii) For what value of θ is the bead in stable equilibrium? Find the frequency of small oscillations of the bead about that point.

12A Dynamics and Relativity

(a) A rocket moves in a straight line with speed $v(t)$ and is subject to no external forces. The rocket is composed of a body of mass M and fuel of mass $m(t)$, which is burnt at constant rate α and the exhaust is ejected with constant speed u relative to the rocket. Show that

$$(M+m)\frac{dv}{dt} - \alpha u = 0\,.$$

Show that the speed of the rocket when all its fuel is burnt is

$$v_0 + u \log\left(1 + \frac{m_0}{M}\right)\,,$$

where v_0 and m_0 are the speed of the rocket and the mass of the fuel at $t = 0$.

(b) A two-stage rocket moves in a straight line and is subject to no external forces. The rocket is initially at rest. The masses of the bodies of the two stages are kM and $(1-k)M$, with $0 \leqslant k \leqslant 1$, and they initially carry masses km_0 and $(1-k)m_0$ of fuel. Both stages burn fuel at a constant rate α when operating and the exhaust is ejected with constant speed u relative to the rocket. The first stage operates first, until all its fuel is burnt. The body of the first stage is then detached with negligible force and the second stage ignites.

Find the speed of the second stage when all its fuel is burnt. For $0 \leqslant k < 1$ compare it with the speed of the rocket in part (a) in the case $v_0 = 0$. Comment on the case $k = 1$.

END OF PAPER