UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IB

Thursday, 4 June, 2015 1:30 pm to 4:30 pm

PAPER 3

Before you begin read these instructions carefully.

Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt at most ***four*** *questions from Section I and at most* ***six*** *questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise, you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles labelled ***A, B, . . . , H*** *according to the examiner letter affixed to each question, including in the same bundle questions from Sections I and II with the same examiner letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Groups, Rings and Modules

State two equivalent conditions for a commutative ring to be *Noetherian*, and prove they are equivalent. Give an example of a ring which is not Noetherian, and explain why it is not Noetherian.

2G Analysis II

Define what is meant by a *uniformly continuous* function f on a subset E of a metric space. Show that every continuous function on a closed, bounded interval is uniformly continuous. [You may assume the Bolzano–Weierstrass theorem.]

Suppose that a function $g : [0, \infty) \to \mathbb{R}$ is continuous and tends to a finite limit at ∞. Is g necessarily uniformly continuous on $[0, \infty)$? Give a proof or a counterexample as appropriate.

3E Metric and Topological Spaces

Define what it means for a topological space X to be (i) *connected* (ii) *path-connected.*

Prove that any path-connected space X is connected. [You may assume the interval $[0, 1]$ is connected.]

Give a counterexample (without justification) to the converse statement.

4B Complex Methods

Find the Fourier transform of the function

$$f(x) = \frac{1}{1 + x^2}, \quad x \in \mathbb{R},$$

using an appropriate contour integration. Hence find the Fourier transform of its derivative, $f'(x)$, and evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{4x^2}{(1 + x^2)^4} dx.$$

5F Geometry

State the sine rule for spherical triangles.

Let Δ be a spherical triangle with vertices A, B, and C, with angles α, β and γ at the respective vertices. Let a, b, and c be the lengths of the edges BC, AC and AB respectively. Show that $b = c$ if and only if $\beta = \gamma$. [You may use the cosine rule for spherical triangles.] Show that this holds if and only if there exists a reflection M such that $M(A) = A$, $M(B) = C$ and $M(C) = B$.

Are there equilateral triangles on the sphere? Justify your answer.

6A Variational Principles

(a) Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be *convex*.

(b) Define the *Legendre transform* $f^*(p)$ of a convex function $f(x)$, where $x \in \mathbb{R}$. Show that $f^*(p)$ is a convex function.

(c) Find the Legendre transform $f^*(p)$ of the function $f(x) = e^x$, and the domain of f^*.

7C Methods

(a) From the defining property of the δ function,

$$\int_{-\infty}^{\infty} \delta(x)\, f(x)\, dx = f(0)\,,$$

for any function f, prove that

(i) $\delta(-x) = \delta(x)$,

(ii) $\delta(ax) = |a|^{-1}\delta(x)$ for $a \in \mathbb{R}$, $a \neq 0$,

(iii) If $g : \mathbb{R} \to \mathbb{R}$, $x \mapsto g(x)$ is smooth and has isolated zeros x_i where the derivative $g'(x_i) \neq 0$, then

$$\delta[g(x)] = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}\,.$$

(b) Show that the function $\gamma(x)$ defined by

$$\gamma(x) = \lim_{s\to 0} \frac{e^{x/s}}{s\left(1 + e^{x/s}\right)^2}\,,$$

is the $\delta(x)$ function.

8D Quantum Mechanics

A quantum-mechanical system has normalised energy eigenstates χ_1 and χ_2 with non-degenerate energies E_1 and E_2 respectively. The observable A has normalised eigenstates,

$$\begin{aligned} \phi_1 &= C(\chi_1 + 2\chi_2)\,, \qquad \text{eigenvalue} = a_1\,, \\ \phi_2 &= C(2\chi_1 - \chi_2)\,, \qquad \text{eigenvalue} = a_2\,, \end{aligned}$$

where C is a positive real constant. Determine C.

Initially, at time $t = 0$, the state of the system is ϕ_1. Write down an expression for $\psi(t)$, the state of the system with $t \geqslant 0$. What is the probability that a measurement of energy at time t will yield E_2?

For the same initial state, determine the probability that a measurement of A at time $t > 0$ will yield a_1 and the probability that it will yield a_2.

9H Markov Chains

Define what is meant by a *communicating class* and a *closed class* in a Markov chain.

A Markov chain $(X_n : n \geqslant 0)$ with state space $\{1, 2, 3, 4\}$ has transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Write down the communicating classes for this Markov chain and state whether or not each class is closed.

If $X_0 = 2$, let N be the smallest n such that $X_n \neq 2$. Find $\mathbb{P}(N = n)$ for $n = 1, 2, \ldots$ and $\mathbb{E}(N)$. Describe the evolution of the chain if $X_0 = 2$.

SECTION II

10E Linear Algebra

Let $A_1, A_2, \ldots, A_k$ be $n \times n$ matrices over a field $\mathbb{F}$. We say $A_1, A_2, \ldots, A_k$ are simultaneously diagonalisable if there exists an invertible matrix P such that $P^{-1}A_iP$ is diagonal for all $1 \leqslant i \leqslant k$. We say the matrices are commuting if $A_iA_j = A_jA_i$ for all i, j.

(i) Suppose $A_1, A_2, \ldots, A_k$ are simultaneously diagonalisable. Prove that they are commuting.

(ii) Define an *eigenspace* of a matrix. Suppose $B_1, B_2, \ldots, B_k$ are commuting $n \times n$ matrices over a field $\mathbb{F}$. Let E denote an eigenspace of B_1. Prove that $B_i(E) \leqslant E$ for all i.

(iii) Suppose $B_1, B_2, \ldots, B_k$ are commuting diagonalisable matrices. Prove that they are simultaneously diagonalisable.

(iv) Are the 2×2 diagonalisable matrices over $\mathbb{C}$ simultaneously diagonalisable? Explain your answer.

11F Groups, Rings and Modules

Can a group of order 55 have 20 elements of order 11? If so, give an example. If not, give a proof, including the proof of any statements you need.

Let G be a group of order pq, with p and q primes, $p > q$. Suppose furthermore that q does not divide $p - 1$. Show that G is cyclic.

12G Analysis II

Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}^m$ to be *differentiable at* $x \in \mathbb{R}^n$ with derivative $Df(x)$.

State and prove the *chain rule* for the derivative of $g \circ f$, where $g : \mathbb{R}^m \to \mathbb{R}^p$ is a differentiable function.

Now let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $g(x) = f(x, c - x)$ where c is a constant. Show that g is differentiable and find its derivative in terms of the partial derivatives of f. Show that if $D_1f(x, y) = D_2f(x, y)$ holds everywhere in $\mathbb{R}^2$, then $f(x, y) = h(x + y)$ for some differentiable function h.

13G Complex Analysis

State the argument principle.

Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a holomorphic injective function. Show that $f'(z) \neq 0$ for each z in U and that $f(U)$ is open.

Stating clearly any theorems that you require, show that for each $a \in U$ and a sufficiently small $r > 0$,

$$g(w) = \frac{1}{2\pi i} \int_{|z-a|=r} \frac{z f'(z)}{f(z) - w} dz$$

defines a holomorphic function on some open disc D about $f(a)$.

Show that g is the inverse for the restriction of f to $g(D)$.

14F Geometry

Let $T : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be a Möbius transformation on the Riemann sphere $\mathbb{C}_\infty$.

(i) Show that T has either one or two fixed points.

(ii) Show that if T is a Möbius transformation corresponding to (under stereographic projection) a rotation of S^2 through some fixed non-zero angle, then T has two fixed points, z_1, z_2, with $z_2 = -1/\bar{z}_1$.

(iii) Suppose T has two fixed points z_1, z_2 with $z_2 = -1/\bar{z}_1$. Show that either T corresponds to a rotation as in (ii), or one of the fixed points, say z_1, is attractive, i.e. $T^n z \to z_1$ as $n \to \infty$ for any $z \neq z_2$.

15C Methods

(i) Consider the Poisson equation $\nabla^2\psi(\mathbf{r}) = f(\mathbf{r})$ with forcing term f on the infinite domain $\mathbb{R}^3$ with $\lim_{|\mathbf{r}|\to\infty} \psi = 0$. Derive the Green's function $G(\mathbf{r}, \mathbf{r}') = -1/(4\pi|\mathbf{r} - \mathbf{r}'|)$ for this equation using the divergence theorem. [You may assume without proof that the divergence theorem is valid for the Green's function.]

(ii) Consider the *Helmholtz equation*

$$\nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = f(\mathbf{r})\,, \qquad (\dagger)$$

where k is a real constant. A Green's function $g(\mathbf{r}, \mathbf{r}')$ for this equation can be constructed from $G(\mathbf{r}, \mathbf{r}')$ of (i) by assuming $g(\mathbf{r}, \mathbf{r}') = U(r)G(\mathbf{r}, \mathbf{r}')$ where $r = |\mathbf{r} - \mathbf{r}'|$ and $U(r)$ is a regular function. Show that $\lim_{r\to 0} U(r) = 1$ and that U satisfies the equation

$$\frac{d^2U}{dr^2} + k^2U(r) = 0\,. \qquad (\ddagger)$$

(iii) Take the Green's function with the specific solution $U(r) = e^{ikr}$ to Eq. ($\ddagger$) and consider the Helmholtz equation ($\dagger$) on the semi-infinite domain $z > 0$, $x, y \in \mathbb{R}$. Use the method of images to construct a Green's function for this problem that satisfies the boundary conditions

$$\frac{\partial g}{\partial z'} = 0 \quad \text{on} \quad z' = 0 \quad \text{and} \quad \lim_{|\mathbf{r}|\to\infty} g(\mathbf{r}, \mathbf{r}') = 0\,.$$

(iv) A solution to the Helmholtz equation on a bounded domain can be constructed in complete analogy to that of the Poisson equation using the Green's function in Green's 3rd identity

$$\psi(\mathbf{r}) = \int_{\partial V} \left[\psi(\mathbf{r}')\frac{\partial g(\mathbf{r}, \mathbf{r}')}{\partial n'} - g(\mathbf{r}, \mathbf{r}')\frac{\partial \psi(\mathbf{r}')}{\partial n'}\right] dS' + \int_V f(\mathbf{r}')g(\mathbf{r}, \mathbf{r}')dV'\,,$$

where V denotes the volume of the domain, ∂V its boundary and $\partial/\partial n'$ the outgoing normal derivative on the boundary. Now consider the homogeneous Helmholtz equation $\nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = 0$ on the domain $z > 0$, $x, y \in \mathbb{R}$ with boundary conditions $\psi(\mathbf{r}) = 0$ at $|\mathbf{r}| \to \infty$ and

$$\left.\frac{\partial \psi}{\partial z}\right|_{z=0} = \begin{cases} 0 & \text{for } \rho > a \\ A & \text{for } \rho \leqslant a \end{cases}$$

where $\rho = \sqrt{x^2 + y^2}$ and A and a are real constants. Construct a solution in integral form to this equation using cylindrical coordinates (z, ρ, φ) with $x = \rho\cos\varphi$, $y = \rho\sin\varphi$.

16D Quantum Mechanics

Define the angular momentum operators $\hat{L}_i$ for a particle in three dimensions in terms of the position and momentum operators $\hat{x}_i$ and $\hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}$. Write down an expression for $[\hat{L}_i, \hat{L}_j]$ and use this to show that $[\hat{L}^2, \hat{L}_i] = 0$ where $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$. What is the significance of these two commutation relations?

Let $\psi(x, y, z)$ be both an eigenstate of $\hat{L}_z$ with eigenvalue zero and an eigenstate of $\hat{L}^2$ with eigenvalue $\hbar^2 l(l+1)$. Show that $(\hat{L}_x + i\hat{L}_y)\psi$ is also an eigenstate of both $\hat{L}_z$ and $\hat{L}^2$ and determine the corresponding eigenvalues.

Find real constants A and B such that

$$\phi(x, y, z) = \left(Az^2 + By^2 - r^2\right) e^{-r}, \qquad r^2 = x^2 + y^2 + z^2,$$

is an eigenfunction of $\hat{L}_z$ with eigenvalue zero and an eigenfunction of $\hat{L}^2$ with an eigenvalue which you should determine. [*Hint: You might like to show that* $\hat{L}_i\, f(r) = 0$.]

17A Electromagnetism

A charge density $\rho = \lambda/r$ fills the region of 3-dimensional space $a < r < b$, where r is the radial distance from the origin and λ is a constant. Compute the electric field in all regions of space in terms of Q, the total charge of the region. Sketch a graph of the magnitude of the electric field versus r (assuming that $Q > 0$).

Now let $\Delta = b - a \to 0$. Derive the surface charge density σ in terms of Δ, a and λ and explain how a finite surface charge density may be obtained in this limit. Sketch the magnitude of the electric field versus r in this limit. Comment on any discontinuities, checking a standard result involving σ for this particular case.

A second shell of equal and opposite total charge is centred on the origin and has a radius $c < a$. Sketch the electric potential of this system, assuming that it tends to 0 as $r \to \infty$.

18B Fluid Dynamics

A source of sound induces a travelling wave of pressure above the free surface of a fluid located in the $z < 0$ domain as

$$p = p_{atm} + p_0 \cos(kx - \omega t),$$

with $p_0 \ll p_{atm}$. Here k and ω are fixed real numbers. We assume that the flow induced in the fluid is irrotational.

(i) State the linearized equation of motion for the fluid and the free surface, $z = h(x,t)$, with all boundary conditions.

(ii) Solve for the velocity potential, $\phi(x,z,t)$, and the height of the free surface, $h(x,t)$. Verify that your solutions are dimensionally correct.

(iii) Interpret physically the behaviour of the solution when $\omega^2 = gk$.

19D Numerical Analysis

Define the QR factorization of an $m \times n$ matrix A. Explain how it can be used to solve the least squares problem of finding the vector $x^* \in \mathbb{R}^n$ which minimises $||\mathsf{A}x - b||$, where $b \in \mathbb{R}^m$, $m > n$, and $||\cdot||$ is the Euclidean norm.

Explain how to construct Q and R by the Gram-Schmidt procedure. Why is this procedure not useful for numerical factorization of large matrices?

Let

$$\mathsf{A} = \begin{bmatrix} 5 & 6 & -14 \\ 5 & 4 & 4 \\ -5 & 2 & -8 \\ 5 & 12 & -18 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Using the Gram-Schmidt procedure find a QR decomposition of A. Hence solve the least squares problem giving both x^* and $||\mathsf{A}x^* - b||$.

20H Statistics

(a) Suppose that $X_1, \ldots, X_n$ are independent identically distributed random variables, each with density $f(x) = \theta \exp(-\theta x)$, $x > 0$ for some unknown $\theta > 0$. Use the generalised likelihood ratio to obtain a size α test of $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$.

(b) A die is loaded so that, if p_i is the probability of face i, then $p_1 = p_2 = \theta_1$, $p_3 = p_4 = \theta_2$ and $p_5 = p_6 = \theta_3$. The die is thrown n times and face i is observed x_i times. Write down the likelihood function for $\theta = (\theta_1, \theta_2, \theta_3)$ and find the maximum likelihood estimate of θ.

Consider testing whether or not $\theta_1 = \theta_2 = \theta_3$ for this die. Find the generalised likelihood ratio statistic Λ and show that

$$2 \log_e \Lambda \approx T, \qquad \text{where } T = \sum_{i=1}^{3} \frac{(o_i - e_i)^2}{e_i},$$

where you should specify o_i and e_i in terms of $x_1, \ldots, x_6$. Explain how to obtain an approximate size 0.05 test using the value of T. Explain what you would conclude (and why) if $T = 2.03$.

21H Optimization

Consider the linear programming problem P:

$$\text{minimise } c^T x \text{ subject to } Ax \geqslant b, \;\; x \geqslant 0,$$

where x and c are in $\mathbb{R}^n$, A is a real $m \times n$ matrix, b is in $\mathbb{R}^m$ and ^{T} denotes transpose. Derive the dual linear programming problem D. Show from first principles that the dual of D is P.

Suppose that $c^T = (6, 10, 11)$, $b^T = (1, 1, 3)$ and $A = \begin{pmatrix} 1 & 3 & 8 \\ 1 & 1 & 2 \\ 2 & 4 & 4 \end{pmatrix}$. Write down the dual D and find the optimal solution of the dual using the simplex algorithm. Hence, or otherwise, find the optimal solution $x^* = (x_1^*, x_2^*, x_3^*)$ of P.

Suppose that c is changed to $\tilde{c} = (6 + \varepsilon_1, 10 + \varepsilon_2, 11 + \varepsilon_3)$. Give necessary and sufficient conditions for x^* still to be the optimal solution of P. If $\varepsilon_1 = \varepsilon_2 = 0$, find the range of values for ε_3 for which x^* is still the optimal solution of P.

END OF PAPER