UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 5 June, 2017 9:00 am to 12:00 pm

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold Cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups

Let w_1, w_2, w_3 be distinct elements of $\mathbb{C} \cup \{\infty\}$. Write down the Möbius map f that sends w_1, w_2, w_3 to $\infty, 0, 1$, respectively. [*Hint: You need to consider four cases.*]

Now let w_4 be another element of $\mathbb{C} \cup \{\infty\}$ distinct from w_1, w_2, w_3. Define the *cross-ratio* $[w_1, w_2, w_3, w_4]$ in terms of f.

Prove that there is a circle or line through w_1, w_2, w_3 and w_4 if and only if the cross-ratio $[w_1, w_2, w_3, w_4]$ is real.

[*You may assume without proof that Möbius maps map circles and lines to circles and lines and also that there is a unique circle or line through any three distinct points of* $\mathbb{C} \cup \{\infty\}$.]

2E Groups

What does it mean to say that H is a *normal subgroup* of the group G? For a normal subgroup H of G define the quotient group G/H. [You do not need to verify that G/H is a group.]

State the Isomorphism Theorem.

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{R},\ ad \neq 0 \right\}$$

be the group of 2×2 invertible upper-triangular real matrices. By considering a suitable homomorphism, show that the subset

$$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\}$$

of G is a normal subgroup of G and identify the quotient G/H.

3B Vector Calculus

Use the change of variables $x = r\cosh\theta$, $y = r\sinh\theta$ to evaluate

$$\int_A y\,dx\,dy\,,$$

where A is the region of the xy-plane bounded by the two line segments:

$y = 0, \quad 0 \leqslant x \leqslant 1\,;$

$5y = 3x, \quad 0 \leqslant x \leqslant \frac{5}{4}\,;$

and the curve

$x^2 - y^2 = 1, \quad x \geqslant 1\,.$

4B Vector Calculus

(a) The two sets of basis vectors $\mathbf{e}_i$ and $\mathbf{e}'_i$ (where $i = 1, 2, 3$) are related by

$$\mathbf{e}'_i = R_{ij}\mathbf{e}_j\,,$$

where R_{ij} are the entries of a rotation matrix. The components of a vector $\mathbf{v}$ with respect to the two bases are given by

$$\mathbf{v} = v_i\mathbf{e}_i = v'_i\mathbf{e}'_i\,.$$

Derive the relationship between v_i and v'_i.

(b) Let $\mathbf{T}$ be a 3×3 array defined in each (right-handed orthonormal) basis. Using part (a), state and prove the quotient theorem as applied to $\mathbf{T}$.

SECTION II

5E Groups

Let N be a normal subgroup of a finite group G of prime index $p = |G : N|$.

By considering a suitable homomorphism, show that if H is a subgroup of G that is not contained in N, then $H \cap N$ is a normal subgroup of H of index p.

Let C be a conjugacy class of G that is contained in N. Prove that C is either a conjugacy class in N or is the disjoint union of p conjugacy classes in N.

[*You may use standard theorems without proof.*]

6E Groups

State Lagrange's theorem. Show that the order of an element x in a finite group G is finite and divides the order of G.

State Cauchy's theorem.

List all groups of order 8 up to isomorphism. Carefully justify that the groups on your list are pairwise non-isomorphic and that any group of order 8 is isomorphic to one on your list. [You may use without proof the Direct Product Theorem and the description of standard groups in terms of generators satisfying certain relations.]

7E Groups

(a) Let G be a finite group acting on a finite set X. State the Orbit-Stabiliser theorem. [Define the terms used.] Prove that

$$\sum_{x \in X} |\mathrm{Stab}(x)| = n|G| \ ,$$

where n is the number of distinct orbits of X under the action of G.

Let $S = \{(g,x) \in G \times X : g \cdot x = x\}$, and for $g \in G$, let $\mathrm{Fix}(g) = \{x \in X : g \cdot x = x\}$. Show that

$$|S| = \sum_{x \in X} |\mathrm{Stab}(x)| = \sum_{g \in G} |\mathrm{Fix}(g)| \ ,$$

and deduce that

$$n = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)| \ . \qquad (*)$$

(b) Let H be the group of rotational symmetries of the cube. Show that H has 24 elements. [If your proof involves calculating stabilisers, then you must carefully verify such calculations.]

Using $(*)$, find the number of distinct ways of colouring the faces of the cube red, green and blue, where two colourings are distinct if one cannot be obtained from the other by a rotation of the cube. [A colouring need not use all three colours.]

8E Groups

Prove that every element of the symmetric group S_n is a product of transpositions. [You may assume without proof that every permutation is the product of disjoint cycles.]

(a) Define the *sign* of a permutation in S_n, and prove that it is well defined. Define the *alternating group* A_n.

(b) Show that S_n is generated by the set $\{(1\ 2), (1\ 2\ 3\ \ldots\ n)\}$.

Given $1 \leqslant k < n$, prove that the set $\{(1\ 1{+}k), (1\ 2\ 3\ \ldots\ n)\}$ generates S_n if and only if k and n are coprime.

9B Vector Calculus

(a) The time-dependent vector field $\mathbf{F}$ is related to the vector field $\mathbf{B}$ by

$$\mathbf{F}(\mathbf{x},t) = \mathbf{B}(\mathbf{z})\,,$$

where $\mathbf{z} = t\mathbf{x}$. Show that

$$(\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{F} = t\,\frac{\partial \mathbf{F}}{\partial t}\,.$$

(b) The vector fields $\mathbf{B}$ and $\mathbf{A}$ satisfy $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. Show that $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$.

(c) The vector field $\mathbf{B}$ satisfies $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. Show that

$$\mathbf{B}(\mathbf{x}) = \boldsymbol{\nabla}\times\big(\mathbf{D}(\mathbf{x})\times\mathbf{x}\big)\,,$$

where

$$\mathbf{D}(\mathbf{x}) = \int_0^1 t\,\mathbf{B}(t\mathbf{x})\,dt\,.$$

10B Vector Calculus

By a suitable choice of $\mathbf{u}$ in the divergence theorem

$$\int_V \boldsymbol{\nabla}\cdot\mathbf{u}\,dV = \int_S \mathbf{u}\cdot d\mathbf{S}\,,$$

show that

$$\int_V \boldsymbol{\nabla}\phi\,dV = \int_S \phi\,d\mathbf{S} \qquad (*)$$

for any continuously differentiable function ϕ.

For the curved surface of the cone

$$\mathbf{x} = (r\cos\theta, r\sin\theta, \sqrt{3}\,r), \qquad 0 \leqslant \sqrt{3}\,r \leqslant 1, \quad 0 \leqslant \theta \leqslant 2\pi,$$

show that $d\mathbf{S} = (\sqrt{3}\,\cos\theta, \sqrt{3}\,\sin\theta, -1)\,r\,dr\,d\theta$.

Verify that $(*)$ holds for this cone and $\phi(x,y,z) = z^2$.

11B Vector Calculus

(a) Let $\mathbf{x} = \mathbf{r}(s)$ be a smooth curve parametrised by arc length s. Explain the meaning of the terms in the equation

$$\frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n}\,,$$

where $\kappa(s)$ is the curvature of the curve.

Now let $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. Show that there is a scalar $\tau(s)$ (the torsion) such that

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n}$$

and derive an expression involving κ and τ for $\dfrac{d\mathbf{n}}{ds}$.

(b) Given a (nowhere zero) vector field $\mathbf{F}$, the *field lines*, or *integral curves*, of $\mathbf{F}$ are the curves parallel to $\mathbf{F}(\mathbf{x})$ at each point $\mathbf{x}$. Show that the curvature κ of the field lines of $\mathbf{F}$ satisfies

$$\frac{\mathbf{F} \times (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{F}}{F^3} = \pm\kappa\mathbf{b}\,, \qquad (*)$$

where $F = |\mathbf{F}|$.

(c) Use $(*)$ to find an expression for the curvature at the point (x, y, z) of the field lines of $\mathbf{F}(x, y, z) = (x, y, -z)$.

12B Vector Calculus

Let S be a piecewise smooth closed surface in $\mathbb{R}^3$ which is the boundary of a volume V.

(a) The smooth functions ϕ and ϕ_1 defined on $\mathbb{R}^3$ satisfy

$$\nabla^2\phi = \nabla^2\phi_1 = 0$$

in V and $\phi(\mathbf{x}) = \phi_1(\mathbf{x}) = f(\mathbf{x})$ on S. By considering an integral of $\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\psi$, where $\psi = \phi - \phi_1$, show that $\phi_1 = \phi$.

(b) The smooth function u defined on $\mathbb{R}^3$ satisfies $u(\mathbf{x}) = f(\mathbf{x}) + C$ on S, where f is the function in part (a) and C is constant. Show that

$$\int_V \boldsymbol{\nabla}u\cdot\boldsymbol{\nabla}u\,dV \geqslant \int_V \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi\,dV$$

where ϕ is the function in part (a). When does equality hold?

(c) The smooth function $w(\mathbf{x}, t)$ satisfies

$$\nabla^2 w = \frac{\partial w}{\partial t}$$

in V and $\dfrac{\partial w}{\partial t} = 0$ on S for all t. Show that

$$\frac{d}{dt}\int_V \boldsymbol{\nabla}w\cdot\boldsymbol{\nabla}w\,dV \leqslant 0$$

with equality only if $\nabla^2 w = 0$ in V.

END OF PAPER