UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Monday, 30 May, 2016 1:30 pm to 4:30 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A****,* ***B****,* ***C****, ...,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1I Number Theory

Define the Riemann zeta function $\zeta(s)$ for $\mathrm{Re}(s) > 1$. State and prove the alternative formula for $\zeta(s)$ as an Euler product. Hence or otherwise show that $\zeta(s) \neq 0$ for $\mathrm{Re}(s) > 1$.

2H Topics in Analysis

By considering the function $\mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$R(a_0, \ldots, a_n) = \sup_{t \in [-1,1]} \left| \sum_{j=0}^{n} a_j t^j \right|,$$

or otherwise, show that there exist $K_n > 0$ and $\delta_n > 0$ such that

$$K_n \sum_{j=0}^{n} |a_j| \geqslant \sup_{t \in [-1,1]} \left| \sum_{j=0}^{n} a_j t^j \right| \geqslant \delta_n \sum_{j=0}^{n} |a_j|$$

for all $a_j \in \mathbb{R}$, $0 \leqslant j \leqslant n$.

Show, quoting carefully any theorems you use, that we must have $\delta_n \to 0$ as $n \to \infty$.

3G Coding and Cryptography

Find the average length of an optimum decipherable binary code for a source that emits five words with probabilities

$$0.25,\ 0.15,\ 0.15,\ 0.2,\ 0.25.$$

Show that, if a source emits N words (with $N \geqslant 2$), and if $l_1, \ldots, l_N$ are the lengths of the codewords in an optimum encoding over the binary alphabet, then

$$l_1 + \cdots + l_N \leqslant \frac{1}{2}(N^2 + N - 2).$$

[*You may assume that an optimum encoding can be given by a Huffman encoding.*]

4F Automata and Formal Languages

State the *pumping lemma* for context-free languages (CFLs). Which of the following are CFLs? Justify your answers.

(i) $\{a^{2n}b^{3n} \mid n \geqslant 3\}$.

(ii) $\{a^{2n}b^{3n}c^{5n} \mid n \geqslant 0\}$.

(iii) $\{a^p \mid p \text{ is a prime}\}$.

Let L, M be CFLs. Show that $L \cup M$ is also a CFL.

5K Statistical Modelling

The body mass index (BMI) of your closest friend is a good predictor of your own BMI. A scientist applies polynomial regression to understand the relationship between these two variables among 200 students in a sixth form college. The R commands

```
> fit.1 <- lm(BMI ~ poly(friendBMI,2,raw=T))
> fit.2 <- lm(BMI ~ poly(friendBMI,3,raw=T))
```

fit the models $Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \varepsilon$ and $Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + \varepsilon$, respectively, with $\varepsilon \sim N(0, \sigma^2)$ in each case.

Setting the parameters `raw` to `FALSE`:

```
> fit.3 <- lm(BMI ~ poly(friendBMI,2,raw=F))
> fit.4 <- lm(BMI ~ poly(friendBMI,3,raw=F))
```

fits the models $Y = \beta_0 + \beta_1 P_1(X) + \beta_2 P_2(X) + \varepsilon$ and $Y = \beta_0 + \beta_1 P_1(X) + \beta_2 P_2(X) + \beta_3 P_3(X) + \varepsilon$, with $\varepsilon \sim N(0, \sigma^2)$. The function P_i is a polynomial of degree i. Furthermore, the design matrix output by the function `poly` with `raw=F` satisfies:

```
> t(poly(friendBMI,3,raw=F))%*%poly(a,3,raw=F)
             1             2             3
1 1.000000e+00  1.288032e-16  3.187554e-17
2 1.288032e-16  1.000000e+00 -6.201636e-17
3 3.187554e-17 -6.201636e-17  1.000000e+00
```

How does the variance of $\hat{\beta}$ differ in the models `fit.2` and `fit.4`? What about the variance of the fitted values $\hat{Y} = X\hat{\beta}$? Finally, consider the output of the commands

```
> anova(fit.1,fit.2)
> anova(fit.3,fit.4)
```

Define the test statistic computed by this function and specify its distribution. Which command yields a higher statistic?

6B Mathematical Biology

Consider an epidemic model where susceptibles are vaccinated at per capita rate v, but immunity (from infection or vaccination) is lost at per capita rate b. The system is given by

$$\begin{aligned}\frac{dS}{dt} &= -rIS + b(N - I - S) - vS\,,\\ \frac{dI}{dt} &= rIS - aI\,,\end{aligned}$$

where $S(t)$ are the susceptibles, $I(t)$ are the infecteds, N is the total population size and all parameters are positive. The basic reproduction ratio $R_0 = rN/a$ satisfies $R_0 > 1$.

Find the critical vaccination rate v_c, in terms of b and R_0, such that the system has an equilibrium with the disease present if $v < v_c$. Show that this equilibrium is stable when it exists.

Find the long-term outcome for S and I if $v > v_c$.

7A Further Complex Methods

Evaluate the integral

$$f(p) = \mathcal{P}\int_{-\infty}^{\infty} dx\,\frac{e^{ipx}}{x^4 - 1}\,,$$

where p is a real number, for (i) $p > 0$ and (ii) $p < 0$.

8E Classical Dynamics

Consider a one-parameter family of transformations $q_i(t) \mapsto Q_i(s,t)$ such that $Q_i(0,t) = q_i(t)$ for all time t, and

$$\frac{\partial}{\partial s} L(Q_i, \dot{Q}_i, t) = 0\,,$$

where L is a Lagrangian and a dot denotes differentiation with respect to t. State and prove Noether's theorem.

Consider the Lagrangian

$$L = \frac{1}{2}(\,\dot{x}^2 + \dot{y}^2 + \dot{z}^2\,) - V(\,x{+}y,\; y{+}z\,)\,,$$

where the potential V is a function of two variables. Find a continuous symmetry of this Lagrangian and construct the corresponding conserved quantity. Use the Euler–Lagrange equations to explicitly verify that the function you have constructed is independent of t.

9C Cosmology

The expansion scale factor, $a(t)$, for an isotropic and spatially homogeneous universe containing material with pressure p and mass density ρ obeys the equations

$$\dot{\rho} + 3(\rho + p)\,\frac{\dot{a}}{a} = 0\,,$$
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3}\,,$$

where the speed of light is set equal to unity, G is Newton's constant, k is a constant equal to 0 or ± 1, and Λ is the cosmological constant. Explain briefly the interpretation of these equations.

Show that these equations imply

$$\frac{\ddot{a}}{a} = -\frac{4\pi G(\rho + 3p)}{3} + \frac{\Lambda}{3}\,.$$

Hence show that a static solution with constant $a = a_{\mathrm{s}}$ exists when $p = 0$ if

$$\Lambda = 4\pi G\rho = \frac{k}{a_{\mathrm{s}}^2}\,.$$

What must the value of k be, if the density ρ is non-zero?

SECTION II

10G Coding and Cryptography

What does it mean to say a binary code C has *length* n, *size* m and *minimum distance* d?

Let $A(n,d)$ be the largest value of m for which there exists an $[n,m,d]$-code. Prove that

$$\frac{2^n}{V(n,d-1)} \leqslant A(n,d) \leqslant \frac{2^n}{V(n,\lfloor (d-1)/2 \rfloor)},$$

where

$$V(n,r) = \sum_{j=0}^{r} \binom{n}{j}.$$

Another bound for $A(n,d)$ is the Singleton bound given by

$$A(n,d) \leqslant 2^{n-d+1}.$$

Prove the Singleton bound and give an example of a linear code of length $n > 1$ that satisfies $A(n,d) = 2^{n-d+1}$.

11F Automata and Formal Languages

(a) Define a *recursive set* and a *recursively enumerable (r.e.) set*. Prove that $E \subseteq \mathbb{N}$ is recursive if and only if both E and $\mathbb{N} \setminus E$ are r.e.

(b) Define *the halting set* $\mathbb{K}$. Prove that $\mathbb{K}$ is r.e. but not recursive.

(c) Let $E_1, E_2, \ldots, E_n$ be r.e. sets. Prove that $\bigcup_{i=1}^{n} E_i$ and $\bigcap_{i=1}^{n} E_i$ are r.e. Show by an example that the union of infinitely many r.e. sets need not be r.e.

(d) Let E be a recursive set and $f : \mathbb{N} \to \mathbb{N}$ a (total) recursive function. Prove that the set $\{f(n) \mid n \in E\}$ is r.e. Is it necessarily recursive? Justify your answer.

[*Any use of Church's thesis in your answer should be explicitly stated.*]

12K Statistical Modelling

(a) Let Y be an n-vector of responses from the linear model $Y = X\beta + \varepsilon$, with $\beta \in \mathbb{R}^p$. The *internally studentized residual* is defined by

$$s_i = \frac{Y_i - x_i^\intercal \hat{\beta}}{\tilde{\sigma}\sqrt{1-p_i}},$$

where $\hat{\beta}$ is the least squares estimate, p_i is the leverage of sample i, and

$$\tilde{\sigma}^2 = \frac{\|Y - X\hat{\beta}\|_2^2}{(n-p)}.$$

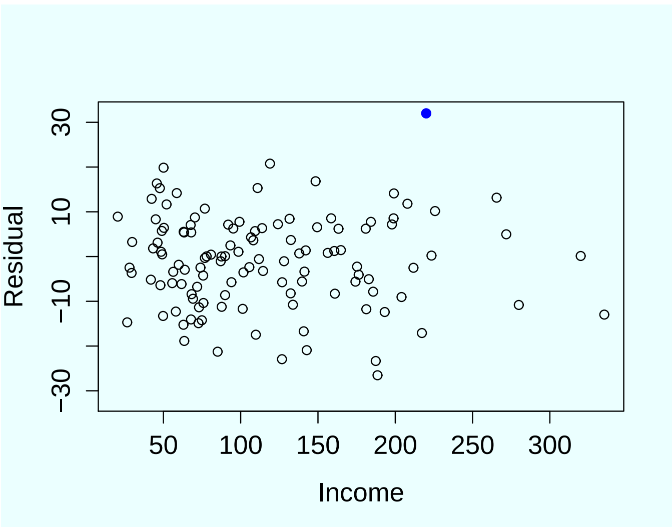
Prove that the joint distribution of $s = (s_1, \ldots, s_n)^\intercal$ is the same in the following two models: (i) $\varepsilon \sim N(0, \sigma I)$, and (ii) $\varepsilon \mid \sigma \sim N(0, \sigma I)$, with $1/\sigma \sim \chi^2_\nu$ (in this model, $\varepsilon_1, \ldots, \varepsilon_n$ are identically t_ν-distributed). [*Hint: A random vector Z is spherically symmetric if for any orthogonal matrix H, $HZ \overset{d}{=} Z$. If Z is spherically symmetric and a.s. nonzero, then $Z/\|Z\|_2$ is a uniform point on the sphere; in addition, any orthogonal projection of Z is also spherically symmetric. A standard normal vector is spherically symmetric.*]

(b) A social scientist regresses the income of 120 Cambridge graduates onto 20 answers from a questionnaire given to the participants in their first year. She notices one questionnaire with very unusual answers, which she suspects was due to miscoding. The sample has a leverage of 0.8. To check whether this sample is an outlier, she computes its *externally studentized residual*,

$$t_i = \frac{Y_i - x_i^\intercal \hat{\beta}}{\tilde{\sigma}_{(i)}\sqrt{1-p_i}} = 4.57,$$

where $\tilde{\sigma}_{(i)}$ is estimated from a fit of all samples except the one in question, (x_i, Y_i). Is this a high leverage point? Can she conclude this sample is an outlier at a significance level of 5%?

(c) After examining the following plot of residuals against the response, the investigator calculates the externally studentized residual of the participant denoted by the black dot, which is 2.33. Can she conclude this sample is an outlier with a significance level of 5%?



13A Further Complex Methods

(a) Legendre's equation for $w(z)$ is

$$(z^2-1)w'' + 2zw' - \ell(\ell+1)w = 0\,, \qquad \text{where} \qquad \ell = 0,\,1,\,2,\,\ldots\,.$$

Let $\mathcal{C}$ be a closed contour. Show by direct substitution that for z within $\mathcal{C}$

$$\int_{\mathcal{C}} dt\, \frac{(t^2-1)^\ell}{(t-z)^{\ell+1}}$$

is a non-trivial solution of Legendre's equation.

(b) Now consider

$$Q_\nu(z) = \frac{1}{4i\sin\nu\pi} \int_{\mathcal{C}'} dt\, \frac{(t^2-1)^\nu}{(t-z)^{\nu+1}}$$

for real $\nu > -1$ and $\nu \neq 0,\,1,\,2,\,\ldots$. The closed contour $\mathcal{C}'$ is defined to start at the origin, wind around $t = 1$ in a counter-clockwise direction, then wind around $t = -1$ in a clockwise direction, then return to the origin, without encircling the point z. Assuming that z does not lie on the real interval $-1 \leqslant x \leqslant 1$, show by deforming $\mathcal{C}'$ onto this interval that functions $Q_\ell(z)$ may be defined as limits of $Q_\nu(z)$ with $\nu \to \ell = 0,\,1,\,2,\,\ldots$.

Find an explicit expression for $Q_0(z)$ and verify that it satisfies Legendre's equation with $\ell = 0$.

14C Cosmology

The distribution function $f(\mathbf{x}, \mathbf{p}, t)$ gives the number of particles in the universe with position in $(\mathbf{x}, \mathbf{x} + \delta\mathbf{x})$ and momentum in $(\mathbf{p}, \mathbf{p} + \delta\mathbf{p})$ at time t. It satisfies the boundary condition that $f \to 0$ as $|\mathbf{x}| \to \infty$ and as $|\mathbf{p}| \to \infty$. Its evolution obeys the Boltzmann equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{d\mathbf{p}}{dt} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} = \left[\frac{df}{dt}\right]_{\mathrm{col}},$$

where the collision term $\left[\frac{df}{dt}\right]_{\mathrm{col}}$ describes any particle production and annihilation that occurs.

The universe expands isotropically and homogeneously with expansion scale factor $a(t)$, so the momenta evolve isotropically with magnitude $p \propto a^{-1}$. Show that the Boltzmann equation simplifies to

$$\frac{\partial f}{\partial t} - \frac{\dot{a}}{a}\mathbf{p} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left[\frac{df}{dt}\right]_{\mathrm{col}}. \qquad (*)$$

The number densities n of particles and $\bar{n}$ of antiparticles are defined in terms of their distribution functions f and $\bar{f}$, and momenta p and $\bar{p}$, by

$$n = \int_0^\infty f\, 4\pi p^2\, dp \quad \text{and} \quad \bar{n} = \int_0^\infty \bar{f}\, 4\pi \bar{p}^2\, d\bar{p}\,,$$

and the collision term may be assumed to be of the form

$$\left[\frac{df}{dt}\right]_{\mathrm{col}} = -\langle \sigma v \rangle \int_0^\infty \bar{f} f\, 4\pi \bar{p}^2\, d\bar{p} + R$$

where $\langle \sigma v \rangle$ determines the annihilation cross-section of particles by antiparticles and R is the production rate of particles.

By integrating equation $(*)$ with respect to the momentum $\mathbf{p}$ and assuming that $\langle \sigma v \rangle$ is a constant, show that

$$\frac{dn}{dt} + 3\frac{\dot{a}}{a}n = -\langle \sigma v \rangle\, n\bar{n} + Q\,,$$

where $Q = \int_0^\infty R\, 4\pi p^2\, dp$. Assuming the same production rate R for antiparticles, write down the corresponding equation satisfied by $\bar{n}$ and show that

$$(n - \bar{n})a^3 = \text{constant}\,.$$

15F Logic and Set Theory

Which of the following statements are true? Justify your answers.

(a) Every ordinal is of the form $\alpha + n$, where α is a limit ordinal and $n \in \omega$.

(b) Every ordinal is of the form $\omega^{\alpha}.m + n$, where α is an ordinal and $m, n \in \omega$.

(c) If $\alpha = \omega.\alpha$, then $\alpha = \omega^{\omega}.\beta$ for some β.

(d) If $\alpha = \omega^{\alpha}$, then α is uncountable.

(e) If $\alpha > 1$ and $\alpha = \alpha^{\omega}$, then α is uncountable.

[*Standard laws of ordinal arithmetic may be assumed, but if you use the Division Algorithm you should prove it.*]

16G Graph Theory

(a) Show that if G is a planar graph then $\chi(G) \leqslant 5$. [You may assume Euler's formula, provided that you state it precisely.]

(b) (i) Prove that if G is a triangle-free planar graph then $\chi(G) \leqslant 4$.

(ii) Prove that if G is a planar graph of girth at least 6 then $\chi(G) \leqslant 3$.

(iii) Does there exist a constant g such that, if G is a planar graph of girth at least g, then $\chi(G) \leqslant 2$? Justify your answer.

17H Galois Theory

(a) Prove that if K is a field and $f \in K[t]$, then there exists a splitting field L of f over K. [You do not need to show uniqueness of L.]

(b) Let K_1 and K_2 be algebraically closed fields of the same characteristic. Show that either K_1 is isomorphic to a subfield of K_2 or K_2 is isomorphic to a subfield of K_1. [For subfields F_i of K_1 and field homomorphisms $\psi_i : F_i \to K_2$ with $i = 1, 2$, we say $(F_1, \psi_1) \leqslant (F_2, \psi_2)$ if F_1 is a subfield of F_2 and $\psi_2|_{F_1} = \psi_1$. You may assume the existence of a maximal pair (F, ψ) with respect to the partial order just defined.]

(c) Give an example of a finite field extension $K \subseteq L$ such that there exist $\alpha, \beta \in L \setminus K$ where α is separable over K but β is not separable over K.

18I Representation Theory

Let N be a normal subgroup of the finite group G. Explain how a (complex) representation of G/N gives rise to an associated representation of G, and briefly describe which representations of G arise this way.

Let G be the group of order 54 which is given by

$$G = \langle a, b : a^9 = b^6 = 1, b^{-1}ab = a^2 \rangle.$$

Find the conjugacy classes of G. By observing that $N_1 = \langle a \rangle$ and $N_2 = \langle a^3, b^2 \rangle$ are normal in G, or otherwise, construct the character table of G.

19F Number Fields

(a) Let $f(X) \in \mathbb{Q}[X]$ be an irreducible polynomial of degree n, $\theta \in \mathbb{C}$ a root of f, and $K = \mathbb{Q}(\theta)$. Show that $disc(f) = \pm N_{K/\mathbb{Q}}(f'(\theta))$.

(b) Now suppose $f(X) = X^n + aX + b$. Write down the matrix representing multiplication by $f'(\theta)$ with respect to the basis $1, \theta, \ldots, \theta^{n-1}$ for K. Hence show that

$$disc(f) = \pm\big((1-n)^{n-1}a^n + n^n b^{n-1}\big).$$

(c) Suppose $f(X) = X^4 + X + 1$. Determine $\mathcal{O}_K$. [You may quote any standard result, as long as you state it clearly.]

20G Algebraic Topology

Let $T = S^1 \times S^1$ be the 2-dimensional torus. Let $\alpha : S^1 \to T$ be the inclusion of the coordinate circle $S^1 \times \{1\}$, and let X be the result of attaching a 2-cell along α.

(a) Write down a presentation for the fundamental group of X (with respect to some basepoint), and identify it with a well-known group.

(b) Compute the simplicial homology of any triangulation of X.

(c) Show that X is not homotopy equivalent to any compact surface.

21I Linear Analysis

(a) State the *closed graph theorem.*

(b) Prove the closed graph theorem assuming the inverse mapping theorem.

(c) Let X, Y, Z be Banach spaces and $T : X \to Y$, $S : Y \to Z$ be linear maps. Suppose that $S \circ T$ is bounded and S is both bounded and injective. Show that T is bounded.

22H Riemann Surfaces

(a) Let $f : R \to S$ be a non-constant holomorphic map between Riemann surfaces. Prove that f takes open sets of R to open sets of S.

(b) Let U be a simply connected domain strictly contained in $\mathbb{C}$. Is there a conformal equivalence between U and $\mathbb{C}$? Justify your answer.

(c) Let R be a compact Riemann surface and $A \subset R$ a discrete subset. Given a non-constant holomorphic function $f : R \setminus A \to \mathbb{C}$, show that $f(R \setminus A)$ is dense in $\mathbb{C}$.

23H Algebraic Geometry

Let k be an algebraically closed field.

(a) Let X and Y be affine varieties defined over k. Given a map $f : X \to Y$, define what it means for f to be a *morphism of affine varieties*.

(b) Let $f : \mathbb{A}^1 \to \mathbb{A}^3$ be the map given by

$$f(t) = (t, t^2, t^3).$$

Show that f is a morphism. Show that the image of f is a closed subvariety of $\mathbb{A}^3$ and determine its ideal.

(c) Let $g : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^7$ be the map given by

$$g\big((s_1,t_1),(s_2,t_2),(s_3,t_3)\big) = (s_1s_2s_3,\ s_1s_2t_3,\ s_1t_2s_3,\ s_1t_2t_3,\ t_1s_2s_3,\ t_1s_2t_3,\ t_1t_2s_3,\ t_1t_2t_3).$$

Show that the image of g is a closed subvariety of $\mathbb{P}^7$.

24G Differential Geometry

Define what is meant by the *regular values* and *critical values* of a smooth map $f : X \to Y$ of manifolds. State the Preimage Theorem and Sard's Theorem.

Suppose now that $\dim X = \dim Y$. If X is compact, prove that the set of regular values is open in Y, but show that this may not be the case if X is non-compact.

Construct an example with $\dim X = \dim Y$ and X compact for which the set of critical values is not a submanifold of Y.

[*Hint: You may find it helpful to consider the case $X = S^1$ and $Y = \mathbf{R}$. Properties of bump functions and the function e^{-1/x^2} may be assumed in this question.*]

25J Probability and Measure

Throughout this question $(E, \mathcal{E}, \mu)$ is a measure space and (f_n), f are measurable functions.

(a) Give the definitions of *pointwise convergence*, *pointwise a.e. convergence*, and *convergence in measure.*

(b) If $f_n \to f$ pointwise a.e., does $f_n \to f$ in measure? Give a proof or a counterexample.

(c) If $f_n \to f$ in measure, does $f_n \to f$ pointwise a.e.? Give a proof or a counterexample.

(d) Now suppose that $(E, \mathcal{E}) = ([0,1], \mathcal{B}([0,1]))$ and that μ is Lebesgue measure on $[0,1]$. Suppose (f_n) is a sequence of Borel measurable functions on $[0,1]$ which converges pointwise a.e. to f.

(i) For each n, k let $E_{n,k} = \bigcup_{m \geqslant n} \{x : |f_m(x) - f(x)| > 1/k\}$. Show that $\lim_{n\to\infty} \mu(E_{n,k}) = 0$ for each $k \in \mathbb{N}$.

(ii) Show that for every $\epsilon > 0$ there exists a set A with $\mu(A) < \epsilon$ so that $f_n \to f$ uniformly on $[0,1] \setminus A$.

(iii) Does (ii) hold with $[0,1]$ replaced by $\mathbb{R}$? Give a proof or a counterexample.

26J Applied Probability

(a) Define a *continuous-time Markov chain* X with infinitesimal generator Q and jump chain Y.

(b) Prove that if a state x is transient for Y, then it is transient for X.

(c) Prove or provide a counterexample to the following: if x is positive recurrent for X, then it is positive recurrent for Y.

(d) Consider the continuous-time Markov chain $(X_t)_{t \geqslant 0}$ on $\mathbb{Z}$ with non-zero transition rates given by

$$q(i, i+1) = 2 \cdot 3^{|i|}, \quad q(i,i) = -3^{|i|+1} \quad \text{and} \quad q(i, i-1) = 3^{|i|}.$$

Determine whether X is transient or recurrent. Let $T_0 = \inf\{t \geqslant J_1 : X_t = 0\}$, where J_1 is the first jump time. Does X have an invariant distribution? Justify your answer. Calculate $\mathbb{E}_0[T_0]$.

(e) Let X be a continuous-time random walk on $\mathbb{Z}^d$ with $q(x) = \|x\|^\alpha \wedge 1$ and $q(x,y) = q(x)/(2d)$ for all $y \in \mathbb{Z}^d$ with $\|y - x\| = 1$. Determine for which values of α the walk is transient and for which it is recurrent. In the recurrent case, determine the range of α for which it is also positive recurrent. [Here $\|x\|$ denotes the Euclidean norm of x.]

27J Principles of Statistics

Derive the maximum likelihood estimator $\widehat{\theta}_n$ based on independent observations $X_1, \ldots, X_n$ that are identically distributed as $N(\theta, 1)$, where the unknown parameter θ lies in the parameter space $\Theta = \mathbb{R}$. Find the limiting distribution of $\sqrt{n}(\widehat{\theta}_n - \theta)$ as $n \to \infty$.

Now define

$$\begin{aligned} \widetilde{\theta}_n &= \widehat{\theta}_n \quad \text{whenever } |\widehat{\theta}_n| > n^{-1/4}, \\ &= 0 \quad \text{otherwise,} \end{aligned}$$

and find the limiting distribution of $\sqrt{n}(\widetilde{\theta}_n - \theta)$ as $n \to \infty$.

Calculate

$$\lim_{n\to\infty} \sup_{\theta\in\Theta} nE_\theta(T_n - \theta)^2$$

for the choices $T_n = \widehat{\theta}_n$ and $T_n = \widetilde{\theta}_n$. Based on the above findings, which estimator T_n of θ would you prefer? Explain your answer.

[*Throughout, you may use standard facts of stochastic convergence, such as the central limit theorem, provided they are clearly stated.*]

28K Stochastic Financial Models

(a) What is a Brownian motion?

(b) State the Brownian reflection principle. State the Cameron–Martin theorem for Brownian motion with constant drift.

(c) Let $(W_t)_{t\geqslant 0}$ be a Brownian motion. Show that

$$\mathbb{P}\left(\max_{0\leqslant s\leqslant t}(W_s + as) \leqslant b\right) = \Phi\left(\frac{b - at}{\sqrt{t}}\right) - e^{2ab}\Phi\left(\frac{-b - at}{\sqrt{t}}\right),$$

where Φ is the standard normal distribution function.

(d) Find

$$\mathbb{P}\left(\max_{u\geqslant t}(W_u + au) \leqslant b\right).$$

29E Dynamical Systems

Consider the dynamical system

$$\begin{aligned} \dot{x} &= x(y-a)\,, \\ \dot{y} &= 1-x-y^2\,, \end{aligned}$$

where $-1 < a < 1$. Find and classify the fixed points of the system.

Use Dulac's criterion with a weighting function of the form $\phi = x^p$ and a suitable choice of p to show that there are no periodic orbits for $a \neq 0$. For the case $a = 0$ use the same weighting function to find a function $V(x,y)$ which is constant on trajectories. [*Hint: $\phi\dot{\mathbf{x}}$ is Hamiltonian.*]

Calculate the stable manifold at $(0,-1)$ correct to quadratic order in x.

Sketch the phase plane for the cases (i) $a = 0$ and (ii) $a = \frac{1}{2}$.

30D Integrable Systems

What does it mean for an evolution equation $u_t = K(x, u, u_x, \ldots)$ to be in *Hamiltonian form*? Define the associated Poisson bracket.

An evolution equation $u_t = K(x, u, u_x, \ldots)$ is said to be *bi-Hamiltonian* if it can be written in Hamiltonian form in two distinct ways, i.e.

$$K = \mathcal{J}\,\delta H_0 = \mathcal{E}\,\delta H_1$$

for Hamiltonian operators $\mathcal{J}, \mathcal{E}$ and functionals H_0, H_1. By considering the sequence $\{H_m\}_{m\geqslant 0}$ defined by the recurrence relation

$$\mathcal{E}\,\delta H_{m+1} = \mathcal{J}\,\delta H_m\,, \qquad (*)$$

show that bi-Hamiltonian systems possess infinitely many first integrals in involution. [You may assume that $(*)$ can always be solved for H_{m+1}, given H_m.]

The Harry Dym equation for the function $u = u(x,t)$ is

$$u_t = \frac{\partial^3}{\partial x^3}\left(u^{-1/2}\right).$$

This equation can be written in Hamiltonian form $u_t = \mathcal{E}\delta H_1$ with

$$\mathcal{E} = 2u\frac{\partial}{\partial x} + u_x\,, \quad H_1[u] = \frac{1}{8}\int u^{-5/2}u_x^2\,\mathrm{d}x\,.$$

Show that the Harry Dym equation possesses infinitely many first integrals in involution. [You need not verify the Jacobi identity if your argument involves a Hamiltonian operator.]

31A Principles of Quantum Mechanics

A particle in one dimension has position and momentum operators $\hat{x}$ and $\hat{p}$ whose eigenstates obey

$$\langle x|x'\rangle = \delta(x-x')\,, \qquad \langle p|p'\rangle = \delta(p-p')\,, \qquad \langle x|p\rangle = (2\pi\hbar)^{-1/2}e^{ixp/\hbar}\,.$$

For a state $|\psi\rangle$, define the position-space and momentum-space wavefunctions $\psi(x)$ and $\tilde{\psi}(p)$ and show how each of these can be expressed in terms of the other.

Write down the translation operator $U(\alpha)$ and check that your expression is consistent with the property $U(\alpha)|x\rangle = |x+\alpha\rangle$. For a state $|\psi\rangle$, relate the position-space and momentum-space wavefunctions for $U(\alpha)|\psi\rangle$ to $\psi(x)$ and $\tilde{\psi}(p)$ respectively.

Now consider a harmonic oscillator with mass m, frequency ω, and annihilation and creation operators

$$a = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)\,, \qquad a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\,.$$

Let $\psi_n(x)$ and $\tilde{\psi}_n(p)$ be the wavefunctions corresponding to the normalised energy eigenstates $|n\rangle$, where $n = 0, 1, 2, \ldots$.

(i) Express $\psi_0(x-\alpha)$ explicitly in terms of the wavefunctions $\psi_n(x)$.

(ii) Given that $\tilde{\psi}_n(p) = f_n(u)\,\tilde{\psi}_0(p)$, where the f_n are polynomials and $u = (2/\hbar m\omega)^{1/2}p$, show that

$$e^{-i\gamma u} = e^{-\gamma^2/2}\sum_{n=0}^{\infty}\frac{\gamma^n}{\sqrt{n!}}\,f_n(u) \quad \text{for any real } \gamma\,.$$

[You may quote standard results for a harmonic oscillator. You may also use, without proof, $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ for operators A and B which each commute with $[A,B]$.]

32A Applications of Quantum Mechanics

A particle in one dimension of mass m and energy $E = \hbar^2 k^2/2m$ $(k > 0)$ is incident from $x = -\infty$ on a potential $V(x)$ with $V(x) \to 0$ as $x \to -\infty$ and $V(x) = \infty$ for $x > 0$. The relevant solution of the time-independent Schrödinger equation has the asymptotic form

$$\psi(x) \sim \exp(ikx) + r(k)\exp(-ikx), \qquad x \to -\infty.$$

Explain briefly why a pole in the reflection amplitude $r(k)$ at $k = i\kappa$ with $\kappa > 0$ corresponds to the existence of a stable bound state in this potential. Indicate why a pole in $r(k)$ just below the real k-axis, at $k = k_0 - i\rho$ with $k_0 \gg \rho > 0$, corresponds to a quasi-stable bound state. Find an approximate expression for the lifetime τ of such a quasi-stable state.

Now suppose that

$$V(x) = \begin{cases} (\hbar^2 U/2m)\,\delta(x+a) & \text{for } x < 0 \\ \infty & \text{for } x > 0 \end{cases}$$

where $U > 0$ and $a > 0$ are constants. Compute the reflection amplitude $r(k)$ in this case and deduce that there are quasi-stable bound states if U is large. Give expressions for the wavefunctions and energies of these states and compute their lifetimes, working to leading non-vanishing order in $1/U$ for each expression.

[You may assume $\psi = 0$ for $x \geqslant 0$ and $\lim_{\epsilon\to 0+}\{\,\psi'(-a+\epsilon) - \psi'(-a-\epsilon)\,\} = U\,\psi(-a)$.]

33C Statistical Physics

Consider an ideal quantum gas with one-particle states $|i\rangle$ of energy ϵ_i. Let $p_i^{(n_i)}$ denote the probability that state $|i\rangle$ is occupied by n_i particles. Here, n_i can take the values 0 or 1 for fermions and any non-negative integer for bosons. The entropy of the gas is given by

$$S = -k_B \sum_i \sum_{n_i} p_i^{(n_i)} \ln p_i^{(n_i)}.$$

(a) Write down the constraints that must be satisfied by the probabilities if the average energy $\langle E\rangle$ and average particle number $\langle N\rangle$ are kept at fixed values.

Show that if S is maximised then

$$p_i^{(n_i)} = \frac{1}{\mathcal{Z}_i} e^{-(\beta\epsilon_i + \gamma)n_i},$$

where β and γ are Lagrange multipliers. What is $\mathcal{Z}_i$?

(b) Insert these probabilities $p_i^{(n_i)}$ into the expression for S, and combine the result with the first law of thermodynamics to find the meaning of β and γ.

(c) Calculate the average occupation number $\langle n_i\rangle = \sum_{n_i} n_i p_i^{(n_i)}$ for a gas of fermions.

34E Electrodynamics

A point particle of charge q and mass m moves in an electromagnetic field with 4-vector potential $A_\mu(x)$, where x^μ is position in spacetime. Consider the action

$$S = -mc\int\Big(-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\Big)^{1/2}d\lambda \;+\; q\int A_\mu\frac{dx^\mu}{d\lambda}\,d\lambda\,, \qquad (*)$$

where λ is an arbitrary parameter along the particle's worldline and $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ is the Minkowski metric.

(a) By varying the action with respect to $x^\mu(\lambda)$, with fixed endpoints, obtain the equation of motion

$$m\frac{du^\mu}{d\tau} = qF^\mu{}_\nu u^\nu\,,$$

where τ is the proper time, $u^\mu = dx^\mu/d\tau$ is the velocity 4-vector, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor.

(b) This particle moves in the field generated by a second point charge Q that is held at rest at the origin of some inertial frame. By choosing a suitable expression for A_μ and expressing the first particle's spatial position in spherical polar coordinates (r,θ,ϕ), show from the action $(*)$ that

$$\mathcal{E} \equiv \dot{t} - \Gamma/r\,,$$
$$\ell c \equiv r^2\dot{\phi}\sin^2\theta$$

are constants, where $\Gamma = -qQ/(4\pi\epsilon_0 mc^2)$ and overdots denote differentiation with respect to τ.

(c) Show that when the motion is in the plane $\theta = \pi/2$,

$$\mathcal{E} + \frac{\Gamma}{r} = \sqrt{1 + \frac{\dot{r}^2}{c^2} + \frac{\ell^2}{r^2}}\,.$$

Hence show that the particle's orbit is bounded if $\mathcal{E} < 1$, and that the particle can reach the origin in finite proper time if $\Gamma > |\ell|$.

35D General Relativity

Consider a family of geodesics with s an affine parameter and V^a the tangent vector on each curve. The *equation of geodesic deviation for a vector field* W^a is

$$\frac{D^2W^a}{Ds^2} = R^a{}_{bcd}V^bV^cW^d\,, \qquad (*)$$

where $\dfrac{D}{Ds}$ denotes the directional covariant derivative $V^b\nabla_b$.

(i) Show that if

$$V^b\,\frac{\partial W^a}{\partial x^b} = W^b\,\frac{\partial V^a}{\partial x^b}$$

then W^a satisfies $(*)$.

(ii) Show that V^a and sV^a satisfy $(*)$.

(iii) Show that if W^a is a Killing vector field, meaning that $\nabla_bW_a + \nabla_aW_b = 0$, then W^a satisfies $(*)$.

(iv) Show that if $W^a = wU^a$ satisfies $(*)$, where w is a scalar field and U^a is a time-like unit vector field, then

$$\frac{d^2w}{ds^2} = (\Omega^2 - K)w\,,$$

$$\text{where} \quad \Omega^2 = -\frac{DU^a}{Ds}\frac{DU_a}{Ds} \quad \text{and} \quad K = R_{abcd}\,U^aV^bV^cU^d\,.$$

[*You may use*: $\nabla_b\nabla_cX^a - \nabla_c\nabla_bX^a = R^a{}_{dbc}X^d$ *for any vector field* X^a.]

36B Fluid Dynamics II

State the vorticity equation and interpret the meaning of each term.

A planar vortex sheet is diffusing in the presence of a perpendicular straining flow. The flow is everywhere of the form $\mathbf{u} = (U(y,t), -Ey, Ez)$, where $U \to \pm U_0$ as $y \to \pm\infty$, and U_0 and $E > 0$ are constants. Show that the vorticity has the form $\boldsymbol{\omega} = \omega(y,t)\mathbf{e}_z$, and obtain a scalar equation describing the evolution of ω.

Explain physically why the solution approaches a steady state in which the vorticity is concentrated near $y = 0$. Use scaling to estimate the thickness δ of the steady vorticity layer as a function of E and the kinematic viscosity ν.

Determine the steady vorticity profile, $\omega(y)$, and the steady velocity profile, $U(y)$.

$\left[\textit{Hint:} \quad \mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-u^2}\mathrm{d}u.\ \right]$

State, with a brief physical justification, why you might expect this steady flow to be unstable to long-wavelength perturbations, defining what you mean by long.

37D Waves

Write down the linearised equations governing motion of an inviscid compressible fluid at uniform entropy. Assuming that the velocity is irrotational, show that it may be derived from a velocity potential $\phi(\mathbf{x}, t)$ satisfying the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \nabla^2 \phi \,,$$

and identify the wave speed c_0. Obtain from these linearised equations the energy-conservation equation

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{I} = 0 \,,$$

and give expressions for the acoustic-energy density E and the acoustic-energy flux $\mathbf{I}$ in terms of ϕ.

Such a fluid occupies a semi-infinite waveguide $x > 0$ of square cross-section $0 < y < a$, $0 < z < a$ bounded by rigid walls. An impenetrable membrane closing the end $x = 0$ makes prescribed small displacements to

$$x = X(y, z, t) \equiv \mathrm{Re}\left[e^{-i\omega t} A(y, z)\right] \,,$$

where $\omega > 0$ and $|A| \ll a, c_0/\omega$. Show that the velocity potential is given by

$$\phi = \mathrm{Re}\left[e^{-i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos\left(\frac{m\pi y}{a}\right) \cos\left(\frac{n\pi z}{a}\right) f_{mn}(x)\right] \,,$$

where the functions $f_{mn}(x)$, including their amplitudes, are to be determined, with the sign of any square roots specified clearly.

If $0 < \omega < \pi c_0/a$, what is the asymptotic behaviour of ϕ as $x \to +\infty$? Using this behaviour and the energy-conservation equation averaged over both time and the cross-section, or otherwise, determine the double-averaged energy flux along the waveguide,

$$\left\langle \, \overline{I_x} \, \right\rangle(x) \equiv \frac{\omega}{2\pi a^2} \int_0^{2\pi/\omega} \int_0^a \int_0^a I_x(x, y, z, t) \; \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}t \,,$$

explaining why this is independent of x.

38B Numerical Analysis

(a) Consider the periodic function

$$f(x) = 5 + 2\cos\left(2\pi x - \frac{\pi}{2}\right) + 3\cos(4\pi x)$$

on the interval $[0,1]$. The N-point discrete Fourier transform of f is defined by

$$F_N(n) = \frac{1}{N}\sum_{k=0}^{N-1} f_k\, \omega_N^{-nk}, \quad n = 0, 1, \ldots, N-1, \tag{$*$}$$

where $\omega_N = e^{2\pi i/N}$ and $f_k = f(k/N)$. Compute $F_4(n)$, $n = 0, \ldots, 3$, using the Fast Fourier Transform (FFT). Compare your result with what you get by computing $F_4(n)$ directly from $(*)$. Carefully explain all your computations.

(b) Now let f be any analytic function on $\mathbb{R}$ that is periodic with period 1. The continuous Fourier transform of f is defined by

$$\hat{f}_n = \int_0^1 f(\tau)\, e^{-2\pi i n\tau}\, d\tau\,, \quad n \in \mathbb{Z}\,.$$

Use integration by parts to show that the Fourier coefficients $\hat{f}_n$ decay spectrally.

Explain what it means for the discrete Fourier transform of f to approximate the continuous Fourier transform with *spectral accuracy*. Prove that it does so.

What can you say about the behaviour of $F_N(N-n)$ as $N \to \infty$ for fixed n?

END OF PAPER