UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part IA 2016

## List of Courses

**Paper 1, Section I**

**3D Analysis I**

What does it mean to say that a sequence of real numbers $(x_n)$ converges to $x$? Suppose that $(x_n)$ converges to $x$. Show that the sequence $(y_n)$ given by

$$y_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

also converges to $x$.

**Paper 1, Section I**

**4F Analysis I**

Let $a_n$ be the number of pairs of integers $(x, y) \in \mathbb{Z}^2$ such that $x^2 + y^2 \leqslant n^2$. What is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$? [You may use the comparison test, provided you state it clearly.]

**Paper 1, Section II**

**9E Analysis I**

State the Bolzano–Weierstrass theorem. Use it to show that a continuous function $f : [a, b] \to \mathbb{R}$ attains a global maximum; that is, there is a real number $c \in [a, b]$ such that $f(c) \geqslant f(x)$ for all $x \in [a, b]$.

A function $f$ is said to attain a local maximum at $c \in \mathbb{R}$ if there is some $\varepsilon > 0$ such that $f(c) \geqslant f(x)$ whenever $|x - c| < \varepsilon$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, and that $f''(x) < 0$ for all $x \in \mathbb{R}$. Show that there is at most one $c \in \mathbb{R}$ at which $f$ attains a local maximum.

If there is a constant $K < 0$ such that $f''(x) < K$ for all $x \in \mathbb{R}$, show that $f$ attains a global maximum. [*Hint: if $g'(x) < 0$ for all $x \in \mathbb{R}$, then $g$ is decreasing.*]

Must $f : \mathbb{R} \to \mathbb{R}$ attain a global maximum if we merely require $f''(x) < 0$ for all $x \in \mathbb{R}$? Justify your answer.

**Paper 1, Section II**
**10E Analysis I**

Let $f : \mathbb{R} \to \mathbb{R}$. We say that $x \in \mathbb{R}$ is a real root of $f$ if $f(x) = 0$. Show that if $f$ is differentiable and has $k$ distinct real roots, then $f'$ has at least $k-1$ real roots. [Rolle's theorem may be used, provided you state it clearly.]

Let $p(x) = \sum_{i=1}^{n} a_i x^{d_i}$ be a polynomial in $x$, where all $a_i \neq 0$ and $d_{i+1} > d_i$. (In other words, the $a_i$ are the nonzero coefficients of the polynomial, arranged in order of increasing power of $x$.) The *number of sign changes* in the coefficients of $p$ is the number of $i$ for which $a_i a_{i+1} < 0$. For example, the polynomial $x^5 - x^3 - x^2 + 1$ has 2 sign changes. Show by induction on $n$ that the number of positive real roots of $p$ is less than or equal to the number of sign changes in its coefficients.

**Paper 1, Section II**
**11D Analysis I**

If $(x_n)$ and $(y_n)$ are sequences converging to $x$ and $y$ respectively, show that the sequence $(x_n + y_n)$ converges to $x + y$.

If $x_n \neq 0$ for all $n$ and $x \neq 0$, show that the sequence $\left(\frac{1}{x_n}\right)$ converges to $\frac{1}{x}$.

(a) Find $\lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right)$.

(b) Determine whether $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}}$ converges.

Justify your answers.

**Paper 1, Section II**
**12F Analysis I**

Let $f : [0,1] \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leqslant |x - y|$ for all $x, y \in [0,1]$.

Show that $f$ is continuous and that for all $\varepsilon > 0$, there exists a piecewise constant function $g$ such that

$$\sup_{x\in[0,1]} |f(x) - g(x)| \leqslant \varepsilon.$$

For all integers $n \geqslant 1$, let $u_n = \int_0^1 f(t)\cos(nt)dt$. Show that the sequence $(u_n)$ converges to 0.

**Paper 2, Section I**
**1A Differential Equations**

(a) Find the solution of the differential equation

$$y'' - y' - 6y = 0$$

that is bounded as $x \to \infty$ and satisfies $y = 1$ when $x = 0$.

(b) Solve the difference equation

$$(y_{n+1} - 2y_n + y_{n-1}) - \frac{h}{2}(y_{n+1} - y_{n-1}) - 6h^2 y_n = 0.$$

Show that if $0 < h \ll 1$, the solution that is bounded as $n \to \infty$ and satisfies $y_0 = 1$ is approximately $(1 - 2h)^n$.

(c) By setting $x = nh$, explain the relation between parts (a) and (b).

**Paper 2, Section I**
**2A Differential Equations**

(a) For each non-negative integer $n$ and positive constant $\lambda$, let

$$I_n(\lambda) = \int_0^\infty x^n e^{-\lambda x} dx.$$

By differentiating $I_n$ with respect to $\lambda$, find its value in terms of $n$ and $\lambda$.

(b) By making the change of variables $x = u + v,\ y = u - v$, transform the differential equation

$$\frac{\partial^2 f}{\partial x \partial y} = 1$$

into a differential equation for $g$, where $g(u, v) = f(x, y)$.

**Paper 2, Section II**

**5A Differential Equations**

(a) Find and sketch the solution of

$$y'' + y = \delta(x - \pi/2),$$

where $\delta$ is the Dirac delta function, subject to $y(0) = 1$ and $y'(0) = 0$.

(b) A bowl of soup, which Sam has just warmed up, cools down at a rate equal to the product of a constant $k$ and the difference between its temperature $T(t)$ and the temperature $T_0$ of its surroundings. Initially the soup is at temperature $T(0) = \alpha T_0$, where $\alpha > 2$.

(i) Write down and solve the differential equation satisfied by $T(t)$.

(ii) At time $t_1$, when the temperature reaches half of its initial value, Sam quickly adds some hot water to the soup, so the temperature increases instantaneously by $\beta$, where $\beta > \alpha T_0/2$. Find $t_1$ and $T(t)$ for $t > t_1$.

(iii) Sketch $T(t)$ for $t > 0$.

(iv) Sam wants the soup to be at temperature $\alpha T_0$ at time $t_2$, where $t_2 > t_1$. What value of $\beta$ should Sam choose to achieve this? Give your answer in terms of $\alpha$, $k$, $t_2$ and $T_0$.

**Paper 2, Section II**
**6A Differential Equations**

(a) The function $y(x)$ satisfies

$$y'' + p(x)y' + q(x)y = 0.$$

(i) Define the *Wronskian* $W(x)$ of two linearly independent solutions $y_1(x)$ and $y_2(x)$. Derive a linear first-order differential equation satisfied by $W(x)$.

(ii) Suppose that $y_1(x)$ is known. Use the Wronskian to write down a first-order differential equation for $y_2(x)$. Hence express $y_2(x)$ in terms of $y_1(x)$ and $W(x)$.

(b) Verify that $y_1(x) = \cos(x^\gamma)$ is a solution of

$$ax^\alpha y'' + bx^{\alpha-1}y' + y = 0,$$

where $a$, $b$, $\alpha$ and $\gamma$ are constants, provided that these constants satisfy certain conditions which you should determine.

Use the method that you described in part (a) to find a solution which is linearly independent of $y_1(x)$.

**Paper 2, Section II**

**7A Differential Equations**

The function $y(x)$ satisfies

$$y'' + p(x)y' + q(x)y = 0.$$

What does it mean to say that the point $x = 0$ is (i) an *ordinary point* and (ii) a *regular singular point* of this differential equation? Explain what is meant by the *indicial equation* at a regular singular point. What can be said about the nature of the solutions in the neighbourhood of a regular singular point in the different cases that arise according to the values of the roots of the indicial equation?

State the nature of the point $x = 0$ of the equation

$$xy'' + (x - m + 1)y' - (m - 1)y = 0. \qquad (*)$$

Set $y(x) = x^\sigma \sum_{n=0}^{\infty} a_n x^n$, where $a_0 \neq 0$, and find the roots of the indicial equation.

(a) Show that one solution of $(*)$ with $m \neq 0, -1, -2, \cdots$ is

$$y(x) = x^m \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \; x^n}{(m+n)(m+n-1)\cdots(m+1)} \right),$$

and find a linearly independent solution in the case when $m$ is not an integer.

(b) If $m$ is a positive integer, show that $(*)$ has a polynomial solution.

(c) What is the form of the general solution of $(*)$ in the case $m = 0$? [You do not need to find the general solution explicitly.]

**Paper 2, Section II**

**8A Differential Equations**

(a) By considering eigenvectors, find the general solution of the equations

$$\begin{aligned}\frac{dx}{dt} &= 2x + 5y,\\ \frac{dy}{dt} &= -x - 2y,\end{aligned} \qquad (\dagger)$$

and show that it can be written in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha \begin{pmatrix} 5\cos t \\ -2\cos t - \sin t \end{pmatrix} + \beta \begin{pmatrix} 5\sin t \\ \cos t - 2\sin t \end{pmatrix},$$

where $\alpha$ and $\beta$ are constants.

(b) For any square matrix $M$, $\exp(M)$ is defined by

$$\exp(M) = \sum_{n=0}^{\infty} \frac{M^n}{n!}.$$

Show that if $M$ has constant elements, the vector equation $\frac{d\mathbf{x}}{dt} = M\mathbf{x}$ has a solution $\mathbf{x} = \exp(Mt)\mathbf{x}_0$, where $\mathbf{x}_0$ is a constant vector. Hence solve $(\dagger)$ and show that your solution is consistent with the result of part (a).

**Paper 4, Section I**

**3B Dynamics and Relativity**

With the help of definitions or equations of your choice, determine the dimensions, in terms of mass ($M$), length ($L$), time ($T$) and charge ($Q$), of the following quantities:

(i) force;

(ii) moment of a force (*i.e.* torque);

(iii) energy;

(iv) Newton's gravitational constant $G$;

(v) electric field $\mathbf{E}$;

(vi) magnetic field $\mathbf{B}$;

(vii) the vacuum permittivity $\epsilon_0$.

**Paper 4, Section I**

**4B Dynamics and Relativity**

The radial equation of motion of a particle moving under the influence of a central force is

$$\ddot{r} - \frac{h^2}{r^3} = -kr^n,$$

where $h$ is the angular momentum per unit mass of the particle, $n$ is a constant, and $k$ is a positive constant.

Show that circular orbits with $r = a$ are possible for any positive value of $a$, and that they are stable to small perturbations that leave $h$ unchanged if $n > -3$.

**Paper 4, Section II**

**9B Dynamics and Relativity**

(a) A rocket, moving non-relativistically, has speed $v(t)$ and mass $m(t)$ at a time $t$ after it was fired. It ejects mass with constant speed $u$ relative to the rocket. Let the total momentum, at time $t$, of the system (rocket and ejected mass) in the direction of the motion of the rocket be $P(t)$. Explain carefully why $P(t)$ can be written in the form

$$P(t) = m(t)\,v(t) - \int_0^t \big(v(\tau) - u\big)\frac{dm(\tau)}{d\tau}d\tau\,. \qquad (*)$$

If the rocket experiences no external force, show that

$$m\frac{dv}{dt} + u\frac{dm}{dt} = 0\,. \qquad (\dagger)$$

Derive the expression corresponding to $(*)$ for the total kinetic energy of the system at time $t$. Show that kinetic energy is not necessarily conserved.

(b) Explain carefully how $(*)$ should be modified for a rocket moving relativistically, given that there are no external forces. Deduce that

$$\frac{d(m\gamma v)}{dt} = \left(\frac{v-u}{1-uv/c^2}\right)\frac{d(m\gamma)}{dt}\,,$$

where $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$ and hence that

$$m\gamma^2\frac{dv}{dt} + u\frac{dm}{dt} = 0\,. \qquad (\ddagger)$$

(c) Show that $(\dagger)$ and $(\ddagger)$ agree in the limit $c \to \infty$. Briefly explain the fact that kinetic energy is not conserved for the non-relativistic rocket, but relativistic energy is conserved for the relativistic rocket.

**Paper 4, Section II**
**10B Dynamics and Relativity**

A particle of unit mass moves with angular momentum $h$ in an attractive central force field of magnitude $\frac{k}{r^2}$, where $r$ is the distance from the particle to the centre and $k$ is a constant. *You may assume* that the equation of its orbit can be written in plane polar coordinates in the form

$$r = \frac{\ell}{1 + e\cos\theta},$$

where $\ell = \frac{h^2}{k}$ and $e$ is the eccentricity. Show that the energy of the particle is

$$\frac{h^2(e^2-1)}{2\ell^2}.$$

A comet moves in a parabolic orbit about the Sun. When it is at its perihelion, a distance $d$ from the Sun, and moving with speed $V$, it receives an impulse which imparts an additional velocity of magnitude $\alpha V$ directly away from the Sun. Show that the eccentricity of its new orbit is $\sqrt{1+4\alpha^2}$, and sketch the two orbits on the same axes.

**Paper 4, Section II**
**11B Dynamics and Relativity**

(a) Alice travels at constant speed $v$ to Alpha Centauri, which is at distance $d$ from Earth. She then turns around (taking very little time to do so), and returns at speed $v$. Bob stays at home. By how much has Bob aged during the journey? By how much has Alice aged? [No justification is required.]

Briefly explain what is meant by the *twin paradox* in this context. Why is it not a paradox?

(b) Suppose instead that Alice's world line is given by

$$-c^2t^2 + x^2 = c^2t_0^2,$$

where $t_0$ is a positive constant. Bob stays at home, at $x = \alpha c t_0$, where $\alpha > 1$. Alice and Bob compare their ages on both occasions when they meet. By how much does Bob age? Show that Alice ages by $2t_0 \cosh^{-1}\alpha$.

**Paper 4, Section II**

**12B Dynamics and Relativity**

State what the vectors $\mathbf{a}$, $\mathbf{r}$, $\mathbf{v}$ and $\boldsymbol{\omega}$ represent in the following equation:

$$\mathbf{a} = \mathbf{g} - 2\,\boldsymbol{\omega} \times \mathbf{v} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\,, \qquad (*)$$

where $\mathbf{g}$ is the acceleration due to gravity.

Assume that the radius of the Earth is $6 \times 10^6$ m, that $|\mathbf{g}| = 10\,\mathrm{ms}^{-2}$, and that there are $9 \times 10^4$ seconds in a day. Use these data to determine roughly the order of magnitude of each term on the right hand side of $(*)$ in the case of a particle dropped from a point at height 20 m above the surface of the Earth.

Taking again $|\mathbf{g}| = 10\,\mathrm{ms}^{-2}$, find the time $T$ of the particle's fall in the absence of rotation.

Use a suitable approximation scheme to show that

$$\mathbf{R} \approx \mathbf{R}_0 - \frac{1}{3}\boldsymbol{\omega} \times \mathbf{g}\,T^3 - \frac{1}{2}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}_0)\,T^2\,,$$

where $\mathbf{R}$ is the position vector of the point at which the particle lands, and $\mathbf{R}_0$ is the position vector of the point at which the particle would have landed in the absence of rotation.

The particle is dropped at latitude 45°. Find expressions for the approximate northerly and easterly displacements of $\mathbf{R}$ from $\mathbf{R}_0$ in terms of $\omega$, $g$, $R_0$ (the magnitudes of $\boldsymbol{\omega}$, $\mathbf{g}$ and $\mathbf{R}_0$, respectively), and $T$. You should ignore the curvature of the Earth's surface.

**Paper 3, Section I**

**1D Groups**

Let $G$ be a group, and let $H$ be a subgroup of $G$. Show that the following are equivalent.

(i) $a^{-1}b^{-1}ab \in H$ for all $a, b \in G$.

(ii) $H$ is a normal subgroup of $G$ and $G/H$ is abelian.

Hence find all abelian quotient groups of the dihedral group $D_{10}$ of order 10.

**Paper 3, Section I**

**2D Groups**

State and prove Lagrange's theorem.

Let $p$ be an odd prime number, and let $G$ be a finite group of order $2p$ which has a normal subgroup of order 2. Show that $G$ is a cyclic group.

**Paper 3, Section II**

**5D Groups**

For each of the following, either give an example or show that none exists.

(i) A non-abelian group in which every non-trivial element has order 2.

(ii) A non-abelian group in which every non-trivial element has order 3.

(iii) An element of $S_9$ of order 18.

(iv) An element of $S_9$ of order 20.

(v) A finite group which is not isomorphic to a subgroup of an alternating group.

**Paper 3, Section II**

**6D Groups**

Define the *sign*, $\text{sgn}(\sigma)$, of a permutation $\sigma \in S_n$ and prove that it is well defined. Show that the function $\text{sgn} : S_n \to \{1, -1\}$ is a homomorphism.

Show that there is an injective homomorphism $\psi : GL_2(\mathbb{Z}/2\mathbb{Z}) \to S_4$ such that $\text{sgn} \circ \psi$ is non-trivial.

Show that there is an injective homomorphism $\phi : S_n \to GL_n(\mathbb{R})$ such that $\det(\phi(\sigma)) = \text{sgn}(\sigma)$.

**Paper 3, Section II**

**7D Groups**

State and prove the orbit-stabiliser theorem.

Let $p$ be a prime number, and $G$ be a finite group of order $p^n$ with $n \geqslant 1$. If $N$ is a non-trivial normal subgroup of $G$, show that $N \cap Z(G)$ contains a non-trivial element.

If $H$ is a proper subgroup of $G$, show that there is a $g \in G \backslash H$ such that $g^{-1}Hg = H$.

[You may use Lagrange's theorem, provided you state it clearly.]

**Paper 3, Section II**

**8D Groups**

Define the *Möbius group* $\mathcal{M}$ and its action on the Riemann sphere $\mathbb{C}_\infty$. [You are not required to verify the group axioms.] Show that there is a surjective group homomorphism $\phi : SL_2(\mathbb{C}) \to \mathcal{M}$, and find the kernel of $\phi$.

Show that if a non-trivial element of $\mathcal{M}$ has finite order, then it fixes precisely two points in $\mathbb{C}_\infty$. Hence show that any finite abelian subgroup of $\mathcal{M}$ is either cyclic or isomorphic to $C_2 \times C_2$.

[You may use standard properties of the Möbius group, provided that you state them clearly.]

**Paper 4, Section I**

**1E Numbers and Sets**

Find a pair of integers $x$ and $y$ satisfying $17x + 29y = 1$. What is the smallest positive integer congruent to $17^{138}$ modulo 29?

**Paper 4, Section I**

**2E Numbers and Sets**

Explain the meaning of the phrase *least upper bound;* state the least upper bound property of the real numbers. Use the least upper bound property to show that a bounded, increasing sequence of real numbers converges.

Suppose that $a_n, b_n \in \mathbb{R}$ and that $a_n \geqslant b_n > 0$ for all $n$. If $\sum_{n=1}^{\infty} a_n$ converges, show that $\sum_{n=1}^{\infty} b_n$ converges.

**Paper 4, Section II**

**5E Numbers and Sets**

(a) Let $S$ be a set. Show that there is no bijective map from $S$ to the power set of $S$. Let $\mathcal{T} = \{(x_n) \mid x_i \in \{0,1\} \text{ for all } i \in \mathbb{N}\}$ be the set of sequences with entries in $\{0,1\}$. Show that $\mathcal{T}$ is uncountable.

(b) Let $A$ be a finite set with more than one element, and let $B$ be a countably infinite set. Determine whether each of the following sets is countable. Justify your answers.

(i) $S_1 = \{f : A \to B \mid f \text{ is injective}\}$.

(ii) $S_2 = \{g : B \to A \mid g \text{ is surjective}\}$.

(iii) $S_3 = \{h : B \to B \mid h \text{ is bijective}\}$.

**Paper 4, Section II**

**6E Numbers and Sets**

Suppose that $a, b \in \mathbb{Z}$ and that $b = b_1 b_2$, where $b_1$ and $b_2$ are relatively prime and greater than 1. Show that there exist unique integers $a_1, a_2, n \in \mathbb{Z}$ such that $0 \leqslant a_i < b_i$ and

$$\frac{a}{b} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + n.$$

Now let $b = p_1^{n_1} \dots p_k^{n_k}$ be the prime factorization of $b$. Deduce that $\frac{a}{b}$ can be written uniquely in the form

$$\frac{a}{b} = \frac{q_1}{p_1^{n_1}} + \cdots + \frac{q_k}{p_k^{n_k}} + n\,,$$

where $0 \leqslant q_i < p_i^{n_i}$ and $n \in \mathbb{Z}$. Express $\frac{a}{b} = \frac{1}{315}$ in this form.

**Paper 4, Section II**

**7E Numbers and Sets**

State the inclusion-exclusion principle.

Let $A = (a_1, a_2, \dots, a_n)$ be a string of $n$ digits, where $a_i \in \{0, 1, \dots, 9\}$. We say that the string $A$ has a run of length $k$ if there is some $j \leqslant n - k + 1$ such that either $a_{j+i} \equiv a_j + i \pmod{10}$ for all $0 \leqslant i < k$ or $a_{j+i} \equiv a_j - i \pmod{10}$ for all $0 \leqslant i < k$. For example, the strings

$$(\underline{0, 1, 2}, 8, 4, 9),\ (3, \underline{9, 8, 7}, 4, 8) \text{ and } (3, \underline{1, 0, 9}, 4, 5)$$

all have runs of length 3 (underlined), but no run in $(3, 1, 2, 1, 1, 2)$ has length $> 2$. How many strings of length 6 have a run of length $\geqslant 3$?

**Paper 4, Section II**

**8E Numbers and Sets**

Define the binomial coefficient $\binom{n}{m}$. Prove directly from your definition that

$$(1+z)^n = \sum_{m=0}^{n} \binom{n}{m} z^m$$

for any complex number $z$.

(a) Using this formula, or otherwise, show that

$$\sum_{k=0}^{3n} (-3)^k \binom{6n}{2k} = 2^{6n}.$$

(b) By differentiating, or otherwise, evaluate $\sum_{m=0}^{n} m \binom{n}{m}$.

Let $S_r(n) = \sum_{m=0}^{n} (-1)^m m^r \binom{n}{m}$, where $r$ is a non-negative integer. Show that $S_r(n) = 0$ for $r < n$. Evaluate $S_n(n)$.

**Paper 2, Section I**

**3F Probability**

Let $X_1, \ldots, X_n$ be independent random variables, all with uniform distribution on $[0,1]$. What is the probability of the event $\{X_1 > X_2 > \cdots > X_{n-1} > X_n\}$?

**Paper 2, Section I**

**4F Probability**

Define the *moment-generating function* $m_Z$ of a random variable $Z$. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with distribution $\mathcal{N}(0,1)$, and let $Z = X_1^2 + \cdots + X_n^2$. For $\theta < 1/2$, show that

$$m_Z(\theta) = (1 - 2\theta)^{-n/2} .$$

**Paper 2, Section II**

**9F Probability**

For any positive integer $n$ and positive real number $\theta$, the Gamma distribution $\Gamma(n, \theta)$ has density $f_\Gamma$ defined on $(0, \infty)$ by

$$f_\Gamma(x) = \frac{\theta^n}{(n-1)!} x^{n-1} e^{-\theta x} .$$

For any positive integers $a$ and $b$, the Beta distribution $B(a,b)$ has density $f_B$ defined on $(0,1)$ by

$$f_B(x) = \frac{(a+b-1)!}{(a-1)!(b-1)!} x^{a-1} (1-x)^{b-1} .$$

Let $X$ and $Y$ be independent random variables with respective distributions $\Gamma(n, \theta)$ and $\Gamma(m, \theta)$. Show that the random variables $X/(X+Y)$ and $X+Y$ are independent and give their distributions.

**Paper 2, Section II**

**10F Probability**

We randomly place $n$ balls in $m$ bins independently and uniformly. For each $i$ with $1 \leqslant i \leqslant m$, let $B_i$ be the number of balls in bin $i$.

(a) What is the distribution of $B_i$? For $i \neq j$, are $B_i$ and $B_j$ independent?

(b) Let $E$ be the number of empty bins, $C$ the number of bins with two or more balls, and $S$ the number of bins with exactly one ball. What are the expectations of $E$, $C$ and $S$?

(c) Let $m = an$, for an integer $a \geqslant 2$. What is $\mathbb{P}(E = 0)$? What is the limit of $\mathbb{E}[E]/m$ when $n \to \infty$?

(d) Instead, let $n = dm$, for an integer $d \geqslant 2$. What is $\mathbb{P}(C = 0)$? What is the limit of $\mathbb{E}[C]/m$ when $n \to \infty$?

**Paper 2, Section II**

**11F Probability**

Let $X$ be a non-negative random variable such that $\mathbb{E}[X^2] > 0$ is finite, and let $\theta \in [0, 1]$.

(a) Show that

$$\mathbb{E}[X\,\mathbb{I}[\{X > \theta\mathbb{E}[X]\}]] \geqslant (1-\theta)\mathbb{E}[X]\,.$$

(b) Let $Y_1$ and $Y_2$ be random variables such that $\mathbb{E}[Y_1^2]$ and $\mathbb{E}[Y_2^2]$ are finite. State and prove the Cauchy–Schwarz inequality for these two variables.

(c) Show that

$$\mathbb{P}(X > \theta\,\mathbb{E}[X]) \geqslant (1-\theta)^2 \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}\,.$$

**Paper 2, Section II**

**12F Probability**

A random graph with $n$ nodes $v_1, \ldots, v_n$ is drawn by placing an edge with probability $p$ between $v_i$ and $v_j$ for all distinct $i$ and $j$, independently. A triangle is a set of three distinct nodes $v_i, v_j, v_k$ that are all connected: there are edges between $v_i$ and $v_j$, between $v_j$ and $v_k$ and between $v_i$ and $v_k$.

(a) Let $T$ be the number of triangles in this random graph. Compute the maximum value and the expectation of $T$.

(b) State the Markov inequality. Show that if $p = 1/n^{\alpha}$, for some $\alpha > 1$, then $\mathbb{P}(T = 0) \to 1$ when $n \to \infty$.

(c) State the Chebyshev inequality. Show that if $p$ is such that $\mathrm{Var}[T]/\mathbb{E}[T]^2 \to 0$ when $n \to \infty$, then $\mathbb{P}(T = 0) \to 0$ when $n \to \infty$.

**Paper 3, Section I**

**3C Vector Calculus**

State the chain rule for the derivative of a composition $t \mapsto f(\mathbf{X}(t))$, where $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{X} : \mathbb{R} \to \mathbb{R}^n$ are smooth.

Consider parametrized curves given by

$$\mathbf{x}(t) = (x(t), y(t)) = (a\cos t, a\sin t)\,.$$

Calculate the tangent vector $\dfrac{d\mathbf{x}}{dt}$ in terms of $x(t)$ and $y(t)$. Given that $u(x,y)$ is a smooth function in the upper half-plane $\{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}$ satisfying

$$x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = u\,,$$

deduce that

$$\frac{d}{dt}\,u\left(x(t), y(t)\right) \;=\; u\left(x(t), y(t)\right).$$

If $u(1,1) = 10$, find $u(-1,1)$.

**Paper 3, Section I**

**4C Vector Calculus**

If $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are vectors in $\mathbb{R}^3$, show that $T_{ij} = v_i w_j$ defines a rank 2 tensor. For which choices of the vectors $\mathbf{v}$ and $\mathbf{w}$ is $T_{ij}$ isotropic?

Write down the most general isotropic tensor of rank 2.

Prove that $\epsilon_{ijk}$ defines an isotropic rank 3 tensor.

**Paper 3, Section II**

**9C Vector Calculus**

What is a *conservative* vector field on $\mathbb{R}^n$?

State Green's theorem in the plane $\mathbb{R}^2$.

(a) Consider a smooth vector field $\mathbf{V} = \big(P(x,y), Q(x,y)\big)$ defined on all of $\mathbb{R}^2$ which satisfies
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0\,.$$
By considering
$$F(x,y) = \int_0^x P(x',0)\,dx' \;+\; \int_0^y Q(x,y')\,dy'$$
or otherwise, show that $\mathbf{V}$ is conservative.

(b) Now let $\mathbf{V} = \big(1+\cos(2\pi x + 2\pi y), 2 + \cos(2\pi x + 2\pi y)\big)$. Show that there exists a smooth function $F(x,y)$ such that $\mathbf{V} = \nabla F$.

Calculate $\int_C \mathbf{V}\cdot d\mathbf{x}$, where $C$ is a smooth curve running from $(0,0)$ to $(m,n) \in \mathbb{Z}^2$. Deduce that there does *not* exist a smooth function $F(x,y)$ which satisfies $\mathbf{V} = \nabla F$ and which is, in addition, periodic with period 1 in each coordinate direction, *i.e.* $F(x,y) = F(x+1,y) = F(x,y+1)$.

**Paper 3, Section II**

**10C Vector Calculus**

Define the *Jacobian* $J[\mathbf{u}]$ of a smooth mapping $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$. Show that if $\mathbf{V}$ is the vector field with components
$$V_i = \frac{1}{3!}\epsilon_{ijk}\epsilon_{abc}\frac{\partial u_a}{\partial x_j}\frac{\partial u_b}{\partial x_k}u_c\,,$$
then $J[\mathbf{u}] = \nabla\cdot\mathbf{V}$. If $\mathbf{v}$ is another such mapping, state the chain rule formula for the derivative of the composition $\mathbf{w}(\mathbf{x}) = \mathbf{u}(\mathbf{v}(\mathbf{x}))$, and hence give $J[\mathbf{w}]$ in terms of $J[\mathbf{u}]$ and $J[\mathbf{v}]$.

Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Let there be given, for each $t \in \mathbb{R}$, a smooth mapping $\mathbf{u}_t : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathbf{u}_t(\mathbf{x}) = \mathbf{x} + t\mathbf{F}(\mathbf{x}) + o(t)$ as $t \to 0$. Show that
$$J[\mathbf{u}_t] \;=\; 1 + tQ(x) + o(t)$$
for some $Q(x)$, and express $Q$ in terms of $\mathbf{F}$. Assuming now that $\mathbf{u}_{t+s}(\mathbf{x}) = \mathbf{u}_t(\mathbf{u}_s(\mathbf{x}))$, deduce that if $\nabla\cdot\mathbf{F} = 0$ then $J[\mathbf{u}_t] = 1$ for all $t \in \mathbb{R}$. What geometric property of the mapping $\mathbf{x} \mapsto \mathbf{u}_t(\mathbf{x})$ does this correspond to?

**Paper 3, Section II**

**11C Vector Calculus**

(a) For smooth scalar fields $u$ and $v$, derive the identity

$$\nabla \cdot (u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u$$

and deduce that

$$\int_{\rho \leqslant |\mathbf{x}| \leqslant r} \left(v\nabla^2 u - u\nabla^2 v\right)\, dV = \int_{|\mathbf{x}|=r} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS - \int_{|\mathbf{x}|=\rho} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right) dS\,.$$

Here $\nabla^2$ is the Laplacian, $\dfrac{\partial}{\partial n} = \mathbf{n}\cdot\nabla$ where $\mathbf{n}$ is the unit outward normal, and $dS$ is the scalar area element.

(b) Give the expression for $\big(\nabla \times \mathbf{V}\big)_i$ in terms of $\epsilon_{ijk}$. Hence show that

$$\nabla \times \big(\nabla \times \mathbf{V}\big) \;=\; \nabla(\nabla \,\cdot \mathbf{V}) \;-\; \nabla^2 \mathbf{V}\,.$$

(c) Assume that if $\nabla^2\varphi \;=\; -\rho$, where $\varphi(\mathbf{x}) \;=\; O(|\mathbf{x}|^{-1})$ and $\nabla\varphi(\mathbf{x}) \;=\; O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}| \to \infty$, then

$$\varphi(\mathbf{x}) \;=\; \int_{\mathbb{R}^3} \frac{\rho(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\, dV\,.$$

The vector fields $\mathbf{B}$ and $\mathbf{J}$ satisfy

$$\nabla \times \mathbf{B} \;=\; \mathbf{J}\,.$$

Show that $\nabla \cdot \mathbf{J} = 0$. In the case that $\mathbf{B} = \nabla \times \mathbf{A}$, with $\nabla \cdot\, \mathbf{A} = 0$, show that

$$\mathbf{A}(\mathbf{x}) \;=\; \int_{\mathbb{R}^3} \frac{\mathbf{J}(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|}\, dV\,, \qquad (*)$$

and hence that

$$\mathbf{B}(\mathbf{x}) \;=\; \int_{\mathbb{R}^3} \frac{\mathbf{J}(\mathbf{y}) \times (\mathbf{x}-\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}|^3}\, dV\,.$$

Verify that $\mathbf{A}$ given by $(*)$ does indeed satisfy $\nabla \cdot\, \mathbf{A} = 0$. [It may be useful to make a change of variables in the right hand side of $(*)$.]

**Paper 3, Section II**
**12C Vector Calculus**

(a) Let

$$\mathbf{F} = (z, x, y)$$

and let $C$ be a circle of radius $R$ lying in a plane with unit normal vector $(a, b, c)$. Calculate $\nabla\times\mathbf{F}$ and use this to compute $\oint_C \mathbf{F}\cdot d\mathbf{x}$. Explain any orientation conventions which you use.

(b) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field such that the matrix with entries $\dfrac{\partial F_j}{\partial x_i}$ is symmetric. Prove that $\oint_C \mathbf{F}\cdot d\mathbf{x} = 0$ for every circle $C \subset \mathbb{R}^3$.

(c) Let $\mathbf{F} = \dfrac{1}{r}(x, y, z)$, where $r = \sqrt{x^2 + y^2 + z^2}$ and let $C$ be the circle which is the intersection of the sphere $(x-5)^2 + (y-3)^2 + (z-2)^2 = 1$ with the plane $3x - 5y - z = 2$. Calculate $\oint_C \mathbf{F}\cdot d\mathbf{x}$.

(d) Let $\mathbf{F}$ be the vector field defined, for $x^2 + y^2 > 0$, by

$$\mathbf{F} = \left( \frac{-y}{x^2 + y^2}\,,\ \frac{x}{x^2 + y^2}\,,\ z \right) .$$

Show that $\nabla \times \mathbf{F} = \mathbf{0}$. Let $C$ be the curve which is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $z = x + 200$. Calculate $\oint_C \mathbf{F}\cdot d\mathbf{x}$.

**Paper 1, Section I**

**1A Vectors and Matrices**

Let $z \in \mathbb{C}$ be a solution of

$$z^2 + bz + 1 = 0,$$

where $b \in \mathbb{R}$ and $|b| \leqslant 2$. For which values of $b$ do the following hold?

(i) $|e^z| < 1$.

(ii) $|e^{iz}| = 1$.

(iii) $\operatorname{Im}(\cosh z) = 0$.

**Paper 1, Section I**

**2C Vectors and Matrices**

Write down the general form of a $2 \times 2$ rotation matrix. Let $R$ be a real $2 \times 2$ matrix with positive determinant such that $|R\mathbf{x}| = |\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^2$. Show that $R$ is a rotation matrix.

Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that any real $2 \times 2$ matrix $A$ which satisfies $AJ = JA$ can be written as $A = \lambda R$, where $\lambda$ is a real number and $R$ is a rotation matrix.

## Paper 1, Section II

### 5A Vectors and Matrices

(a) Use suffix notation to prove that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

(b) Show that the equation of the plane through three non-colinear points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}),$$

where $\mathbf{r}$ is the position vector of a point in this plane.

Find a unit vector normal to the plane in the case $\mathbf{a} = (2,0,1)$, $\mathbf{b} = (0,4,0)$ and $\mathbf{c} = (1,-1,2)$.

(c) Let $\mathbf{r}$ be the position vector of a point in a given plane. The plane is a distance $d$ from the origin and has unit normal vector $\mathbf{n}$, where $\mathbf{n} \cdot \mathbf{r} \geqslant 0$. Write down the equation of this plane.

This plane intersects the sphere with centre at $\mathbf{p}$ and radius $q$ in a circle with centre at $\mathbf{m}$ and radius $\rho$. Show that

$$\mathbf{m} - \mathbf{p} = \gamma\, \mathbf{n}.$$

Find $\gamma$ in terms of $q$ and $\rho$. Hence find $\rho$ in terms of $\mathbf{n}$, $d$, $\mathbf{p}$ and $q$.

**Paper 1, Section II**
**6B Vectors and Matrices**

The $n \times n$ real symmetric matrix $M$ has eigenvectors of unit length $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, where $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Prove that the eigenvalues are real and that $\mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$.

Let $\mathbf{x}$ be any (real) unit vector. Show that

$$\mathbf{x}^{\mathrm{T}} M \mathbf{x} \leqslant \lambda_1 .$$

What can be said about $\mathbf{x}$ if $\mathbf{x}^{\mathrm{T}} M \mathbf{x} = \lambda_1$?

Let $S$ be the set of all (real) unit vectors of the form

$$\mathbf{x} = (0, x_2, \ldots, x_n) .$$

Show that $\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 \in S$ for some $\alpha_1, \alpha_2 \in \mathbb{R}$. Deduce that

$$\operatorname*{Max}_{\mathbf{x} \in S} \mathbf{x}^{\mathrm{T}} M \mathbf{x} \geqslant \lambda_2 .$$

The $(n-1) \times (n-1)$ matrix $A$ is obtained by removing the first row and the first column of $M$. Let $\mu$ be the greatest eigenvalue of $A$. Show that

$$\lambda_1 \geqslant \mu \geqslant \lambda_2 .$$

**Paper 1, Section II**
**7B Vectors and Matrices**

What does it mean to say that a matrix can be diagonalised? Given that the $n \times n$ real matrix $M$ has $n$ eigenvectors satisfying $\mathbf{e}_a \cdot \mathbf{e}_b = \delta_{ab}$, explain how to obtain the diagonal form $\Lambda$ of $M$. Prove that $\Lambda$ is indeed diagonal. Obtain, with proof, an expression for the trace of $M$ in terms of its eigenvalues.

The elements of $M$ are given by

$$M_{ij} = \begin{cases} 0 & \text{for } i = j , \\ 1 & \text{for } i \neq j . \end{cases}$$

Determine the elements of $M^2$ and hence show that, if $\lambda$ is an eigenvalue of $M$, then

$$\lambda^2 = (n-1) + (n-2)\lambda .$$

Assuming that $M$ can be diagonalised, give its diagonal form.

**Paper 1, Section II**

**8C Vectors and Matrices**

(a) Show that the equations

$$\begin{aligned} 1 + s + t &= a \\ 1 - s + t &= b \\ 1 - 2t &= c \end{aligned}$$

determine $s$ and $t$ uniquely if and only if $a + b + c = 3$.

Write the following system of equations

$$\begin{aligned} 5x + 2y - z &= 1 + s + t \\ 2x + 5y - z &= 1 - s + t \\ -x - y + 8z &= 1 - 2t \end{aligned}$$

in matrix form $A\mathbf{x} = \mathbf{b}$. Use Gaussian elimination to solve the system for $x, y$, and $z$. State a relationship between the rank and the kernel of a matrix. What is the rank and what is the kernel of $A$?

For which values of $x, y$, and $z$ is it possible to solve the above system for $s$ and $t$?

(b) Define a *unitary* $n \times n$ matrix. Let $A$ be a real symmetric $n \times n$ matrix, and let $I$ be the $n \times n$ identity matrix. Show that $|(A + iI)\mathbf{x}|^2 = |A\mathbf{x}|^2 + |\mathbf{x}|^2$ for arbitrary $\mathbf{x} \in \mathbb{C}^n$, where $|\mathbf{x}|^2 = \sum_{j=1}^n |x_j|^2$. Find a similar expression for $|(A - iI)\mathbf{x}|^2$. Prove that $(A - iI)(A + iI)^{-1}$ is well-defined and is a unitary matrix.