UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 2 June, 2016 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1E Groups, Rings and Modules

Let G be a group of order n. Define what is meant by a *permutation representation* of G. Using such representations, show G is isomorphic to a subgroup of the symmetric group S_n. Assuming G is non-abelian simple, show G is isomorphic to a subgroup of A_n. Give an example of a permutation representation of S_3 whose kernel is A_3.

2G Analysis II

(a) Let X be a subset of $\mathbb{R}$. What does it mean to say that a sequence of functions $f_n \colon X \to \mathbb{R}$ $(n \in \mathbb{N})$ is *uniformly convergent*?

(b) Which of the following sequences of functions are uniformly convergent? Justify your answers.

(i) $f_n \colon (0,1) \to \mathbb{R}$, $\qquad f_n(x) = \dfrac{1-x^n}{1-x}$.

(ii) $f_n \colon (0,1) \to \mathbb{R}$, $\qquad f_n(x) = \displaystyle\sum_{k=1}^{n} \frac{1}{k^2} x^k$.

(iii) $f_n \colon \mathbb{R} \to \mathbb{R}$, $\qquad f_n(x) = x/n$.

(iv) $f_n \colon [0,\infty) \to \mathbb{R}$, $\qquad f_n(x) = xe^{-nx}$.

3E Metric and Topological Spaces

Let X be a topological space and $A \subseteq X$ be a subset. A *limit point* of A is a point $x \in X$ such that any open neighbourhood U of x intersects A. Show that A is closed if and only if it contains all its limit points. Explain what is meant by the *interior* $\mathrm{Int}(A)$ and the *closure* $\overline{A}$ of A. Show that if A is connected, then $\overline{A}$ is connected.

4A Complex Methods

The function $f(x)$ has Fourier transform

$$\tilde{f}(k) \;=\; \int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \frac{-2ki}{p^2+k^2},$$

where $p > 0$ is a real constant. Using contour integration, calculate $f(x)$ for $x < 0$. [Jordan's lemma and the residue theorem may be used without proof.]

5F Geometry

(a) State Euler's formula for a triangulation of a sphere.

(b) A sphere is decomposed into hexagons and pentagons with precisely three edges at each vertex. Determine the number of pentagons.

6C Variational Principles

Two points A and B are located on the curved surface of the circular cylinder of radius R with axis along the $z-$axis. We denote their locations by (R, ϕ_A, z_A) and (R, ϕ_B, z_B) using cylindrical polar coordinates and assume $\phi_A \neq \phi_B$, $z_A \neq z_B$. A path $\phi(z)$ is drawn on the cylinder to join A and B. Show that the path of minimum distance between the points A and B is a helix, and determine its pitch. [For a helix with axis parallel to the z axis, the pitch is the change in z after one complete helical turn.]

7A Methods

Calculate the Green's function $G(x;\xi)$ given by the solution to

$$\frac{d^2G}{dx^2} = \delta(x-\xi); \quad G(0;\xi) = \frac{dG}{dx}(1;\xi) = 0,$$

where $\xi \in (0,1)$, $x \in (0,1)$ and $\delta(x)$ is the Dirac δ-function. Use this Green's function to calculate an explicit solution $y(x)$ to the boundary value problem

$$\frac{d^2y}{dx^2} = xe^{-x},$$

where $x \in (0,1)$, and $y(0) = y'(1) = 0$.

8B Quantum Mechanics

(a) Consider a quantum particle moving in one space dimension, in a time-independent real potential $V(x)$. For a wavefunction $\psi(x,t)$, define the *probability density* $\rho(x,t)$ and *probability current* $j(x,t)$ and show that

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0.$$

(b) Suppose now that $V(x) = 0$ and $\psi(x,t) = (e^{ikx} + Re^{-ikx})e^{-iEt/\hbar}$, where $E = \hbar^2k^2/(2m)$, k and m are real positive constants, and R is a complex constant. Compute the probability current for this wavefunction. Interpret the terms in ψ and comment on how this relates to the computed expression for the probability current.

9H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain such that $X_0 = i$. Prove that

$$\sum_{n=0}^{\infty} \mathbb{P}_i(X_n = i) = \frac{1}{\mathbb{P}_i(X_n \neq i \text{ for all } n \geqslant 1)}$$

where $1/0 = +\infty$. [You may use the strong Markov property without proof.]

SECTION II

10F Linear Algebra

Let $\alpha : V \to V$ be a linear transformation defined on a finite dimensional inner product space V over $\mathbb{C}$. Recall that α is normal if α and its adjoint α^* commute. Show that α being normal is equivalent to each of the following statements:

(i) $\alpha = \alpha_1 + i\alpha_2$ where α_1, α_2 are self-adjoint operators and $\alpha_1\alpha_2 = \alpha_2\alpha_1$;

(ii) there is an orthonormal basis for V consisting of eigenvectors of α;

(iii) there is a polynomial g with complex coefficients such that $\alpha^* = g(\alpha)$.

11E Groups, Rings and Modules

(a) Define what is meant by an *algebraic integer* α. Show that the ideal

$$I = \{h \in \mathbb{Z}[x] \mid h(\alpha) = 0\}$$

in $\mathbb{Z}[x]$ is generated by a monic irreducible polynomial f. Show that $\mathbb{Z}[\alpha]$, considered as a $\mathbb{Z}$-module, is freely generated by n elements where $n = \deg f$.

(b) Assume $\alpha \in \mathbb{C}$ satisfies $\alpha^5 + 2\alpha + 2 = 0$. Is it true that the ideal (5) in $\mathbb{Z}[\alpha]$ is a prime ideal? Is there a ring homomorphism $\mathbb{Z}[\alpha] \to \mathbb{Z}[\sqrt{-1}]$? Justify your answers.

(c) Show that the only unit elements of $\mathbb{Z}[\sqrt{-5}]$ are 1 and -1. Show that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

12G Analysis II

Let X be a metric space.

(a) What does it mean to say that a function $f \colon X \to \mathbb{R}$ is *uniformly continuous*? What does it mean to say that f is *Lipschitz*? Show that if f is Lipschitz then it is uniformly continuous. Show also that if $(x_n)_n$ is a Cauchy sequence in X, and f is uniformly continuous, then the sequence $(f(x_n))_n$ is convergent.

(b) Let $f \colon X \to \mathbb{R}$ be continuous, and X be sequentially compact. Show that f is uniformly continuous. Is f necessarily Lipschitz? Justify your answer.

(c) Let Y be a dense subset of X, and let $g \colon Y \to \mathbb{R}$ be a continuous function. Show that there exists at most one continuous function $f \colon X \to \mathbb{R}$ such that for all $y \in Y$, $f(y) = g(y)$. Prove that if g is uniformly continuous, then such a function f exists, and is uniformly continuous.

[A subset $Y \subset X$ is *dense* if for any nonempty open subset $U \subset X$, the intersection $U \cap Y$ is nonempty.]

13G Complex Analysis

(a) Prove Cauchy's theorem for a triangle.

(b) Write down an expression for the winding number $I(\gamma, a)$ of a closed, piecewise continuously differentiable curve γ about a point $a \in \mathbb{C}$ which does not lie on γ.

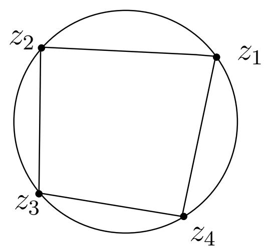
(c) Let $U \subset \mathbb{C}$ be a domain, and $f\colon U \to \mathbb{C}$ a holomorphic function with no zeroes in U. Suppose that for infinitely many positive integers k the function f has a holomorphic k-th root. Show that there exists a holomorphic function $F\colon U \to \mathbb{C}$ such that $f = \exp F$.

14F Geometry

(a) Define the *cross-ratio* $[z_1, z_2, z_3, z_4]$ of four distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C} \cup \{\infty\}$. Show that the cross-ratio is invariant under Möbius transformations. Express $[z_2, z_1, z_3, z_4]$ in terms of $[z_1, z_2, z_3, z_4]$.

(b) Show that $[z_1, z_2, z_3, z_4]$ is real if and only if z_1, z_2, z_3, z_4 lie on a line or circle in $\mathbb{C} \cup \{\infty\}$.

(c) Let z_1, z_2, z_3, z_4 lie on a circle in $\mathbb{C}$, given in anti-clockwise order as depicted.



Show that $[z_1, z_2, z_3, z_4]$ is a negative real number, and that $[z_2, z_1, z_3, z_4]$ is a positive real number greater than 1. Show that $|[z_1, z_2, z_3, z_4]| + 1 = |[z_2, z_1, z_3, z_4]|$. Use this to deduce Ptolemy's relation on lengths of edges and diagonals of the inscribed 4-gon:

$$|z_1 - z_3||z_2 - z_4| = |z_1 - z_2||z_3 - z_4| + |z_2 - z_3||z_4 - z_1|.$$

15B Methods

(a) Show that the Fourier transform of $f(x) = e^{-a^2x^2}$, for $a > 0$, is

$$\tilde{f}(k) = \frac{\sqrt{\pi}}{a} e^{-\frac{k^2}{4a^2}} ,$$

stating clearly any properties of the Fourier transform that you use.
[*Hint: You may assume that* $\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$.]

(b) Consider now the Cauchy problem for the diffusion equation in one space dimension, i.e. solving for $\theta(x,t)$ satisfying:

$$\frac{\partial \theta}{\partial t} = D \frac{\partial^2 \theta}{\partial x^2} \quad \text{with } \theta(x,0) = g(x),$$

where D is a positive constant and $g(x)$ is specified. Consider the following property of a solution:
Property P: If the initial data $g(x)$ is positive and it is non-zero only within a bounded region (i.e. there is a constant α such that $\theta(x,0) = 0$ for all $|x| > \alpha$), then for any $\epsilon > 0$ (however small) and β (however large) the solution $\theta(\beta, \epsilon)$ can be non-zero, i.e. the solution can become non-zero arbitrarily far away after an arbitrarily short time.

Does Property P hold for solutions of the diffusion equation? Justify your answer (deriving any expression for the solution $\theta(x,t)$ that you use).

(c) Consider now the wave equation in one space dimension:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

with given initial data $u(x,0) = \phi(x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$ (and c is a constant).
Does Property P (with $g(x)$ and $\theta(\beta,\epsilon)$ now replaced by $\phi(x)$ and $u(\beta,\epsilon)$ respectively) hold for solutions of the wave equation? Justify your answer again as above.

16B Quantum Mechanics

The spherically symmetric bound state wavefunctions $\psi(r)$ for the Coulomb potential $V = -e^2/(4\pi\epsilon_0 r)$ are normalisable solutions of the equation

$$\frac{d^2\psi}{dr^2} + \frac{2}{r}\frac{d\psi}{dr} + \frac{2\lambda}{r}\psi = -\frac{2mE}{\hbar^2}\psi.$$

Here $\lambda = (me^2)/(4\pi\epsilon_0\hbar^2)$ and $E < 0$ is the energy of the state.

(a) By writing the wavefunction as $\psi(r) = f(r)\exp(-Kr)$, for a suitable constant K that you should determine, show that there are normalisable wavefunctions $\psi(r)$ only for energies of the form

$$E = \frac{-me^4}{32\pi^2\epsilon_0^2\hbar^2 N^2},$$

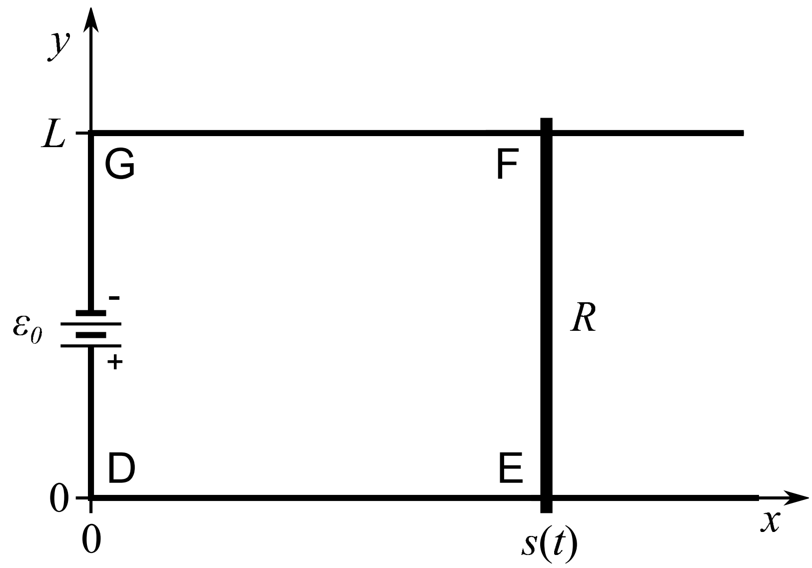
with N being a positive integer.

(b) The energies in (a) reproduce the predictions of the Bohr model of the hydrogen atom. How do the wavefunctions above compare to the assumptions in the Bohr model?

17D Electromagnetism

(a) State Faraday's law of induction for a moving circuit in a time-dependent magnetic field and define all the terms that appear.

(b) Consider a rectangular circuit DEFG in the $z = 0$ plane as shown in the diagram below. There are two rails parallel to the x-axis for $x > 0$ starting at D at $(x, y) = (0, 0)$ and G at $(0, L)$. A battery provides an electromotive force $\mathcal{E}_0$ between D and G driving current in a positive sense around DEFG. The circuit is completed with a bar resistor of resistance R, length L and mass m that slides without friction on the rails; it connects E at $(s(t), 0)$ and F at $(s(t), L)$. The rest of the circuit has no resistance. The circuit is in a constant uniform magnetic field B_0 parallel to the z-axis.
[In parts (i)-(iv) you can neglect any magnetic field due to current flow.]



(i) Calculate the current in the bar and indicate its direction on a diagram of the circuit.

(ii) Find the force acting on the bar.

(iii) If the initial velocity and position of the bar are respectively $\dot{s}(0) = v_0 > 0$ and $s(0) = s_0 > 0$, calculate $\dot{s}(t)$ and $s(t)$ for $t > 0$.

(iv) If $\mathcal{E}_0 = 0$, find the total energy dissipated in the circuit after $t = 0$ and verify that total energy is conserved.

(v) Describe qualitatively the effect of the magnetic field caused by the induced current flowing in the circuit when $\mathcal{E}_0 = 0$.

18C Fluid Dynamics

A layer of thickness h_1 of a fluid of density ρ_1 is located above a layer of thickness h_2 of a fluid of density $\rho_2 > \rho_1$. The two-fluid system is bounded by two impenetrable surfaces at $y = h_1$ and $y = -h_2$ and is assumed to be two-dimensional (i.e. independent of z). The fluid is subsequently perturbed, and the interface between the two fluids is denoted $y = \eta(x,t)$.

(a) Assuming irrotational motion in each fluid, state the equations and boundary conditions satisfied by the flow potentials, φ_1 and φ_2.

(b) The interface is perturbed by small-amplitude waves of the form $\eta = \eta_0 e^{i(kx-\omega t)}$, with $\eta_0 k \ll 1$. State the equations and boundary conditions satisfied by the linearised system.

(c) Calculate the dispersion relation of the waves relating the frequency ω to the wavenumber k.

19D Numerical Analysis

(a) Determine real quadratic functions $a(x), b(x), c(x)$ such that the interpolation formula,

$$f(x) \approx a(x)f(0) + b(x)f(2) + c(x)f(3)\,,$$

is exact when $f(x)$ is any real polynomial of degree 2.

(b) Use this formula to construct approximations for $f(5)$ and $f'(1)$ which are exact when $f(x)$ is any real polynomial of degree 2. Calculate these approximations for $f(x) = x^3$ and comment on your answers.

(c) State the Peano kernel theorem and define the *Peano kernel* $K(\theta)$. Use this theorem to find the minimum values of the constants α and β such that

$$\left| f(1) - \tfrac{1}{3}\big[f(0) + 3f(2) - f(3)\big] \right| \leqslant \alpha \max_{\xi\in[0,3]} \left| f^{(2)}(\xi) \right|,$$

and

$$\left| f(1) - \tfrac{1}{3}\big[f(0) + 3f(2) - f(3)\big] \right| \leqslant \beta \, \|f^{(2)}\|_1\,,$$

where $f \in C^2[0,3]$. Check that these inequalities hold for $f(x) = x^3$.

20H Statistics

Let $X_1, \ldots, X_n$ be independent samples from the Poisson distribution with mean θ.

(a) Compute the maximum likelihood estimator of θ. Is this estimator biased?

(b) Under the assumption that n is very large, use the central limit theorem to find an approximate 95% confidence interval for θ. [You may use the notation z_α for the number such that $\mathbb{P}(Z \geqslant z_\alpha) = \alpha$ for a standard normal $Z \sim N(0,1)$.]

(c) Now suppose the parameter θ has the $\Gamma(k, \lambda)$ prior distribution. What is the posterior distribution? What is the Bayes point estimator for θ for the quadratic loss function $L(\theta, a) = (\theta - a)^2$? Let X_{n+1} be another independent sample from the same distribution. Given $X_1, \ldots, X_n$, what is the posterior probability that $X_{n+1} = 0$?
[*Hint: The density of the* $\Gamma(k, \lambda)$ *distribution is* $f(x; k, \lambda) = \lambda^k x^{k-1} e^{-\lambda x} / \Gamma(k)$, *for* $x > 0$.]

21H Optimization

(a) State and prove the Lagrangian sufficiency theorem.

(b) Let $n \geqslant 1$ be a given constant, and consider the problem:

$$\text{minimise} \sum_{i=1}^{n} \left(2y_i^2 + x_i^2\right) \text{ subject to } x_i = 1 + \sum_{k=1}^{i} y_k \text{ for all } i = 1, \ldots, n.$$

Find, with proof, constants a, b, A, B such that the optimal solution is given by

$$x_i = a2^i + b2^{-i} \text{ and } y_i = A2^i + B2^{-i}, \text{ for all } i = 1, \ldots, n.$$

END OF PAPER