UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Monday, 1 June, 2015 1:30 pm to 4:30 pm

PAPER 1

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on ***one side*** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, *. . . ,* ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1H Number Theory

Define the Legendre symbol $\left(\frac{a}{p}\right)$. State and prove Euler's criterion, assuming if you wish the existence of primitive roots mod p.

By considering the prime factors of n^2+4 for n an odd integer, prove that there are infinitely many primes p with $p \equiv 5 \pmod 8$.

2I Topics in Analysis

Let Ω be a non-empty bounded open subset of $\mathbb{R}^2$ with closure $\bar{\Omega}$ and boundary $\partial\Omega$. Let $\phi : \bar{\Omega} \to \mathbb{R}$ be continuous with ϕ twice differentiable on Ω.

(i) Why does ϕ have a maximum on $\bar{\Omega}$?

(ii) If $\epsilon > 0$ and $\nabla^2\phi \geqslant \epsilon$ on Ω, show that ϕ has a maximum on $\partial\Omega$.

(iii) If $\nabla^2\phi \geqslant 0$ on Ω, show that ϕ has a maximum on $\partial\Omega$.

(iv) If $\nabla^2\phi = 0$ on Ω and $\phi = 0$ on $\partial\Omega$, show that $\phi = 0$ on $\bar{\Omega}$.

3G Coding and Cryptography

Let $\mathcal{A}$ be a finite alphabet. Explain what is meant by saying that a binary code $c : \mathcal{A} \to \{0,1\}^*$ has *minimum distance* δ. If c is such a binary code with minimum distance δ, show that c is $\delta - 1$ error-detecting and $\lfloor \frac{1}{2}(\delta - 1) \rfloor$ error-correcting.

Show that it is possible to construct a code that has minimum distance δ for any integer $\delta > 0$.

4J Statistical Modelling

The outputs $Y_1, \ldots, Y_n$ of a particular process are positive and are believed to be related to p-vectors of covariates $x_1, \ldots, x_n$ according to the following model

$$\log(Y_i) = \mu + x_i^T\beta + \varepsilon_i.$$

In this model ε_i are i.i.d. $N(0, \sigma^2)$ random variables where $\sigma > 0$ is known. It is not possible to measure the output directly, but we can detect whether the output is greater than or less than or equal to a certain known value $c > 0$. If

$$Z_i = \begin{cases} 1 & \text{if } Y_i > c \\ 0 & \text{if } Y_i \leqslant c, \end{cases}$$

show that a probit regression model can be used for the data (Z_i, x_i), $i = 1, \ldots, n$.

How can we recover μ and β from the parameters of the probit regression model?

5E Mathematical Biology

The population density $n(a,t)$ of individuals of age a at time t satisfies

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu(a)\, n(a,t), \qquad n(0,t) = \int_0^\infty b(a)\, n(a,t)\, da$$

where $\mu(a)$ is the age-dependent death rate and $b(a)$ is the birth rate per individual of age a. Show that this may be solved with a similarity solution of the form $n(a,t) = e^{\gamma t} r(a)$ if the growth rate γ satisfies $\phi(\gamma) = 1$ where

$$\phi(\gamma) = \int_0^\infty b(a)\, e^{-\gamma a - \int_0^a \mu(s)\, ds}\, da.$$

Suppose now that the birth rate is given by $b(a) = Ba^p e^{-\lambda a}$ with $B, \lambda > 0$ and p is a positive integer, and the death rate is constant in age (i.e. $\mu(a) = \mu$). Find the average number of offspring per individual.

Find the similarity solution, and find the threshold B^* for the birth parameter B so that $B > B^*$ corresponds to a growing population.

6B Further Complex Methods

Evaluate the real integral

$$\int_0^\infty \frac{x^{1/2} \ln x}{1 + x^2}\, dx$$

where $x^{1/2}$ is taken to be the positive square root.

What is the value of

$$\int_0^\infty \frac{x^{1/2}}{1 + x^2}\, dx\ ?$$

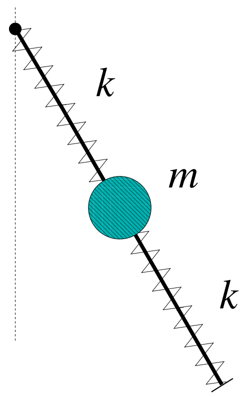
7D Classical Dynamics

(a) The action for a one-dimensional dynamical system with a generalized coordinate q and Lagrangian L is given by

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt\,.$$

State the principle of least action and derive the Euler–Lagrange equation.

(b) A planar spring-pendulum consists of a light rod of length l and a bead of mass m, which is able to slide along the rod without friction and is attached to the ends of the rod by two identical springs of force constant k as shown in the figure. The rod is pivoted at one end and is free to swing in a vertical plane under the influence of gravity.



(i) Identify suitable generalized coordinates and write down the Lagrangian of the system.

(ii) Derive the equations of motion.

8C Cosmology

Consider three galaxies O, A and B with position vectors $\mathbf{r}_O$, $\mathbf{r}_A$ and $\mathbf{r}_B$ in a homogeneous universe. Assuming they move with non-relativistic velocities $\mathbf{v}_O = \mathbf{0}$, $\mathbf{v}_A$ and $\mathbf{v}_B$, show that spatial homogeneity implies that the velocity field $\mathbf{v}(\mathbf{r})$ satisfies

$$\mathbf{v}(\mathbf{r}_B - \mathbf{r}_A) = \mathbf{v}(\mathbf{r}_B - \mathbf{r}_O) - \mathbf{v}(\mathbf{r}_A - \mathbf{r}_O)\,,$$

and hence that $\mathbf{v}$ is linearly related to $\mathbf{r}$ by

$$v_i = \sum_{j=1}^{3} H_{ij} r_j\,,$$

where the components of the matrix H_{ij} are independent of $\mathbf{r}$.

Suppose the matrix H_{ij} has the form

$$H_{ij} = \frac{D}{t}\begin{pmatrix} 5 & -1 & -2 \\ 1 & 5 & -1 \\ 2 & 1 & 5 \end{pmatrix}\,,$$

with $D > 0$ constant. Describe the kinematics of the cosmological expansion.

SECTION II

9G Coding and Cryptography

Define the *Hamming code*. Show that it is a perfect, linear, 1-error correcting code.

I wish to send a message through a noisy channel to a friend. The message consists of a large number $N = 1,000$ of letters from a 16-letter alphabet $\mathcal{A}$. When my friend has decoded the message, she can tell whether there have been any errors. If there have, she asks me to send the message again and this is repeated until she has received the message without error. For each individual binary digit that is transmitted, there is independently a small probability $p = 0.001$ of an error.

(a) Suppose that I encode my message by writing each letter as a 4-bit binary string. The whole message is then $4N$ bits long. What is the probability P that the entire message is transmitted without error? How many times should I expect to transmit the message until my friend receives it without error?

(b) As an alternative, I use the Hamming code to encode each letter of $\mathcal{A}$ as a 7-bit binary string. What is the probability that my friend can decode a single 7-bit string correctly? Deduce that the probability Q that the entire message is correctly decoded is given approximately by

$$Q \simeq (1 - 21p^2)^N \simeq \exp(-21Np^2)\,.$$

Which coding method is better?

10J Statistical Modelling

An experiment is conducted where scientists count the numbers of each of three different strains of fleas that are reproducing in a controlled environment. Varying concentrations of a particular toxin that impairs reproduction are administered to the fleas. The results of the experiment are stored in a data frame `fleas` in `R`, whose first few rows are given below.

```
> fleas[1:3, ]
  number  conc strain
1     81 0.250      0
2     93 0.250      2
3    102 0.875      1
```

The full dataset has 80 rows. The first column provides the number of fleas, the second provides the concentration of the toxin and the third specifies the strain of the flea as factors 0, 1 or 2. Strain 0 is the common flea and strains 1 and 2 have been genetically modified in a way thought to increase their ability to reproduce in the presence of the toxin.

This question continues on the next page

10J Statistical Modelling (continued)

Explain and interpret the R commands and (abbreviated) output below. In particular, you should describe the model being fitted, briefly explain how the standard errors are calculated, and comment on the hypothesis tests being described in the summary.

```
> fit1 <- glm(number ~ conc*strain, data=fleas, family=poisson)
> summary(fit1)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)   4.47171    0.03849 116.176  < 2e-16 ***
conc         -0.28700    0.06727  -4.266 1.99e-05 ***
strain1       0.09381    0.05483   1.711 0.087076 .
strain2       0.12157    0.05591   2.175 0.029666 *
conc:strain1  0.34215    0.09178   3.728 0.000193 ***
conc:strain2  0.02385    0.09789   0.244 0.807510
```

Explain and motivate the following R code in the light of the output above. Briefly explain the differences between the models fitted below, and the model corresponding to `fit1`.

```
> strain_grp <- fleas$strain
> levels(strain_grp)
[1] "0" "1" "2"
> levels(strain_grp) <- c(0, 1, 0)
> fit2 <- glm(number ~ conc + strain + conc:strain_grp,
+ data=fleas, family=poisson)
> fit3 <- glm(number ~ conc*strain_grp, data=fleas, family=poisson)
```

Denote by M_1, M_2, M_3 the three models being fitted in sequence above. Explain the hypothesis tests comparing the models to each other that can be performed using the output from the following R code.

```
> c(fit1$dev, fit2$dev, fit3$dev)
[1] 56.87 56.93 76.98
> qchisq(0.95, df = 1)
[1] 3.84
```

Use these numbers to comment on the most appropriate model for the data.

11B Further Complex Methods

Consider the differential equation

$$xy'' + (a-x)y' - by = 0 \qquad (*)$$

where a and b are constants with $\mathrm{Re}\,(b) > 0$ and $\mathrm{Re}\,(a-b) > 0$. Laplace's method for finding solutions involves writing

$$y(x) = \int_C e^{xt} f(t)\, dt$$

for some suitable contour C and some suitable function $f(t)$. Determine $f(t)$ for the equation $(*)$ and use a clearly labelled diagram to specify contours C giving two independent solutions when x is real in each of the cases $x > 0$ and $x < 0$.

Now let $a = 3$ and $b = 1$. Find explicit expressions for two independent solutions to $(*)$. Find, in addition, a solution $y(x)$ with $y(0) = 1$.

12C Cosmology

A closed universe contains black-body radiation, has a positive cosmological constant Λ, and is governed by the equation

$$\frac{\dot{a}^2}{a^2} = \frac{\Gamma}{a^4} - \frac{1}{a^2} + \frac{\Lambda}{3},$$

where $a(t)$ is the scale factor and Γ is a positive constant. Using the substitution $y = a^2$ and the boundary condition $y(0) = 0$, deduce the boundary condition for $\dot{y}(0)$ and show that

$$\ddot{y} = \frac{4\Lambda}{3} y - 2$$

and hence that

$$a^2(t) = \frac{3}{2\Lambda}\left[1 - \cosh\left(\sqrt{\frac{4\Lambda}{3}}\, t\right) + \lambda \sinh\left(\sqrt{\frac{4\Lambda}{3}}\, t\right)\right].$$

Express the constant λ in terms of Λ and Γ.

Sketch the graphs of $a(t)$ for the cases $\lambda > 1$ and $0 < \lambda < 1$.

13I Logic and Set Theory

State and prove the Completeness Theorem for Propositional Logic.

[You do *not* need to give definitions of the various terms involved. You may assume the Deduction Theorem, provided that you state it precisely.]

State the Compactness Theorem and the Decidability Theorem, and deduce them from the Completeness Theorem.

Let S consist of the propositions $p_{n+1} \Rightarrow p_n$ for $n = 1, 2, 3, \ldots$. Does S prove p_1? Justify your answer. [Here $p_1, p_2, p_3, \ldots$ are primitive propositions.]

14I Graph Theory

(a) What does it mean to say that a graph G is *strongly regular with parameters* (k, a, b)?

(b) Let G be an incomplete, strongly regular graph with parameters (k, a, b) and of order n. Suppose $b \geqslant 1$. Show that the numbers

$$\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(b-a) - 2k}{\sqrt{(a-b)^2 + 4(k-b)}}\right)$$

are integers.

(c) Suppose now that G is an incomplete, strongly regular graph with parameters $(k, 0, 3)$. Show that $|G| \in \{6, 162\}$.

15F Representation Theory

(a) Let G be a finite group and let $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ be a representation of G. Suppose that there are elements g, h in G such that the matrices $\rho(g)$ and $\rho(h)$ do not commute. Use Maschke's theorem to prove that ρ is irreducible.

(b) Let n be a positive integer. You are given that the *dicyclic* group

$$G_{4n} = \langle a, b : a^{2n} = 1,\ a^n = b^2,\ b^{-1}ab = a^{-1} \rangle$$

has order $4n$.

(i) Show that if ϵ is any $(2n)$th root of unity in $\mathbb{C}$, then there is a representation of G_{4n} over $\mathbb{C}$ which sends

$$a \mapsto \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ \epsilon^n & 0 \end{pmatrix}.$$

(ii) Find all the irreducible representations of G_{4n}.

(iii) Find the character table of G_{4n}.

[*Hint: You may find it helpful to consider the cases n odd and n even separately.*]

16H Number Fields

(a) Let K be a number field, and f a monic polynomial whose coefficients are in $\mathcal{O}_K$. Let M be a field containing K and $\alpha \in M$. Show that if $f(\alpha) = 0$, then α is an algebraic integer.

Hence conclude that if $h \in K[x]$ is monic, with $h^n \in \mathcal{O}_K[x]$, then $h \in \mathcal{O}_K[x]$.

(b) Compute an integral basis for $\mathcal{O}_{\mathbb{Q}(\alpha)}$ when the minimum polynomial of α is $x^3 - x - 4$.

17F Galois Theory

(i) Let $K \subseteq L$ be a field extension and $f \in K[t]$ be irreducible of positive degree. Prove the theorem which states that there is a 1-1 correspondence

$$\mathrm{Root}_f(L) \longleftrightarrow \mathrm{Hom}_K\left(\frac{K[t]}{\langle f\rangle}, L\right).$$

(ii) Let K be a field and $f \in K[t]$. What is a splitting field for f? What does it mean to say f is separable? Show that every $f \in K[t]$ is separable if K is a finite field.

(iii) The primitive element theorem states that if $K \subseteq L$ is a finite separable field extension, then $L = K(\alpha)$ for some $\alpha \in L$. Give the proof of this theorem assuming K is infinite.

18H Algebraic Topology

State carefully a version of the Seifert–van Kampen theorem for a cover of a space by two closed sets.

Let X be the space obtained by gluing together a Möbius band M and a torus $T = S^1 \times S^1$ along a homeomorphism of the boundary of M with $S^1 \times \{1\} \subset T$. Find a presentation for the fundamental group of X, and hence show that it is infinite and non-abelian.

19G Linear Analysis

(a) Let $(e_n)_{n=1}^\infty$ be an orthonormal basis of an inner product space X. Show that for all $x \in X$ there is a unique sequence $(a_n)_{n=1}^\infty$ of scalars such that $x = \sum_{n=1}^\infty a_n e_n$.

Assume now that X is a Hilbert space and that $(f_n)_{n=1}^\infty$ is another orthonormal basis of X. Prove that there is a unique bounded linear map $U\colon X \to X$ such that $U(e_n) = f_n$ for all $n \in \mathbb{N}$. Prove that this map U is unitary.

(b) Let $1 \leqslant p < \infty$ with $p \neq 2$. Show that no subspace of ℓ_2 is isomorphic to ℓ_p. [*Hint: Apply the generalized parallelogram law to suitable vectors.*]

20F Riemann Surfaces

Let $f : R \to S$ be a non-constant holomorphic map between compact connected Riemann surfaces and let $B \subset S$ denote the set of branch points. Show that the map $f : R \setminus f^{-1}(B) \to S \setminus B$ is a regular covering map.

Given $w \in S \setminus B$ and a closed curve γ in $S \setminus B$ with initial and final point w, explain how this defines a permutation of the (finite) set $f^{-1}(w)$. Show that the group H obtained from all such closed curves is a transitive subgroup of the full symmetric group of the fibre $f^{-1}(w)$.

Find the group H for $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ where $f(z) = z^3/(1-z^2)$.

21F Algebraic Geometry

Let k be an algebraically closed field.

(i) Let X and Y be affine varieties defined over k. Given a map $f : X \to Y$, define what it means for f to be a morphism of affine varieties.

(ii) With X, Y still affine varieties over k, show that there is a one-to-one correspondence between $\mathrm{Hom}(X, Y)$, the set of morphisms between X and Y, and $\mathrm{Hom}(A(Y), A(X))$, the set of k-algebra homomorphisms between $A(Y)$ and $A(X)$.

(iii) Let $f : \mathbb{A}^2 \to \mathbb{A}^4$ be given by $f(t,u) = (u, t, t^2, tu)$. Show that the image of f is an affine variety X, and find a set of generators for $I(X)$.

22G Differential Geometry

Let $\Omega \subset \mathbb{R}^2$ be a domain (connected open subset) with boundary $\partial\Omega$ a continuously differentiable simple closed curve. Denoting by $A(\Omega)$ the area of Ω and $l(\partial\Omega)$ the length of the curve $\partial\Omega$, state and prove the isoperimetric inequality relating $A(\Omega)$ and $l(\partial\Omega)$ with optimal constant, including the characterization for equality. [You may appeal to Wirtinger's inequality as long as you state it precisely.]

Does the result continue to hold if the boundary $\partial\Omega$ is allowed finitely many points at which it is not differentiable? Briefly justify your answer by giving either a counterexample or an indication of a proof.

23J Probability and Measure

(a) Define the following concepts: a π-system, a d-system and a σ-algebra.

(b) State the Dominated Convergence Theorem.

(c) Does the set function

$$\mu(A) = \begin{cases} 0 & \text{for } A \text{ bounded,} \\ 1 & \text{for } A \text{ unbounded,} \end{cases}$$

furnish an example of a Borel measure?

(d) Suppose $g\colon [0,1] \to [0,1]$ is a measurable function. Let $f\colon [0,1] \to \mathbb{R}$ be continuous with $f(0) \leqslant f(1)$. Show that the limit

$$\lim_{n\to\infty} \int_0^1 f(\,g(x)^n)\,dx$$

exists and lies in the interval $[f(0), f(1)]$.

24K Applied Probability

(a) Give the definition of a birth and death chain in terms of its generator. Show that a measure π is invariant for a birth and death chain if and only if it solves the detailed balance equations.

(b) There are s servers in a post office and a single queue. Customers arrive as a Poisson process of rate λ and the service times at each server are independent and exponentially distributed with parameter μ. Let X_t denote the number of customers in the post office at time t. Find conditions on λ, μ and s for X to be positive recurrent, null recurrent and transient, justifying your answers.

25J Principles of Statistics

Consider a normally distributed random vector $X \in \mathbb{R}^p$ modelled as $X \sim N(\theta, I_p)$ where $\theta \in \mathbb{R}^p$, I_p is the $p \times p$ identity matrix, and where $p \geqslant 3$. Define the *Stein estimator* $\hat{\theta}_{STEIN}$ of θ.

Prove that $\hat{\theta}_{STEIN}$ dominates the estimator $\tilde{\theta} = X$ for the risk function induced by quadratic loss

$$\ell(a, \theta) = \sum_{i=1}^{p} (a_i - \theta_i)^2, \quad a \in \mathbb{R}^p.$$

Show however that the worst case risks coincide, that is, show that

$$\sup_{\theta \in \mathbb{R}^p} E_\theta \, \ell(X, \theta) = \sup_{\theta \in \mathbb{R}^p} E_\theta \, \ell(\hat{\theta}_{STEIN}, \theta).$$

[You may use Stein's lemma without proof, provided it is clearly stated.]

26K Stochastic Financial Models

(i) What does it mean to say that $(X_n, \mathcal{F}_n)_{n \geqslant 0}$ is a martingale?

(ii) If Y is an integrable random variable and $Y_n = E[\, Y \mid \mathcal{F}_n \,]$, prove that $(Y_n, \mathcal{F}_n)$ is a martingale. [Standard facts about conditional expectation may be used without proof provided they are clearly stated.] When is it the case that the limit $\lim_{n \to \infty} Y_n$ exists almost surely?

(iii) An urn contains initially one red ball and one blue ball. A ball is drawn at random and then returned to the urn with a new ball of the *other* colour. This process is repeated, adding one ball at each stage to the urn. If the number of red balls after n draws and replacements is X_n, and the number of blue balls is Y_n, show that $M_n = h(X_n, Y_n)$ is a martingale, where

$$h(x, y) = (x - y)(x + y - 1).$$

Does this martingale converge almost surely?

27C Asymptotic Methods

(a) State the integral expression for the gamma function $\Gamma(z)$, for $\mathrm{Re}(z) > 0$, and express the integral

$$\int_0^\infty t^{\gamma-1}\, e^{it}\, dt\,, \qquad 0 < \gamma < 1\,,$$

in terms of $\Gamma(\gamma)$. Explain why the constraints on γ are necessary.

(b) Show that

$$\int_0^\infty \frac{e^{-kt^2}}{(t^2+t)^{\frac{1}{4}}}\, dt \sim \sum_{m=0}^{\infty} \frac{a_m}{k^{\alpha+\beta m}}\,, \qquad k \to \infty\,,$$

for some constants a_m, α and β. Determine the constants α and β, and express a_m in terms of the gamma function.

State without proof the basic result needed for the rigorous justification of the above asymptotic formula.

[You may use the identity:

$$(1+z)^\alpha = \sum_{m=0}^{\infty} c_m z^m, \qquad c_m = \frac{\Gamma(\alpha+1)}{m!\,\Gamma(\alpha+1-m)}\,, \quad |z| < 1\,.]$$

28B Dynamical Systems

(a) What is a Lyapunov function?

Consider the dynamical system for $\mathbf{x}(t) = \big(x(t), y(t)\big)$ given by

$$\begin{aligned} \dot{x} &= -x + y + x(x^2 + y^2)\,, \\ \dot{y} &= -y - 2x + y(x^2 + y^2)\,. \end{aligned}$$

Prove that the origin is asymptotically stable (quoting carefully any standard results that you use).

Show that the domain of attraction of the origin includes the region $x^2 + y^2 < r_1^2$ where the maximum possible value of r_1 is to be determined.

Show also that there is a region $E = \{\mathbf{x} \,|\, x^2 + y^2 > r_2^2\}$ such that $\mathbf{x}(0) \in E$ implies that $|\mathbf{x}(t)|$ increases without bound. Explain your reasoning carefully. Find the smallest possible value of r_2.

(b) Now consider the dynamical system

$$\begin{aligned} \dot{x} &= x - y - x(x^2 + y^2)\,, \\ \dot{y} &= y + 2x - y(x^2 + y^2)\,. \end{aligned}$$

Prove that this system has a periodic solution (again, quoting carefully any standard results that you use).

Demonstrate that this periodic solution is unique.

29D Integrable Systems

Let $u_t = K(x, u, u_x, \ldots)$ be an evolution equation for the function $u = u(x,t)$. Assume u and all its derivatives decay rapidly as $|x| \to \infty$. What does it mean to say that the evolution equation for u can be written in *Hamiltonian form*?

The modified KdV (mKdV) equation for u is

$$u_t + u_{xxx} - 6u^2 u_x = 0\,.$$

Show that small amplitude solutions to this equation are dispersive.

Demonstrate that the mKdV equation can be written in Hamiltonian form and define the associated Poisson bracket $\{\ ,\ \}$ on the space of functionals of u. Verify that the Poisson bracket is linear in each argument and anti-symmetric.

Show that a functional $I = I[u]$ is a first integral of the mKdV equation if and only if $\{I, H\} = 0$, where $H = H[u]$ is the Hamiltonian.

Show that if u satisfies the mKdV equation then

$$\frac{\partial}{\partial t}\left(u^2\right) + \frac{\partial}{\partial x}\left(2uu_{xx} - u_x^2 - 3u^4\right) = 0\,.$$

Using this equation, show that the functional

$$I[u] = \int u^2\, dx$$

Poisson-commutes with the Hamiltonian.

30E Partial Differential Equations

(a) State the Cauchy–Kovalevskaya theorem, and explain for which values of $a \in \mathbb{R}$ it implies the existence of solutions to the Cauchy problem

$$xu_x + yu_y + au_z = u\,, \qquad u(x,y,0) = f(x,y)\,,$$

where f is real analytic. Using the method of characteristics, solve this problem for these values of a, and comment on the behaviour of the characteristics as a approaches any value where the non-characteristic condition fails.

(b) Consider the Cauchy problem

$$u_y = v_x\,, \qquad v_y = -u_x$$

with initial data $u(x,0) = f(x)$ and $v(x,0) = 0$ which are 2π-periodic in x. Give an example of a sequence of smooth solutions (u_n, v_n) which are also 2π-periodic in x whose corresponding initial data $u_n(x,0) = f_n(x)$ and $v_n(x,0) = 0$ are such that $\int_0^{2\pi} |f_n(x)|^2 dx \to 0$ while $\int_0^{2\pi} |u_n(x,y)|^2 dx \to \infty$ for non-zero y as $n \to \infty$.

Comment on the significance of this in relation to the concept of *well-posedness.*

31A Principles of Quantum Mechanics

If A and B are operators which each commute with their commutator $[A, B]$, show that

$$F(\lambda) \;=\; e^{\lambda A}\, e^{\lambda B}\, e^{-\lambda(A+B)} \qquad \text{satisfies} \qquad F'(\lambda) \;=\; \lambda\, [A, B]\, F(\lambda)\,.$$

By solving this differential equation for $F(\lambda)$, deduce that

$$e^{A}\, e^{B} \;=\; e^{\frac{1}{2}[A,B]}\, e^{A+B}\,.$$

The annihilation and creation operators for a harmonic oscillator of mass m and frequency ω are defined by

$$a = \sqrt{\frac{m\omega}{2\hbar}}\Big(\hat{x} + \frac{i}{m\omega}\hat{p}\Big)\,, \quad a^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\Big(\hat{x} - \frac{i}{m\omega}\hat{p}\Big)\,.$$

Write down an expression for the general normalised eigenstate $|n\rangle$ $(n = 0, 1, 2, \ldots)$ of the oscillator Hamiltonian H in terms of the ground state $|0\rangle$. What is the energy eigenvalue E_n of the state $|n\rangle$?

Suppose the oscillator is now subject to a small perturbation so that it is described by the modified Hamiltonian $H + \varepsilon V(\hat{x})$ with $V(\hat{x}) = \cos(\mu\hat{x})$. Show that

$$V(\hat{x}) \;=\; \tfrac{1}{2}\, e^{-\gamma^2/2} \left(e^{i\gamma\, a^{\dagger}} e^{i\gamma\, a} + e^{-i\gamma\, a^{\dagger}} e^{-i\gamma\, a} \right)\,,$$

where γ is a constant, to be determined. Hence show that to $O(\varepsilon^2)$ the shift in the ground state energy as a result of the perturbation is

$$\varepsilon\, e^{-\mu^2\hbar/4m\omega} \;-\; \varepsilon^2\, e^{-\mu^2\hbar/2m\omega}\, \frac{1}{\hbar\omega} \sum_{p=1}^{\infty} \frac{1}{(2p)!\, 2p} \Big(\frac{\mu^2\hbar}{2m\omega}\Big)^{2p}\,.$$

[Standard results of perturbation theory may be quoted without proof.]

32A Applications of Quantum Mechanics

Define the Rayleigh–Ritz quotient $R[\psi]$ for a normalisable state $|\psi\rangle$ of a quantum system with Hamiltonian H. Given that the spectrum of H is discrete and that there is a unique ground state of energy E_0, show that $R[\psi] \geqslant E_0$ and that equality holds if and only if $|\psi\rangle$ is the ground state.

A simple harmonic oscillator (SHO) is a particle of mass m moving in one dimension subject to the potential

$$V(x) \;=\; \frac{1}{2}m\omega^2 x^2\,.$$

Estimate the ground state energy E_0 of the SHO by using the ground state wavefunction for a particle in an infinite potential well of width a, centred on the origin (the potential is $U(x) = 0$ for $|x| < a/2$ and $U(x) = \infty$ for $|x| > a/2$). Take a as the variational parameter.

Perform a similar estimate for the energy E_1 of the first excited state of the SHO by using the first excited state of the infinite potential well as a trial wavefunction.

Is the estimate for E_1 necessarily an upper bound? Justify your answer.

$$\left[\text{You may use}: \quad \int_{-\pi/2}^{\pi/2} y^2\cos^2 y\,dy \;=\; \frac{\pi}{4}\Big(\frac{\pi^2}{6}-1\Big) \quad\text{and}\quad \int_{-\pi}^{\pi} y^2\sin^2 y\,dy \;=\; \pi\Big(\frac{\pi^2}{3}-\frac{1}{2}\Big)\,.\right]$$

33C Statistical Physics

(a) Define the canonical partition function Z for a system with energy levels E_n, where n labels states, given that the system is in contact with a heat reservoir at temperature T. What is the probability $p(n)$ that the system occupies state n? Starting from an expression for the entropy $S = k_B\, \partial\,(T \ln Z)\,/\,\partial T$, deduce that

$$S = -k_B \sum_n p(n)\, \ln p(n)\,. \qquad (*)$$

(b) Consider an ensemble consisting of W copies of the system in part (a) with W very large, so that there are $W p(n)$ members of the ensemble in state n. Starting from an expression for the number of ways in which this can occur, find the entropy S_W of the ensemble and hence re-derive the expression $(*)$. [You may assume Stirling's formula $\ln X! \approx X \ln X - X$ for X large.]

(c) Consider a system of N non-interacting particles at temperature T. Each particle has q internal states with energies

$$0\,,\ \mathcal{E}\,,\ 2\mathcal{E}\,,\ \ldots\,,\ (q{-}1)\mathcal{E}\,.$$

Assuming that the internal states are the only relevant degrees of freedom, calculate the total entropy of the system. Find the limiting values of the entropy as $T \to 0$ and $T \to \infty$ and comment briefly on your answers.

34A Electrodynamics

Briefly explain how to interpret the components of the relativistic stress–energy tensor in terms of the density and flux of energy and momentum in some inertial frame.

(i) The stress–energy tensor of the electromagnetic field is

$$T^{\mu\nu}_{\text{em}} = \frac{1}{\mu_0}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right),$$

where $F_{\mu\nu}$ is the field strength, $\eta_{\mu\nu}$ is the Minkowski metric, and μ_0 is the permeability of free space. Show that $\partial_\mu T^{\mu\nu}_{\text{em}} = -F^{\nu}{}_{\mu}J^{\mu}$, where J^μ is the current 4-vector.

[Maxwell's equations are $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$ and $\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0$.]

(ii) A fluid consists of point particles of rest mass m and charge q. The fluid can be regarded as a continuum, with 4-velocity $u^\mu(x)$ depending on the position x in spacetime. For each x there is an inertial frame S_x in which the fluid particles at x are at rest. By considering components in S_x, show that the fluid has a current 4-vector field

$$J^\mu = q\, n_0 u^\mu ,$$

and a stress–energy tensor

$$T^{\mu\nu}_{\text{fluid}} = m\, n_0 u^\mu u^\nu ,$$

where $n_0(x)$ is the proper number density of particles (the number of particles per unit spatial volume in S_x in a small region around x). Write down the Lorentz 4-force on a fluid particle at x. By considering the resulting 4-acceleration of the fluid, show that the total stress–energy tensor is conserved, i.e.

$$\partial_\mu \left(T^{\mu\nu}_{\text{em}} + T^{\mu\nu}_{\text{fluid}}\right) = 0 .$$

35D General Relativity

A vector field ξ^a is said to be a *conformal Killing vector field* of the metric g_{ab} if

$$\xi_{(a;b)} = \tfrac{1}{2}\psi g_{ab} \qquad (*)$$

for some scalar field ψ. It is a *Killing vector field* if $\psi = 0$.

(a) Show that $(*)$ is equivalent to

$$\xi^c g_{ab,c} + \xi^c{}_{,a}\, g_{bc} + \xi^c{}_{,b}\, g_{ac} = \psi\, g_{ab}\,.$$

(b) Show that if ξ^a is a conformal Killing vector field of the metric g_{ab}, then ξ^a is a Killing vector field of the metric $\mathrm{e}^{2\phi} g_{ab}$, where ϕ is any function that obeys

$$2\xi^c \phi_{,c} + \psi = 0\,.$$

(c) Use part (b) to find an example of a metric with coordinates t, x, y and z (where $t > 0$) for which (t, x, y, z) are the contravariant components of a Killing vector field. [*Hint: You may wish to start by considering what happens in Minkowski space.*]

36E Fluid Dynamics II

(i) In a Newtonian fluid, the deviatoric stress tensor is linearly related to the velocity gradient so that the total stress tensor is

$$\sigma_{ij} = -p\delta_{ij} + A_{ijkl}\frac{\partial u_k}{\partial x_l}.$$

Show that for an incompressible isotropic fluid with a symmetric stress tensor we necessarily have

$$A_{ijkl}\frac{\partial u_k}{\partial x_l} = 2\mu e_{ij},$$

where μ is a constant which we call the dynamic viscosity and e_{ij} is the symmetric part of $\partial u_i/\partial x_j$.

(ii) Consider Stokes flow due to the translation of a rigid sphere S_a of radius a so that the sphere exerts a force $\mathbf{F}$ on the fluid. At distances much larger than the radius of the sphere, the instantaneous velocity and pressure fields are

$$u_i(\mathbf{x}) = \frac{1}{8\mu\pi}\left(\frac{F_i}{r} + \frac{F_m x_m x_i}{r^3}\right), \quad p(\mathbf{x}) = \frac{1}{4\pi}\frac{F_m x_m}{r^3},$$

where $\mathbf{x}$ is measured with respect to an origin located at the centre of the sphere, and $r = |\mathbf{x}|$.

Consider a sphere S_R of radius $R \gg a$ instantaneously concentric with S_a. By explicitly computing the tractions and integrating them, show that the force $\mathbf{G}$ exerted by the fluid located in $r > R$ on S_R is constant and independent of R, and evaluate it.

(iii) Explain why the Stokes equations in the absence of body forces can be written as

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0.$$

Show that by integrating this equation in the fluid volume located instantaneously between S_a and S_R, you can recover the result in (ii) directly.

37B Waves

An acoustic plane wave (not necessarily harmonic) travels at speed c_0 in the direction $\hat{\mathbf{k}}$, where $|\hat{\mathbf{k}}| = 1$, through an inviscid, compressible fluid of unperturbed density ρ_0. Show that the velocity $\tilde{\mathbf{u}}$ is proportional to the perturbation pressure $\tilde{p}$, and find $\tilde{\mathbf{u}}/\tilde{p}$. Define the *acoustic intensity* $\mathbf{I}$.

A harmonic acoustic plane wave with wavevector $\mathbf{k} = k(\cos\theta, \sin\theta, 0)$ and unit-amplitude perturbation pressure is incident from $x < 0$ on a thin elastic membrane at unperturbed position $x = 0$. The regions $x < 0$ and $x > 0$ are both occupied by gas with density ρ_0 and sound speed c_0. The kinematic boundary conditions at the membrane are those appropriate for an inviscid fluid, and the (linearized) dynamic boundary condition is

$$m\frac{\partial^2 X}{\partial t^2} - T\frac{\partial^2 X}{\partial y^2} + \big[\tilde{p}(0, y, t)\big]_-^+ = 0$$

where T and m are the tension and mass per unit area of the membrane, and $x = X(y, t)$ (with $|kX| \ll 1$) is its perturbed position. Find the amplitudes of the reflected and transmitted pressure perturbations, expressing your answers in terms of the dimensionless parameter

$$\alpha = \frac{\rho_0 c_0^2}{k\cos\theta(mc_0^2 - T\sin^2\theta)}\,.$$

Hence show that the time-averaged energy flux in the x-direction is conserved across the membrane.

38E Numerical Analysis

(a) The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right) \quad \text{in} \quad 0 \leqslant x \leqslant 1, \quad t \geqslant 0,$$

with the initial condition $u(x, 0) = \phi(x)$ in $0 \leqslant x \leqslant 1$ and zero boundary conditions at $x = 0$ and $x = 1$, is solved by the finite-difference method

$$u_m^{n+1} = u_m^n + \mu\left[a_{m-\frac{1}{2}}u_{m-1}^n - (a_{m-\frac{1}{2}} + a_{m+\frac{1}{2}})u_m^n + a_{m+\frac{1}{2}}u_{m+1}^n\right], \qquad m = 1, 2, \ldots, M,$$

where $\mu = k/h^2$, $k = \Delta t$, $h = 1/(M+1)$, $u_m^n \approx u(mh, nk)$, and $a_\alpha = a(\alpha h)$.

Assuming that the function a and the exact solution are sufficiently smooth, prove that the exact solution satisfies the numerical scheme with error $O(h^3)$ for constant μ.

(b) For the problem in part (a), assume that there exist $0 < a_- < a_+ < \infty$ such that $a_- \leqslant a(x) \leqslant a_+$ for all $0 \leqslant x \leqslant 1$. State (without proof) the Gershgorin theorem and prove that the method is stable for $0 < \mu \leqslant 1/(2a_+)$.

END OF PAPER