17C Methods

Describe the method of characteristics to construct solutions for 1st-order, homogeneous, linear partial differential equations

$$\alpha(x,y)\frac{\partial u}{\partial x}+\beta(x,y)\frac{\partial u}{\partial y}=0\,,$$

with initial data prescribed on a curve $x_0(\sigma)$, $y_0(\sigma)$: $u(x_0(\sigma),y_0(\sigma))=h(\sigma)$.

Consider the partial differential equation (here the two independent variables are time t and spatial direction x)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0\,,$$

with initial data $u(t=0,x)=e^{-x^2}$.

(i) Calculate the characteristic curves of this equation and show that u remains constant along these curves. Qualitatively sketch the characteristics in the (x,t) diagram, i.e. the x axis is the horizontal and the t axis is the vertical axis.

(ii) Let $\tilde{x}_0$ denote the x value of a characteristic at time $t=0$ and thus label the characteristic curves. Let $\tilde{x}$ denote the x value at time t of a characteristic with given $\tilde{x}_0$. By showing that $\partial\tilde{x}/\partial\tilde{x}_0$ becomes a non-monotonic function of $\tilde{x}_0$ (at fixed t) at times $t>\sqrt{e/2}$, or otherwise, show that $\tilde{x}(\tilde{x}_0)$ has a local minimum or maximum. Qualitatively sketch snapshots of the solution $u(t,x)$ for a few fixed values of $t\in[0,\sqrt{e/2}]$ and briefly interpret the onset of the non-monotonic behaviour of $\tilde{x}(\tilde{x}_0)$ at $t=\sqrt{e/2}$.