UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Friday, 2 June, 2017 1:30 pm to 4:30 pm

PAPER 2

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on **one** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold Cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1C Differential Equations

(a) The numbers $z_1, z_2, \ldots$ satisfy

$$z_{n+1} = z_n + c_n \qquad (n \geqslant 1),$$

where $c_1, c_2, \ldots$ are given constants. Find z_{n+1} in terms of $c_1, c_2, \ldots, c_n$ and z_1.

(b) The numbers $x_1, x_2, \ldots$ satisfy

$$x_{n+1} = a_n x_n + b_n \qquad (n \geqslant 1),$$

where $a_1, a_2, \ldots$ are given non-zero constants and $b_1, b_2, \ldots$ are given constants. Let $z_1 = x_1$ and $z_{n+1} = x_{n+1}/U_n$, where $U_n = a_1 a_2 \cdots a_n$. Calculate $z_{n+1} - z_n$, and hence find x_{n+1} in terms of x_1, $b_1, \ldots, b_n$ and $U_1, \ldots, U_n$.

2C Differential Equations

Consider the function

$$f(x,y) = \frac{x}{y} + \frac{y}{x} - \frac{(x-y)^2}{a^2}$$

defined for $x > 0$ and $y > 0$, where a is a non-zero real constant. Show that (λ, λ) is a stationary point of f for each $\lambda > 0$. Compute the Hessian and its eigenvalues at (λ, λ).

3F Probability

Let X be a non-negative integer-valued random variable such that $0 < \mathbb{E}(X^2) < \infty$. Prove that

$$\frac{\mathbb{E}(X)^2}{\mathbb{E}(X^2)} \leqslant \mathbb{P}(X > 0) \leqslant \mathbb{E}(X).$$

[You may use any standard inequality.]

4F Probability

Let X and Y be real-valued random variables with joint density function

$$f(x,y) = \begin{cases} xe^{-x(y+1)} & \text{if } x \geqslant 0 \text{ and } y \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the conditional probability density function of Y given X.

(ii) Find the expectation of Y given X.

SECTION II

5C Differential Equations

The current $I(t)$ at time t in an electrical circuit subject to an applied voltage $V(t)$ obeys the equation

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}\,,$$

where R, L and C are the constant resistance, inductance and capacitance of the circuit with $R \geqslant 0$, $L > 0$ and $C > 0$.

(a) In the case $R = 0$ and $V(t) = 0$, show that there exist time-periodic solutions of frequency ω_0, which you should find.

(b) In the case $V(t) = H(t)$, the Heaviside function, calculate, subject to the condition

$$R^2 > \frac{4L}{C}\,,$$

the current for $t \geqslant 0$, assuming it is zero for $t < 0$.

(c) If $R > 0$ and $V(t) = \sin\omega_0 t$, where ω_0 is as in part (a), show that there is a time-periodic solution $I_0(t)$ of period $T = 2\pi/\omega_0$ and calculate its maximum value I_M.

(i) Calculate the energy dissipated in each period, i.e., the quantity

$$D = \int_0^T RI_0(t)^2\,dt\,.$$

Show that the quantity defined by

$$Q = \frac{2\pi}{D} \times \frac{LI_M^2}{2}$$

satisfies $Q\omega_0 RC = 1$.

(ii) Write down explicitly the general solution $I(t)$ for all $R > 0$, and discuss the relevance of $I_0(t)$ to the large time behaviour of $I(t)$.

6C Differential Equations

(a) Consider the system

$$\frac{dx}{dt} = x(1-x) - xy$$

$$\frac{dy}{dt} = \frac{1}{8}\, y(4x-1)$$

for $x(t) \geqslant 0$, $y(t) \geqslant 0$. Find the critical points, determine their type and explain, with the help of a diagram, the behaviour of solutions for large positive times t.

(b) Consider the system

$$\frac{dx}{dt} = y + (1-x^2-y^2)x$$

$$\frac{dy}{dt} = -x + (1-x^2-y^2)y$$

for $(x(t), y(t)) \in \mathbb{R}^2$. Rewrite the system in polar coordinates by setting $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$, and hence describe the behaviour of solutions for large positive and large negative times.

7C Differential Equations

Let y_1 and y_2 be two solutions of the differential equation

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0\,, \qquad -\infty < x < \infty,$$

where p and q are given. Show, using the Wronskian, that

- *either* there exist α and β, not both zero, such that $\alpha y_1(x) + \beta y_2(x)$ vanishes for all x,
- *or* given x_0, A and B, there exist a and b such that $y(x) = ay_1(x) + by_2(x)$ satisfies the conditions $y(x_0) = A$ and $y'(x_0) = B$.

Find power series y_1 and y_2 such that an arbitrary solution of the equation

$$y''(x) = xy(x)$$

can be written as a linear combination of y_1 and y_2.

8C Differential Equations

(a) Solve $\dfrac{dz}{dt} = z^2$ subject to $z(0) = z_0$. For which z_0 is the solution finite for all $t \in \mathbb{R}$?

Let a be a positive constant. By considering the lines $y = a(x - x_0)$ for constant x_0, or otherwise, show that any solution of the equation

$$\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial y} = 0$$

is of the form $f(x, y) = F(y - ax)$ for some function F.

Solve the equation

$$\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial y} = f^2$$

subject to $f(0, y) = g(y)$ for a given function g. For which g is the solution bounded on $\mathbb{R}^2$?

(b) By means of the change of variables $X = \alpha x + \beta y$ and $T = \gamma x + \delta y$ for appropriate real numbers $\alpha, \beta, \gamma, \delta$, show that the equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} = 0 \qquad (*)$$

can be transformed into the wave equation

$$\frac{1}{c^2}\frac{\partial^2 F}{\partial T^2} - \frac{\partial^2 F}{\partial X^2} = 0,$$

where F is defined by $f(x, y) = F(\alpha x + \beta y, \gamma x + \delta y)$. Hence write down the general solution of $(*)$.

9F Probability

For a positive integer N, $p \in [0,1]$, and $k \in \{0,1,\ldots,N\}$, let

$$p_k(N,p) = \binom{N}{k} p^k (1-p)^{N-k}.$$

(a) For fixed N and p, show that $p_k(N,p)$ is a probability mass function on $\{0,1,\ldots,N\}$ and that the corresponding probability distribution has mean Np and variance $Np(1-p)$.

(b) Let $\lambda > 0$. Show that, for any $k \in \{0,1,2,\ldots\}$,

$$\lim_{N\to\infty} p_k(N,\lambda/N) = \frac{e^{-\lambda}\lambda^k}{k!}. \qquad (*)$$

Show that the right-hand side of $(*)$ is a probability mass function on $\{0,1,2,\ldots\}$.

(c) Let $p \in (0,1)$ and let $a,b \in \mathbb{R}$ with $a < b$. For all N, find integers $k_a(N)$ and $k_b(N)$ such that

$$\sum_{k=k_a(N)}^{k_b(N)} p_k(N,p) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}x^2}\,dx \qquad \text{as } N \to \infty.$$

[You may use the Central Limit Theorem.]

10F Probability

(a) For any random variable X and $\lambda > 0$ and $t > 0$, show that

$$\mathbb{P}(X > t) \leqslant \mathbb{E}(e^{\lambda X})e^{-\lambda t}.$$

For a standard normal random variable X, compute $\mathbb{E}(e^{\lambda X})$ and deduce that

$$\mathbb{P}(X > t) \leqslant e^{-\frac{1}{2}t^2}.$$

(b) Let $\mu, \lambda > 0$, $\mu \neq \lambda$. For independent random variables X and Y with distributions $\mathrm{Exp}(\lambda)$ and $\mathrm{Exp}(\mu)$, respectively, compute the probability density functions of $X+Y$ and $\min\{X,Y\}$.

11F Probability

Let $\beta > 0$. The *Curie–Weiss Model* of ferromagnetism is the probability distribution defined as follows. For $n \in \mathbb{N}$, define random variables $S_1, \ldots, S_n$ with values in $\{\pm 1\}$ such that the probabilities are given by

$$\mathbb{P}(S_1 = s_1, \ldots, S_n = s_n) = \frac{1}{Z_{n,\beta}} \exp\left(\frac{\beta}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j\right)$$

where $Z_{n,\beta}$ is the normalisation constant

$$Z_{n,\beta} = \sum_{s_1 \in \{\pm 1\}} \cdots \sum_{s_n \in \{\pm 1\}} \exp\left(\frac{\beta}{2n} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j\right).$$

(a) Show that $\mathbb{E}(S_i) = 0$ for any i.

(b) Show that $\mathbb{P}(S_2 = +1 | S_1 = +1) \geqslant \mathbb{P}(S_2 = +1)$. [You may use $\mathbb{E}(S_i S_j) \geqslant 0$ for all i, j without proof.]

(c) Let $M = \frac{1}{n} \sum_{i=1}^{n} S_i$. Show that M takes values in $E_n = \{-1 + \frac{2k}{n} : k = 0, \ldots, n\}$, and that for each $m \in E_n$ the number of possible values of $(S_1, \ldots, S_n)$ such that $M = m$ is

$$\frac{n!}{\left(\frac{1+m}{2} n\right)! \left(\frac{1-m}{2} n\right)!}.$$

Find $\mathbb{P}(M = m)$ for any $m \in E_n$.

12F Probability

(a) Let $k \in \{1, 2, \dots\}$. For $j \in \{0, \dots, k+1\}$, let D_j be the first time at which a simple symmetric random walk on $\mathbb{Z}$ with initial position j at time 0 hits 0 or $k+1$. Show $\mathbb{E}(D_j) = j(k+1-j)$. [If you use a recursion relation, you do not need to prove that its solution is unique.]

(b) Let (S_n) be a simple symmetric random walk on $\mathbb{Z}$ starting at 0 at time $n = 0$. For $k \in \{1, 2, \dots\}$, let T_k be the first time at which (S_n) has visited k distinct vertices. In particular, $T_1 = 0$. Show $\mathbb{E}(T_{k+1} - T_k) = k$ for $k \geqslant 1$. [You may use without proof that, conditional on $S_{T_k} = i$, the random variables $(S_{T_k+n})_{n \geqslant 0}$ have the distribution of a simple symmetric random walk starting at i.]

(c) For $n \geqslant 3$, let $\mathbb{Z}_n$ be the circle graph consisting of vertices $0, \dots, n-1$ and edges between k and $k+1$ where n is identified with 0. Let (Y_i) be a simple random walk on $\mathbb{Z}_n$ starting at time 0 from 0. Thus $Y_0 = 0$ and conditional on Y_i the random variable Y_{i+1} is $Y_i \pm 1$ with equal probability (identifying $k+n$ with k).

The *cover time* T of the simple random walk on $\mathbb{Z}_n$ is the first time at which the random walk has visited all vertices. Show that $\mathbb{E}(T) = n(n-1)/2$.

END OF PAPER