UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Thursday, 4 June, 2015 9:00 am to 12:00 noon

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *...,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

What does it mean to say that a positive definite binary quadratic form is *reduced*? Find the three smallest positive integers properly represented by each of the forms $f(x,y) = 3x^2 + 8xy + 9y^2$ and $g(x,y) = 15x^2 + 34xy + 20y^2$. Show that every odd integer represented by some positive definite binary quadratic form with discriminant -44 is represented by at least one of the forms f and g.

2I Topics in Analysis

Let K be a compact subset of $\mathbb{C}$ with path-connected complement. If $w \notin K$ and $\epsilon > 0$, show that there is a polynomial P such that

$$\left| P(z) - \frac{1}{w-z} \right| \leqslant \epsilon$$

for all $z \in K$.

3G Coding and Cryptography

Let A be a random variable that takes each value a in the finite alphabet $\mathcal{A}$ with probability $p(a)$. Show that, if each $l(a)$ is an integer greater than 0 and $\sum 2^{-l(a)} \leqslant 1$, then there is a decodable binary code $c : \mathcal{A} \to \{0,1\}^*$ with each codeword $c(a)$ having length $l(a)$.

Prove that, for any decodable code $c : \mathcal{A} \to \{0,1\}^*$, we have

$$H(A) \leqslant \mathbb{E}\, l(A)$$

where $H(A)$ is the entropy of the random variable A. When is there equality in this inequality?

4J Statistical Modelling

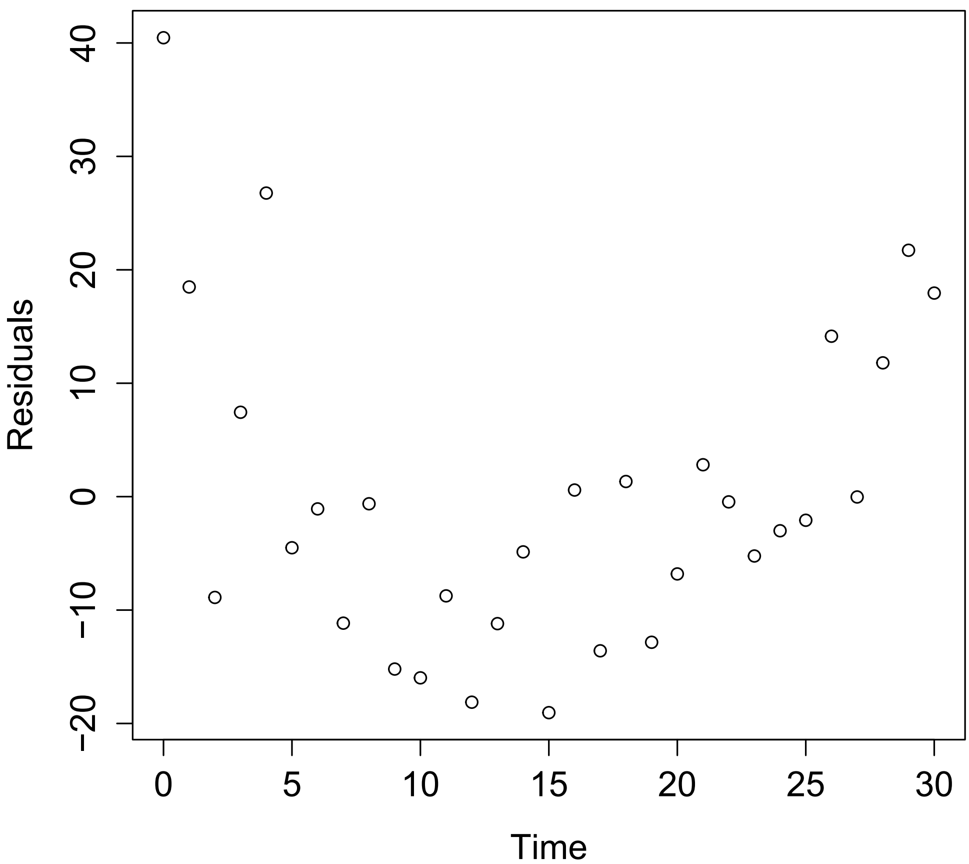
Data are available on the number of counts (atomic disintegration events that take place within a radiation source) recorded with a Geiger counter at a nuclear plant. The counts were registered at each second over a 30 second period for a short-lived, man-made radioactive compound. The first few rows of the dataset are displayed below.

```
> geiger[1:3, ]
  Time Counts
1    0  750.0
2    1  725.2
3    2  695.0
```

Describe the model being fitted with the following `R` command.

```
> fit1 <- lm(Counts ~ Time, data=geiger)
```

Below is a plot against time of the residuals from the model fitted above.



Referring to the plot, suggest how the model could be improved, and write out the `R` code for fitting this new model. Briefly describe how one could test in `R` whether the new model is to be preferred over the old model.

5E Mathematical Biology

The number of a certain type of annual plant in year n is given by x_n. Each plant produces k seeds that year and then dies before the next year. The proportion of seeds that germinate to produce a new plant the next year is given by $e^{-\gamma x_n}$ where $\gamma > 0$. Explain briefly why the system can be described by

$$x_{n+1} = k\, x_n e^{-\gamma x_n} \, .$$

Give conditions on k for a stable positive equilibrium of the plant population size to be possible.

Winters become milder and now a proportion s of all plants survive each year ($s \in (0,1)$). Assume that plants produce seeds as before while they are alive. Show that a wider range of k now gives a stable positive equilibrium population.

6B Further Complex Methods

Define what is meant by the *Cauchy principal value* in the particular case

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - a^2}\, dx \, ,$$

where the constant a is real and strictly positive. Evaluate this expression explicitly, stating clearly any standard results involving contour integrals that you use.

7D Classical Dynamics

(a) Consider a particle of mass m that undergoes periodic motion in a one-dimensional potential $V(q)$. Write down the Hamiltonian $H(p,q)$ for the system. Explain what is meant by the *angle–action variables* (θ, I) of the system and write down the integral expression for the action variable I.

(b) For $V(q) = \frac{1}{2} m\omega^2 q^2$ and fixed total energy E, describe the shape of the trajectories in phase-space. By using the expression for the area enclosed by the trajectory, or otherwise, find the action variable I in terms of ω and E. Hence describe how E changes with ω if ω varies slowly with time. Justify your answer.

8C Cosmology

What is the *flatness problem*? Show by reference to the Friedmann equation how a period of accelerated expansion of the scale factor $a(t)$ in the early stages of the universe can solve the flatness problem if $\rho + 3P < 0$, where ρ is the mass density and P is the pressure.

In the very early universe, where we can neglect the spatial curvature and the cosmological constant, there is a homogeneous scalar field ϕ with a vacuum potential energy

$$V(\phi) = m^2\phi^2\,,$$

and the Friedmann energy equation (in units where $8\pi G = 1$) is

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,,$$

where H is the Hubble parameter. The field ϕ obeys the evolution equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0\,.$$

During inflation, ϕ evolves slowly after starting from a large initial value ϕ_i at $t = 0$. State what is meant by the *slow-roll approximation*. Show that in this approximation,

$$\begin{aligned} \phi(t) &= \phi_i - \frac{2}{\sqrt{3}}mt\,, \\ a(t) &= a_i \exp\left[\frac{m\phi_i}{\sqrt{3}}t - \frac{1}{3}m^2t^2\right] = a_i \exp\left[\frac{\phi_i^2 - \phi^2(t)}{4}\right]\,, \end{aligned}$$

where a_i is the initial value of a.

As $\phi(t)$ decreases from its initial value ϕ_i, what is its approximate value when the slow-roll approximation fails?

SECTION II

9H Number Theory

Let θ be a real number with continued fraction expansion $[a_0, a_1, a_2, \ldots]$. Define the convergents p_n/q_n (by means of recurrence relations) and show that for $\beta > 0$ we have

$$[a_0, a_1, \ldots, a_{n-1}, \beta] = \frac{\beta p_{n-1} + p_{n-2}}{\beta q_{n-1} + q_{n-2}}.$$

Show that

$$\left|\theta - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

and deduce that $p_n/q_n \to \theta$ as $n \to \infty$.

By computing a suitable continued fraction expansion, find solutions in positive integers x and y to each of the equations $x^2 - 53y^2 = 4$ and $x^2 - 53y^2 = -7$.

10I Topics in Analysis

Let $\alpha > 0$. By considering the set E_m consisting of those $f \in C([0,1])$ for which there exists an $x \in [0,1]$ with $|f(x+h) - f(x)| \leqslant m|h|^\alpha$ for all $x + h \in [0,1]$, or otherwise, give a Baire category proof of the existence of continuous functions f on $[0,1]$ such that

$$\limsup_{h \to 0} |h|^{-\alpha} |f(x+h) - f(x)| = \infty$$

at each $x \in [0,1]$.

Are the following statements true? Give reasons.

(i) There exists an $f \in C([0,1])$ such that

$$\limsup_{h \to 0} |h|^{-\alpha} |f(x+h) - f(x)| = \infty$$

for each $x \in [0,1]$ and each $\alpha > 0$.

(ii) There exists an $f \in C([0,1])$ such that

$$\limsup_{h \to 0} |h|^{-\alpha} |f(x+h) - f(x)| = \infty$$

for each $x \in [0,1]$ and each $\alpha \geqslant 0$.

11E Mathematical Biology

A fungal disease is introduced into an isolated population of frogs. Without disease, the normalised population size x would obey the logistic equation $\dot{x} = x(1-x)$, where the dot denotes differentiation with respect to time. The disease causes death at rate d and there is no recovery. The disease transmission rate is β and, in addition, offspring of infected frogs are infected from birth.

(a) Briefly explain why the population sizes x and y of uninfected and infected frogs respectively now satisfy

$$\begin{aligned} \dot{x} &= x\,[\,1 - x - (1+\beta)y\,] \\ \dot{y} &= y\,[\,(1-d) - (1-\beta)x - y\,]\,. \end{aligned}$$

(b) The population starts at the disease-free population size ($x = 1$) and a small number of infected frogs are introduced. Show that the disease will successfully invade if and only if $\beta > d$.

(c) By finding all the equilibria in $x \geqslant 0$, $y \geqslant 0$ and considering their stability, find the long-term outcome for the frog population. State the relationships between d and β that distinguish different final populations.

(d) Plot the long-term steady *total* population size as a function of d for fixed β, and note that an intermediate mortality rate is actually the most harmful. Explain why this is the case.

12C Cosmology

Massive particles and antiparticles each with mass m and respective number densities $n(t)$ and $\bar{n}(t)$ are present at time t in the radiation era of an expanding universe with zero curvature and no cosmological constant. Assuming they interact with cross-section σ at speed v, explain, by identifying the physical significance of each of the terms, why the evolution of $n(t)$ is described by

$$\frac{dn}{dt} = -3\,\frac{\dot{a}}{a}\,n - \langle\sigma v\rangle\, n\bar{n} + P(t)\,,$$

where the expansion scale factor of the universe is $a(t)$, and where the meaning of $P(t)$ should be briefly explained. Show that

$$(n - \bar{n})a^3 = \text{ constant}\,.$$

Assuming initial particle-antiparticle symmetry, show that

$$\frac{d(na^3)}{dt} = \langle\sigma v\rangle\,(n_{\rm eq}^2 - n^2)a^3\,,$$

where $n_{\rm eq}$ is the equilibrium number density at temperature T.

Let $Y = n/T^3$ and $x = m/T$. Show that

$$\frac{dY}{dx} = -\frac{\lambda}{x^2}(Y^2 - Y_{\rm eq}^2)\,,$$

where $\lambda = m^3\,\langle\sigma v\rangle\,/H_m$ and H_m is the Hubble expansion rate when $T = m$.

When $x > x_f \simeq 10$, the number density n can be assumed to be depleted only by annihilations. If λ is constant, show that as $x \to \infty$ at late time, Y approaches a constant value given by

$$Y = \frac{x_f}{\lambda}\,.$$

Why do you expect weakly interacting particles to survive in greater numbers than strongly interacting particles?

13I Logic and Set Theory

(i) State and prove Zorn's Lemma. [You may assume Hartogs' Lemma.] Where in your proof have you made use of the Axiom of Choice?

(ii) Let $<$ be a partial ordering on a set X. Prove carefully that $<$ may be extended to a total ordering of X.

What does it mean to say that $<$ is *well-founded*?

If $<$ has an extension that is a well-ordering, must $<$ be well-founded? If $<$ is well-founded, must every total ordering extending it be a well-ordering? Justify your answers.

14I Graph Theory

(a) Let G be a graph. What is a *Hamilton cycle* in G? What does it mean to say that G is *Hamiltonian*?

(b) Let G be a graph of order $n \geqslant 3$ satisfying $\delta(G) \geqslant \frac{n}{2}$. Show that G is Hamiltonian. For each $n \geqslant 3$, exhibit a non-Hamiltonian graph G_n of order n with $\delta(G_n) = \lceil \frac{n}{2} \rceil - 1$.

(c) Let H be a bipartite graph with $n \geqslant 2$ vertices in each class satisfying $\delta(H) > \frac{n}{2}$. Show that H is Hamiltonian. For each $n \geqslant 2$, exhibit a non-Hamiltonian bipartite graph H_n with n vertices in each class and $\delta(H_n) = \lfloor \frac{n}{2} \rfloor$.

15F Representation Theory

(a) State Mackey's theorem, defining carefully all the terms used in the statement.

(b) Let G be a finite group and suppose that G acts on the set Ω.

If $n \in \mathbb{N}$, we say that the action of G on Ω is *n-transitive* if Ω has at least n elements and for every pair of n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ such that the a_i are distinct elements of Ω and the b_i are distinct elements of Ω, there exists $g \in G$ with $ga_i = b_i$ for every i.

(i) Let Ω have at least n elements, where $n \geqslant 1$ and let $\omega \in \Omega$. Show that G acts n-transitively on Ω if and only if G acts transitively on Ω and the stabiliser G_ω acts $(n-1)$-transitively on $\Omega \setminus \{\omega\}$.

(ii) Show that the permutation module $\mathbb{C}\Omega$ can be decomposed as

$$\mathbb{C}\Omega = \mathbb{C}_G \oplus V,$$

where $\mathbb{C}_G$ is the trivial module and V is some $\mathbb{C}G$-module.

(iii) Assume that $|\Omega| \geqslant 2$, so that $V \neq 0$. Prove that V is irreducible if and only if G acts 2-transitively on Ω. In that case show also that V is not the trivial representation. [*Hint: Pick any orbit of G on Ω; it is isomorphic as a G-set to G/H for some subgroup $H \leqslant G$. Consider the induced character* $\mathrm{Ind}_H^G 1_H$.]

16F Galois Theory

Let $f \in \mathbb{Q}[t]$ be of degree $n > 0$, with no repeated roots, and let L be a splitting field for f.

(i) Show that f is irreducible if and only if for any $\alpha, \beta \in \mathrm{Root}_f(L)$ there is $\phi \in \mathrm{Gal}(L/\mathbb{Q})$ such that $\phi(\alpha) = \beta$.

(ii) Explain how to define an injective homomorphism $\tau : \mathrm{Gal}(L/\mathbb{Q}) \to S_n$. Find an example in which the image of τ is the subgroup of S_3 generated by $(2\ 3)$. Find another example in which τ is an isomorphism onto S_3.

(iii) Let $f(t) = t^5 - 3$ and assume f is irreducible. Find a chain of subgroups of $\mathrm{Gal}(L/\mathbb{Q})$ that shows it is a solvable group. [You may quote without proof any theorems from the course, provided you state them clearly.]

17H Algebraic Topology

Let K and L be simplicial complexes. Explain what is meant by a *simplicial approximation* to a continuous map $f : |K| \to |L|$. State the simplicial approximation theorem, and define the homomorphism induced on homology by a continuous map between triangulable spaces. [You do not need to show that the homomorphism is well-defined.]

Let $h : S^1 \to S^1$ be given by $z \mapsto z^n$ for a positive integer n, where S^1 is considered as the unit complex numbers. Compute the map induced by h on homology.

18G Linear Analysis

State and prove the Baire Category Theorem. [Choose any version you like.]

An *isometry* from a metric space (M, d) to another metric space (N, e) is a function $\varphi \colon M \to N$ such that $e(\varphi(x), \varphi(y)) = d(x, y)$ for all $x, y \in M$. Prove that there exists no isometry from the Euclidean plane ℓ_2^2 to the Banach space c_0 of sequences converging to 0. [*Hint: Assume* $\varphi \colon \ell_2^2 \to c_0$ *is an isometry. For* $n \in \mathbb{N}$ *and* $x \in \ell_2^2$ *let* $\varphi_n(x)$ *denote the* n^{th} *coordinate of* $\varphi(x)$*. Consider the sets* F_n *consisting of all pairs* (x, y) *with* $\|x\|_2 = \|y\|_2 = 1$ *and* $\|\varphi(x) - \varphi(y)\|_\infty = |\varphi_n(x) - \varphi_n(y)|$.]

Show that for each $n \in \mathbb{N}$ there is a linear isometry $\ell_1^n \to c_0$.

19F Riemann Surfaces

Let $\wp(z)$ denote the Weierstrass $\wp$-function with respect to a lattice $\Lambda \subset \mathbb{C}$ and let f be an even elliptic function with periods Λ. Prove that there exists a rational function Q such that $f(z) = Q(\wp(z))$. If we write $Q(w) = p(w)/q(w)$ where p and q are coprime polynomials, find the degree of f in terms of the degrees of the polynomials p and q. Describe all even elliptic functions of degree two. Justify your answers. [You may use standard properties of the Weierstrass $\wp$-function.]

20F Algebraic Geometry

(i) Let X be an affine variety. Define the *tangent space* of X at a point P. Say what it means for the variety to be singular at P.

(ii) Find the singularities of the surface in $\mathbb{P}^3$ given by the equation

$$xyz + yzw + zwx + wxy = 0.$$

(iii) Consider $C = Z(x^2 - y^3) \subseteq \mathbb{A}^2$. Let $X \to \mathbb{A}^2$ be the blowup of the origin. Compute the proper transform of C in X, and show it is non-singular.

21G Differential Geometry

Show that the surface S of revolution $x^2 + y^2 = \cosh^2 z$ in $\mathbb{R}^3$ is homeomorphic to a cylinder and has everywhere negative Gaussian curvature. Show moreover the existence of a closed geodesic on S.

Let $S \subset \mathbb{R}^3$ be an arbitrary embedded surface which is homeomorphic to a cylinder and has everywhere negative Gaussian curvature. By using a suitable version of the Gauss–Bonnet theorem, show that S contains at most one closed geodesic. [If required, appropriate forms of the Jordan curve theorem in the plane may also be used without proof.]

22J Probability and Measure

(a) Let $(E, \mathcal{E}, \mu)$ be a measure space. What does it mean to say that $T\colon E \to E$ is a *measure-preserving transformation*? What does it mean to say that a set $A \in \mathcal{E}$ is *invariant under* T? Show that the class of invariant sets forms a σ-algebra.

(b) Take E to be $[0, 1)$ with Lebesgue measure on its Borel σ-algebra. Show that the baker's map $T : [0, 1) \to [0, 1)$ defined by

$$T(x) = 2x - \lfloor 2x \rfloor$$

is measure-preserving.

(c) Describe in detail the construction of the canonical model for sequences of independent random variables having a given distribution m.

Define the Bernoulli shift map and prove it is a measure-preserving ergodic transformation.

[You may use without proof other results concerning sequences of independent random variables proved in the course, provided you state these clearly.]

23K Applied Probability

(i) Let X be a Poisson process of parameter λ. Let Y be obtained by taking each point of X and, independently of the other points, keeping it with probability p. Show that Y is another Poisson process and find its intensity. Show that for every fixed t the random variables Y_t and $X_t - Y_t$ are independent.

(ii) Suppose we have n bins, and balls arrive according to a Poisson process of rate 1. Upon arrival we choose a bin uniformly at random and place the ball in it. We let M_n be the maximum number of balls in any bin at time n. Show that

$$\mathbb{P}\left(M_n \geqslant (1+\epsilon)\frac{\log n}{\log\log n}\right) \to 0 \quad \text{as } n \to \infty.$$

[You may use the fact that if ξ is a Poisson random variable of mean 1, then

$$\mathbb{P}(\xi \geqslant x) \leqslant \exp(x - x\log x).]$$

24J Principles of Statistics

Define what it means for an estimator $\hat{\theta}$ of an unknown parameter θ to be *consistent.*

Let S_n be a sequence of random real-valued continuous functions defined on $\mathbb{R}$ such that, as $n \to \infty$, $S_n(\theta)$ converges to $S(\theta)$ in probability for every $\theta \in \mathbb{R}$, where $S : \mathbb{R} \to \mathbb{R}$ is non-random. Suppose that for some $\theta_0 \in \mathbb{R}$ and every $\varepsilon > 0$ we have

$$S(\theta_0 - \varepsilon) < 0 < S(\theta_0 + \varepsilon),$$

and that S_n has exactly one zero $\hat{\theta}_n$ for every $n \in \mathbb{N}$. Show that $\hat{\theta}_n \to^P \theta_0$ as $n \to \infty$, and deduce from this that the maximum likelihood estimator (MLE) based on observations $X_1, \ldots, X_n$ from a $N(\theta, 1)$, $\theta \in \mathbb{R}$ model is consistent.

Now consider independent observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$ of bivariate normal random vectors

$$\mathbf{X}_i = (X_{1i}, X_{2i})^T \sim N_2\left[(\mu_i, \mu_i)^T, \sigma^2 I_2\right], \quad i = 1, \ldots, n,$$

where $\mu_i \in \mathbb{R}$, $\sigma > 0$ and I_2 is the 2×2 identity matrix. Find the MLE $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_n)^T$ of $\mu = (\mu_1, \ldots, \mu_n)^T$ and show that the MLE of σ^2 equals

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n s_i^2, \quad s_i^2 = \frac{1}{2}[(X_{1i} - \hat{\mu}_i)^2 + (X_{2i} - \hat{\mu}_i)^2].$$

Show that $\hat{\sigma}^2$ is *not* consistent for estimating σ^2. Explain briefly why the MLE fails in this model.

[You may use the Law of Large Numbers without proof.]

25K Optimization and Control

A burglar having wealth x may retire, or go burgling another night, in either of towns 1 or 2. If he burgles in town i then with probability $p_i = 1 - q_i$ he will, independently of previous nights, be caught, imprisoned and lose all his wealth. If he is not caught then his wealth increases by 0 or $2a_i$, each with probability 1/2 and independently of what happens on other nights. Values of p_i and a_i are the same every night. He wishes to maximize his expected wealth at the point he retires, is imprisoned, or s nights have elapsed.

Using the dynamic programming equation

$$F_s(x) = \max\Big\{ x,\ q_1 E F_{s-1}(x + R_1),\ q_2 E F_{s-1}(x + R_2) \Big\}$$

with R_j, $F_0(x)$ appropriately defined, prove that there exists an optimal policy under which he burgles another night if and only if his wealth is less than $x^* = \max_i\{a_i q_i / p_i\}$.

Suppose $q_1 > q_2$ and $q_1 a_1 > q_2 a_2$. Prove that he should never burgle in town 2.

[*Hint: Suppose $x < x^*$, there are s nights to go, and it has been shown that he ought not burgle in town 2 if less than s nights remain. For the case $a_2 > a_1$, separately consider subcases $x + 2a_2 \geqslant x^*$ and $x + 2a_2 < x^*$. An interchange argument may help.*]

26K Stochastic Financial Models

A single-period market consists of n assets whose prices at time t are denoted by $S_t = (S_t^1, \ldots, S_t^n)^T$, $t = 0, 1$, and a riskless bank account bearing interest rate r. The value of S_0 is given, and $S_1 \sim N(\mu, V)$. An investor with utility $U(x) = -\exp(-\gamma x)$ wishes to choose a portfolio of the available assets so as to maximize the expected utility of her wealth at time 1. Find her optimal investment.

What is the *market portfolio* for this problem? What is the *beta* of asset i? Derive the Capital Asset Pricing Model, that

$$\text{Excess return of asset } i = \text{Excess return of market portfolio} \times \beta_i.$$

The Sharpe ratio of a portfolio θ is defined to be the excess return of the portfolio θ divided by the standard deviation of the portfolio θ. If ρ_i is the correlation of the return on asset i with the return on the market portfolio, prove that

$$\text{Sharpe ratio of asset } i = \text{Sharpe ratio of market portfolio} \times \rho_i.$$

27C Asymptotic Methods

Show that

$$\int_0^1 e^{ikt^3}\,dt = I_1 - I_2\,, \qquad k > 0\,,$$

where I_1 is an integral from 0 to ∞ along the line $\arg(z) = \frac{\pi}{6}$ and I_2 is an integral from 1 to ∞ along a steepest-descent contour C which you should determine.

By employing in the integrals I_1 and I_2 the changes of variables $u = -iz^3$ and $u = -i(z^3 - 1)$, respectively, compute the first two terms of the large k asymptotic expansion of the integral above.

28B Dynamical Systems

Consider the dynamical system

$$\begin{aligned} \dot{x} &= -\mu + x^2 - y, \\ \dot{y} &= y(a - x), \end{aligned}$$

where a is to be regarded as a fixed real constant and μ as a real parameter.

Find the fixed points of the system and determine the stability of the system linearized about the fixed points. Hence identify the values of μ at given a where bifurcations occur.

Describe informally the concepts of centre manifold theory and apply it to analyse the bifurcations that occur in the above system with $a = 1$. In particular, for each bifurcation derive an equation for the dynamics on the extended centre manifold and hence classify the bifurcation.

What can you say, without further detailed calculation, about the case $a = 0$?

29D Integrable Systems

Let $L = L(t)$ and $A = A(t)$ be real $N \times N$ matrices, with L symmetric and A antisymmetric. Suppose that

$$\frac{dL}{dt} = LA - AL\,.$$

Show that all eigenvalues of the matrix $L(t)$ are t-independent. Deduce that the coefficients of the polynomial

$$P(x) = \det(xI - L(t))$$

are first integrals of the system.

What does it mean for a $2n$-dimensional Hamiltonian system to be *integrable*? Consider the *Toda system* with coordinates (q_1, q_2, q_3) obeying

$$\frac{d^2 q_i}{dt^2} = \mathrm{e}^{q_{i-1}-q_i} - \mathrm{e}^{q_i - q_{i+1}}\,, \quad i = 1,\, 2,\, 3$$

where here and throughout the subscripts are to be determined modulo 3 so that $q_4 \equiv q_1$ and $q_0 \equiv q_3$. Show that

$$H(q_i, p_i) \;=\; \frac{1}{2}\sum_{i=1}^{3} p_i^2 \;+\; \sum_{i=1}^{3} \mathrm{e}^{\,q_i - q_{i+1}}$$

is a Hamiltonian for the Toda system.

Set $a_i = \frac{1}{2}\exp\left(\frac{q_i - q_{i+1}}{2}\right)$ and $b_i = -\frac{1}{2}p_i$. Show that

$$\frac{da_i}{dt} = (b_{i+1} - b_i)\, a_i\,, \quad \frac{db_i}{dt} = 2\left(a_i^2 - a_{i-1}^2\right), \quad i = 1,\, 2,\, 3\,.$$

Is this coordinate transformation canonical?

By considering the matrices

$$L = \begin{pmatrix} b_1 & a_1 & a_3 \\ a_1 & b_2 & a_2 \\ a_3 & a_2 & b_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -a_1 & a_3 \\ a_1 & 0 & -a_2 \\ -a_3 & a_2 & 0 \end{pmatrix},$$

or otherwise, compute three independent first integrals of the Toda system. [Proof of independence is not required.]

30E Partial Differential Equations

(a) Show that if $f \in \mathcal{S}(\mathbb{R}^n)$ is a Schwartz function and u is a tempered distribution which solves

$$-\Delta u + m^2 u = f$$

for some constant $m \neq 0$, then there exists a number $C > 0$ which depends only on m, such that $\|u\|_{H^{s+2}} \leqslant C\|f\|_{H^s}$ for any $s \geqslant 0$. Explain briefly why this inequality remains valid if f is only assumed to be in $H^s(\mathbb{R}^n)$.

Show that if $\epsilon > 0$ is given then $\|v\|_{H^1}^2 \leqslant \epsilon\|v\|_{H^2}^2 + \frac{1}{4\epsilon}\|v\|_{H^0}^2$ for any $v \in H^2(\mathbb{R}^n)$.

[*Hint: The inequality* $a \leqslant \epsilon a^2 + \frac{1}{4\epsilon}$ *holds for any positive* ϵ *and* $a \in \mathbb{R}$.]

Prove that if u is a smooth bounded function which solves

$$-\Delta u + m^2 u = u^3 + \alpha \cdot \nabla u$$

for some constant vector $\alpha \in \mathbb{R}^n$ and constant $m \neq 0$, then there exists a number $C' > 0$ such that $\|u\|_{H^2} \leqslant C'$ and C' depends only on $m, \alpha, \|u\|_{L^\infty}, \|u\|_{L^2}$.

[You may use the fact that, for non-negative s, the Sobolev space of functions

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \|f\|_{H^s}^2 = \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^s |\hat{f}(\xi)|^2 \, d\xi < \infty\}.]$$

(b) Let $u(x,t)$ be a smooth real-valued function, which is 2π-periodic in x and satisfies the equation

$$u_t = u^2 u_{xx} + u^3 .$$

Give a complete proof that if $u(x,0) > 0$ for all x then $u(x,t) > 0$ for all x and $t > 0$.

31A Principles of Quantum Mechanics

Let $|\, j,\, m\,\rangle$ denote the normalised joint eigenstates of $\mathbf{J}^2$ and J_3, where $\mathbf{J}$ is the angular momentum operator for a quantum system. State clearly the possible values of the quantum numbers j and m and write down the corresponding eigenvalues in units with $\hbar = 1$.

Consider two quantum systems with angular momentum states $|\, \frac{1}{2},\, r\,\rangle$ and $|\, j,\, m\,\rangle$. The eigenstates corresponding to their combined angular momentum can be written as

$$|\, J,\, M\,\rangle \;=\; \sum_{r\,,\,m} C^{J\,M}_{r\,m}\,|\, \tfrac{1}{2},\, r\,\rangle\,|\, j,\, m\,\rangle\;,$$

where $C^{J\,M}_{r\,m}$ are Clebsch–Gordan coefficients for addition of angular momenta $\frac{1}{2}$ and j. What are the possible values of J and what is a necessary condition relating r, m and M in order that $C^{J\,M}_{r\,m} \neq 0$?

Calculate the values of $C^{J\,M}_{r\,m}$ for $j = 2$ and for all $M \geqslant \frac{3}{2}$. Use the sign convention that $C^{J\,J}_{r\,m} > 0$ when m takes its maximum value.

A particle X with spin $\frac{3}{2}$ and intrinsic parity η_X is at rest. It decays into two particles A and B with spin $\frac{1}{2}$ and spin 0, respectively. Both A and B have intrinsic parity -1. The relative orbital angular momentum quantum number for the two particle system is ℓ. What are the possible values of ℓ for the cases $\eta_X = +1$ and $\eta_X = -1$?

Suppose particle X is prepared in the state $|\, \frac{3}{2},\, \frac{3}{2}\,\rangle$ before it decays. Calculate the probability P for particle A to be found in the state $|\, \frac{1}{2},\, \frac{1}{2}\,\rangle$, given that $\eta_X = +1$.

What is the probability P if instead $\eta_X = -1$?

[Units with $\hbar = 1$ should be used throughout. You may also use without proof

$$J_-|\, j,\, m\,\rangle \;=\; \sqrt{(j+m)(j-m+1)}\;|\, j,\, m-1\,\rangle\,.]$$

32A Applications of Quantum Mechanics

A particle of mass m and energy $E = -\hbar^2\kappa^2/2m < 0$ moves in one dimension subject to a periodic potential

$$V(x) = -\frac{\hbar^2\lambda}{m}\sum_{\ell=-\infty}^{\infty}\delta\left(x-\ell a\right) \qquad \text{with} \qquad \lambda > 0\,.$$

Determine the corresponding Floquet matrix $\mathcal{M}$. [You may assume without proof that for the Schrödinger equation with potential $\alpha\,\delta(x)$ the wavefunction $\psi(x)$ is continuous at $x = 0$ and satisfies $\psi'(0+) - \psi'(0-) = (2m\alpha/\hbar^2)\,\psi(0)$.]

Explain briefly, with reference to Bloch's theorem, how restrictions on the energy of a Bloch state can be derived from $\mathcal{M}$. Deduce that for the potential $V(x)$ above, κ is confined to a range whose boundary values are determined by

$$\tanh\left(\frac{\kappa a}{2}\right) = \frac{\kappa}{\lambda} \qquad \text{and} \qquad \coth\left(\frac{\kappa a}{2}\right) = \frac{\kappa}{\lambda}\,.$$

Sketch the left-hand and right-hand sides of each of these equations as functions of $y = \kappa a/2$. Hence show that there is exactly one allowed band of negative energies with either (i) $E_- \leqslant E < 0$ or (ii) $E_- \leqslant E \leqslant E_+ < 0$ and determine the values of λa for which each of these cases arise. [You should not attempt to evaluate the constants $E_\pm$.]

Comment briefly on the limit $a \to \infty$ with λ fixed.

33C Statistical Physics

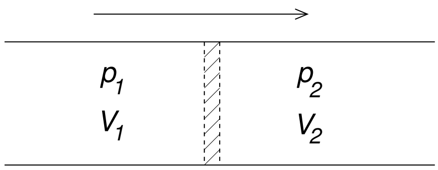
(a) A sample of gas has pressure p, volume V, temperature T and entropy S.

(i) Use the first law of thermodynamics to derive the Maxwell relation

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p .$$

(ii) Define the heat capacity at constant pressure C_p and the enthalpy H and show that $C_p = (\partial H/\partial T)_p$.

(b) Consider a perfectly insulated pipe with a throttle valve, as shown.



Gas initially occupying volume V_1 on the left is forced slowly through the valve at constant pressure p_1. A constant pressure p_2 is maintained on the right and the final volume occupied by the gas after passing through the valve is V_2.

(i) Show that the enthalpy H of the gas is unchanged by this process.

(ii) The Joule–Thomson coefficient is defined to be $\mu = (\partial T/\partial p)_H$. Show that

$$\mu = \frac{V}{C_p}\left[\frac{T}{V}\left(\frac{\partial V}{\partial T}\right)_p - 1\right] .$$

[You may assume the identity $(\partial y/\partial x)_u = -(\partial u/\partial x)_y \,/\, (\partial u/\partial y)_x$.]

(iii) Suppose that the gas obeys an equation of state

$$p = k_B T\left[\frac{N}{V} + B_2(T)\frac{N^2}{V^2}\right]$$

where N is the number of particles. Calculate μ to first order in N/V and hence derive a condition on $\frac{d}{dT}\left(\frac{B_2(T)}{T}\right)$ for obtaining a positive Joule–Thomson coefficient.

34A Electrodynamics

(i) Consider the action

$$S = -\frac{1}{4\mu_0 c}\int \left(F_{\mu\nu}F^{\mu\nu} + 2\lambda^2 A_\mu A^\mu\right)\, d^4x + \frac{1}{c}\int A_\mu J^\mu \, d^4x \,,$$

where $A_\mu(x)$ is a 4-vector potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, $J^\mu(x)$ is a conserved current, and $\lambda \geqslant 0$ is a constant. Derive the field equation

$$\partial_\mu F^{\mu\nu} - \lambda^2 A^\nu = -\mu_0 J^\nu \,.$$

For $\lambda = 0$ the action S describes standard electromagnetism. Show that in this case the theory is invariant under gauge transformations of the field $A_\mu(x)$, which you should define. Is the theory invariant under these same gauge transformations when $\lambda > 0$?

Show that when $\lambda > 0$ the field equation above implies

$$\partial_\mu \partial^\mu A^\nu - \lambda^2 A^\nu = -\mu_0 J^\nu \,. \qquad (*)$$

Under what circumstances does $(*)$ hold in the case $\lambda = 0$?

(ii) Now suppose that $A_\mu(x)$ and $J_\mu(x)$ obeying $(*)$ reduce to static 3-vectors $\mathbf{A}(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$ in some inertial frame. Show that there is a solution

$$\mathbf{A}(\mathbf{x}) = -\mu_0 \int G(\,|\mathbf{x}-\mathbf{x}'|\,)\, \mathbf{J}(\mathbf{x}')\, d^3\mathbf{x}'$$

for a suitable Green's function $G(R)$ with $G(R) \to 0$ as $R \to \infty$. Determine $G(R)$ for any $\lambda \geqslant 0$. [*Hint: You may find it helpful to consider first the case* $\lambda = 0$ *and then the case* $\lambda > 0$, *using the result* $\nabla^2\Big(\dfrac{1}{R} f(R)\Big) = \nabla^2\Big(\dfrac{1}{R}\Big) f(R) + \dfrac{1}{R} f''(R)$, *where* $R = |\,\mathbf{x}-\mathbf{x}'\,|$.]

If $\mathbf{J}(\mathbf{x})$ is zero outside some bounded region, comment on the effect of the value of λ on the behaviour of $\mathbf{A}(\mathbf{x})$ when $|\mathbf{x}|$ is large. [No further detailed calculations are required.]

35D General Relativity

Let $\Gamma^a{}_{bc}$ be the Levi-Civita connection and $R^a{}_{bcd}$ the Riemann tensor corresponding to a metric $g_{ab}(x)$, and let $\widetilde{\Gamma}^a{}_{bc}$ be the Levi-Civita connection and $\widetilde{R}^a{}_{bcd}$ the Riemann tensor corresponding to a metric $\tilde{g}_{ab}(x)$. Let $T^a{}_{bc} = \widetilde{\Gamma}^a{}_{bc} - \Gamma^a{}_{bc}$.

(a) Show that $T^a{}_{bc}$ is a tensor.

(b) Using local inertial coordinates for the metric g_{ab}, or otherwise, show that

$$\widetilde{R}^a{}_{bcd} - R^a{}_{bcd} = 2T^a{}_{b[d;c]} - 2T^a{}_{e[d}T^e{}_{c]b}$$

holds in all coordinate systems, where the semi-colon denotes covariant differentiation using the connection $\Gamma^a{}_{bc}$. [You may assume that $R^a{}_{bcd} = 2\Gamma^a{}_{b[d,c]} - 2\Gamma^a{}_{e[d}\Gamma^e{}_{c]b}$.]

(c) In the case that $T^a{}_{bc} = \ell^a g_{bc}$ for some vector field ℓ^a, show that $\widetilde{R}_{bd} = R_{bd}$ if and only if

$$\ell_{b;d} + \ell_b \ell_d = 0\,.$$

(d) Using the result that $\ell_{[a;b]} = 0$ if and only if $\ell_a = \phi_{,a}$ for some scalar field ϕ, show that the condition on ℓ_a in part (c) can be written as

$$k_{a;b} = 0$$

for a certain covector field k_a, which you should define.

36E Fluid Dynamics II

Consider a three-dimensional high-Reynolds number jet without swirl induced by a force $\mathbf{F} = F\mathbf{e}_z$ imposed at the origin in a fluid at rest. The velocity in the jet, described using cylindrical coordinates (r, θ, z), is assumed to remain steady and axisymmetric, and described by a boundary layer analysis.

(i) Explain briefly why the flow in the jet can be described by the boundary layer equations

$$u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = \nu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right).$$

(ii) Show that the momentum flux in the jet, $F = \int_S \rho u_z^2 dS$, where S is an infinite surface perpendicular to $\mathbf{e}_z$, is not a function of z. Combining this result with scalings from the boundary layer equations, derive the scalings for the unknown width $\delta(z)$ and typical velocity $U(z)$ of the jet as functions of z and the other parameters of the problem (ρ, ν, F).

(iii) Solving for the flow using a self-similar Stokes streamfunction

$$\psi(r, z) = U(z)\delta^2(z) f(\eta), \quad \eta = r/\delta(z),$$

show that $f(\eta)$ satisfies the differential equation

$$ff' - \eta(f'^2 + ff'') = f' - \eta f'' + \eta^2 f'''.$$

What boundary conditions should be applied to this equation? Give physical reasons for them.

[*Hint: In cylindrical coordinates for axisymmetric incompressible flow* $\big(u_r(r,z), 0, u_z(r,z)\big)$ *you are given the incompressibility condition as*

$$\frac{1}{r} \frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0,$$

the z*-component of the Navier–Stokes equation as*

$$\rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{\partial^2 u_z}{\partial z^2} \right],$$

and the relationship between the Stokes streamfunction, $\psi(r,z)$*, and the velocity components as*

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \quad u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}. \quad]$$

37B Waves

Derive the ray-tracing equations for the quantities $\mathrm{d}k_i/\mathrm{d}t$, $\mathrm{d}\omega/\mathrm{d}t$ and $\mathrm{d}x_i/\mathrm{d}t$ during wave propagation through a slowly varying medium with local dispersion relation $\omega = \Omega(\mathbf{k}, \mathbf{x}, t)$, explaining the meaning of the notation $\mathrm{d}/\mathrm{d}t$.

The dispersion relation for water waves is $\Omega^2 = g\kappa \tanh(\kappa h)$, where h is the water depth, $\kappa^2 = k^2 + l^2$, and k and l are the components of $\mathbf{k}$ in the horizontal x and y directions. Water waves are incident from an ocean occupying $x > 0$, $-\infty < y < \infty$ onto a beach at $x = 0$. The undisturbed water depth is $h(x) = \alpha x^p$, where α, p are positive constants and α is sufficiently small that the depth can be assumed to be slowly varying. Far from the beach, the waves are planar with frequency ω_∞ and with crests making an acute angle θ_∞ with the shoreline.

Obtain a differential equation (with k defined implicitly) for a ray $y = y(x)$ and show that near the shore the ray satisfies

$$y - y_0 \sim Ax^q$$

where A and q should be found. Sketch the shape of the wavecrests near the shoreline for the case $p < 2$.

38E Numerical Analysis

(a) Given the finite-difference recurrence

$$\sum_{k=r}^{s} a_k u_{m+k}^{n+1} = \sum_{k=r}^{s} b_k u_{m+k}^{n}, \quad m \in \mathbb{Z},\ n \in \mathbb{Z}^+,$$

that discretises a Cauchy problem, the *amplification factor* is defined by

$$H(\theta) = \left(\sum_{k=r}^{s} b_k e^{ik\theta}\right) \Big/ \left(\sum_{k=r}^{s} a_k e^{ik\theta}\right).$$

Show how $H(\theta)$ acts on the Fourier transform $\hat{u}^n$ of u^n. Hence prove that the method is stable if and only if $|H(\theta)| \leqslant 1$ for all $\theta \in [-\pi, \pi]$.

(b) The two-dimensional diffusion equation

$$u_t = u_{xx} + cu_{yy}$$

for some scalar constant $c > 0$ is discretised with the forward Euler scheme

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \mu(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n} + cu_{i,j+1}^{n} - 2cu_{i,j}^{n} + cu_{i,j-1}^{n}).$$

Using Fourier stability analysis, find the range of values $\mu > 0$ for which the scheme is stable.

END OF PAPER