UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Thursday, 1 June, 2017 9:00 am to 12:00 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Section II questions also carry an alpha or beta quality mark and Section I questions carry a beta quality mark.

Candidates may obtain credit from attempts on ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. Of the Section II questions, no more than three may be on the same course.*

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold Cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1A Vectors and Matrices

Consider $z \in \mathbb{C}$ with $|z| = 1$ and $\arg z = \theta$, where $\theta \in [0, \pi)$.

(a) Prove algebraically that the modulus of $1 + z$ is $2 \cos \frac{1}{2}\theta$ and that the argument is $\frac{1}{2}\theta$. Obtain these results geometrically using the Argand diagram.

(b) Obtain corresponding results algebraically and geometrically for $1 - z$.

2C Vectors and Matrices

Let A and B be real $n \times n$ matrices.

Show that $(AB)^T = B^T A^T$.

For any square matrix, the *matrix exponential* is defined by the series

$$e^A = I + \sum_{k=1}^{\infty} \frac{A^k}{k!} .$$

Show that $(e^A)^T = e^{A^T}$. [You are not required to consider issues of convergence.]

Calculate, in terms of A and A^T, the matrices Q_0, Q_1 and Q_2 in the series for the matrix product

$$e^{tA} \, e^{tA^T} = \sum_{k=0}^{\infty} Q_k t^k , \quad \text{where } \; t \in \mathbb{R}.$$

Hence obtain a relation between A and A^T which necessarily holds if e^{tA} is an orthogonal matrix.

3F Analysis I

Given an increasing sequence of non-negative real numbers $(a_n)_{n=1}^{\infty}$, let

$$s_n = \frac{1}{n} \sum_{k=1}^{n} a_k .$$

Prove that if $s_n \to x$ as $n \to \infty$ for some $x \in \mathbb{R}$ then also $a_n \to x$ as $n \to \infty$.

4E Analysis I

Show that if the power series $\sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{C})$ converges for some fixed $z = z_0$, then it converges absolutely for every z satisfying $|z| < |z_0|$.

Define the *radius of convergence* of a power series.

Give an example of $v \in \mathbb{C}$ and an example of $w \in \mathbb{C}$ such that $|v| = |w| = 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{v^n}{n}$ converges and $\displaystyle\sum_{n=1}^{\infty} \frac{w^n}{n}$ diverges. [You may assume results about standard series without proof.] Use this to find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$.

SECTION II

5A Vectors and Matrices

(a) Define the *vector product* $\mathbf{x} \times \mathbf{y}$ of the vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$. Use suffix notation to prove that

$$\mathbf{x} \times (\mathbf{x} \times \mathbf{y}) = \mathbf{x}\,(\mathbf{x} \cdot \mathbf{y}) - \mathbf{y}\,(\mathbf{x} \cdot \mathbf{x}).$$

(b) The vectors $\mathbf{x}_{n+1}$ $(n = 0, 1, 2, \ldots)$ are defined by $\mathbf{x}_{n+1} = \lambda\,\mathbf{a} \times \mathbf{x}_n$, where $\mathbf{a}$ and $\mathbf{x}_0$ are fixed vectors with $|\mathbf{a}| = 1$ and $\mathbf{a} \times \mathbf{x}_0 \neq \mathbf{0}$, and λ is a positive constant.

(i) Write $\mathbf{x}_2$ as a linear combination of $\mathbf{a}$ and $\mathbf{x}_0$. Further, for $n \geqslant 1$, express $\mathbf{x}_{n+2}$ in terms of λ and $\mathbf{x}_n$. Show, for $n \geqslant 1$, that $|\mathbf{x}_n| = \lambda^n\,|\mathbf{a} \times \mathbf{x}_0|$.

(ii) Let X_n be the point with position vector $\mathbf{x}_n$ $(n = 0, 1, 2, \ldots)$. Show that $X_1, X_2, \ldots$ lie on a pair of straight lines.

(iii) Show that the line segment $X_n X_{n+1}$ $(n \geqslant 1)$ is perpendicular to $X_{n+1} X_{n+2}$. Deduce that $X_n X_{n+1}$ is parallel to $X_{n+2} X_{n+3}$.

Show that $\mathbf{x}_n \to \mathbf{0}$ as $n \to \infty$ if $\lambda < 1$, and give a sketch to illustrate the case $\lambda = 1$.

(iv) The straight line through the points X_{n+1} and X_{n+2} makes an angle θ with the straight line through the points X_n and X_{n+3}. Find $\cos\theta$ in terms of λ.

6B Vectors and Matrices

(a) Show that the eigenvalues of any real $n \times n$ square matrix A are the same as the eigenvalues of A^T.

The eigenvalues of A are λ_1, λ_2, ..., λ_n and the eigenvalues of A^TA are μ_1, μ_2, ..., μ_n. Determine, by means of a proof or a counterexample, whether the following are necessary valid:

(i) $\sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} \lambda_i^2$;

(ii) $\prod_{i=1}^{n} \mu_i = \prod_{i=1}^{n} \lambda_i^2$.

(b) The 3×3 matrix B is given by

$$B = I + \mathbf{m}\mathbf{n}^T ,$$

where $\mathbf{m}$ and $\mathbf{n}$ are orthogonal real unit vectors and I is the 3×3 identity matrix.

(i) Show that $\mathbf{m} \times \mathbf{n}$ is an eigenvector of B, and write down a linearly independent eigenvector. Find the eigenvalues of B and determine whether B is diagonalisable.

(ii) Find the eigenvectors and eigenvalues of B^TB.

7B Vectors and Matrices

(a) Show that a square matrix A is anti-symmetric if and only if $\mathbf{x}^TA\mathbf{x} = 0$ for every vector $\mathbf{x}$.

(b) Let A be a real anti-symmetric $n \times n$ matrix. Show that the eigenvalues of A are imaginary or zero, and that the eigenvectors corresponding to distinct eigenvalues are orthogonal (in the sense that $\mathbf{x}^\dagger\mathbf{y} = 0$, where the dagger denotes the hermitian conjugate).

(c) Let A be a non-zero real 3×3 anti-symmetric matrix. Show that there is a real non-zero vector $\mathbf{a}$ such that $A\mathbf{a} = \mathbf{0}$.

Now let $\mathbf{b}$ be a real vector orthogonal to $\mathbf{a}$. Show that $A^2\mathbf{b} = -\theta^2\mathbf{b}$ for some real number θ.

The matrix e^A is defined by the exponential series $I + A + \frac{1}{2!}A^2 + \cdots$. Express $e^A\mathbf{a}$ and $e^A\mathbf{b}$ in terms of $\mathbf{a}, \mathbf{b}$, $A\mathbf{b}$ and θ.

[*You are not required to consider issues of convergence.*]

8C Vectors and Matrices

(a) Given $\mathbf{y} \in \mathbb{R}^3$ consider the linear transformation T which maps

$$\mathbf{x} \mapsto T\mathbf{x} = (\mathbf{x} \cdot \mathbf{e}_1)\,\mathbf{e}_1 + \mathbf{x} \times \mathbf{y}\,.$$

Express T as a matrix with respect to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and determine the rank and the dimension of the kernel of T for the cases (i) $\mathbf{y} = c_1\mathbf{e}_1$, where c_1 is a fixed number, and (ii) $\mathbf{y} \cdot \mathbf{e}_1 = 0$.

(b) Given that the equation

$$A\,B\,\mathbf{x} = \mathbf{d}\,,$$

where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & 1 \\ -3 & -2 & 1 \\ 1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ k \end{pmatrix},$$

has a solution, show that $4k = 1$.

9D Analysis I

(a) State the Intermediate Value Theorem.

(b) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *differentiable* at a point $a \in \mathbb{R}$. If f is differentiable everywhere on $\mathbb{R}$, must f' be continuous everywhere? Justify your answer.

State the Mean Value Theorem.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Let $a, b \in \mathbb{R}$ with $a < b$.
If $f'(a) \leqslant y \leqslant f'(b)$, prove that there exists $c \in [a, b]$ such that $f'(c) = y$. [*Hint: consider the function g defined by*

$$g(x) = \frac{f(x) - f(a)}{x - a}$$

if $x \neq a$ and $g(a) = f'(a)$.]

If additionally $f(a) \leqslant 0 \leqslant f(b)$, deduce that there exists $d \in [a, b]$ such that $f'(d) + f(d) = y$.

10D Analysis I

Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : (a,b) \to \mathbb{R}$.

(a) Define what it means for f to be *continuous* at $y_0 \in (a,b)$.

f is said to have a *local minimum* at $c \in (a,b)$ if there is some $\varepsilon > 0$ such that $f(c) \leqslant f(x)$ whenever $x \in (a,b)$ and $|x - c| < \varepsilon$.

If f has a local minimum at $c \in (a,b)$ and f is differentiable at c, show that $f'(c) = 0$.

(b) f is said to be *convex* if

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y)$$

for every $x, y \in (a,b)$ and $\lambda \in [0,1]$. If f is convex, $r \in \mathbb{R}$ and $\big[y_0 - |r|\,,\, y_0 + |r|\big] \subset (a,b)$, prove that

$$(1+\lambda) f(y_0) - \lambda f(y_0 - r) \leqslant f(y_0 + \lambda r) \leqslant (1-\lambda) f(y_0) + \lambda f(y_0 + r)$$

for every $\lambda \in [0,1]$.

Deduce that if f is convex then f is continuous.

If f is convex and has a local minimum at $c \in (a,b)$, prove that f has a global minimum at c, i.e., that $f(x) \geqslant f(c)$ for every $x \in (a,b)$. [*Hint: argue by contradiction.*] Must f be differentiable at c? Justify your answer.

11F Analysis I

(a) Let $(x_n)_{n=1}^{\infty}$ be a non-negative and decreasing sequence of real numbers. Prove that $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k x_{2^k}$ converges.

(b) For $s \in \mathbb{R}$, prove that $\sum_{n=1}^{\infty} n^{-s}$ converges if and only if $s > 1$.

(c) For any $k \in \mathbb{N}$, prove that

$$\lim_{n\to\infty} 2^{-n} n^k = 0.$$

(d) The sequence $(a_n)_{n=0}^{\infty}$ is defined by $a_0 = 1$ and $a_{n+1} = 2^{a_n}$ for $n \geqslant 0$. For any $k \in \mathbb{N}$, prove that

$$\lim_{n\to\infty} \frac{2^{n^k}}{a_n} = 0.$$

12E Analysis I

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function defined on the closed, bounded interval $[a,b]$ of $\mathbb{R}$. Suppose that for every $\varepsilon > 0$ there is a dissection $\mathcal{D}$ of $[a,b]$ such that $S_{\mathcal{D}}(f) - s_{\mathcal{D}}(f) < \varepsilon$, where $s_{\mathcal{D}}(f)$ and $S_{\mathcal{D}}(f)$ denote the lower and upper Riemann sums of f for the dissection $\mathcal{D}$. Deduce that f is Riemann integrable. [You may assume without proof that $s_{\mathcal{D}}(f) \leqslant S_{\mathcal{D}'}(f)$ for all dissections $\mathcal{D}$ and $\mathcal{D}'$ of $[a,b]$.]

Prove that if $f\colon [a,b] \to \mathbb{R}$ is continuous, then f is Riemann integrable.

Let $g\colon (0,1] \to \mathbb{R}$ be a bounded continuous function. Show that for any $\lambda \in \mathbb{R}$, the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} g(x) & \text{if } 0 < x \leqslant 1 \ , \\ \lambda & \text{if } x = 0 \ , \end{cases}$$

is Riemann integrable.

Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function with one-sided derivatives at the endpoints. Suppose that the derivative f' is (bounded and) Riemann integrable. Show that

$$\int_a^b f'(x)\, dx = f(b) - f(a) \ .$$

[*You may use the Mean Value Theorem without proof.*]

END OF PAPER