UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part IA

Monday, 1 June, 2015 9:00 am to 12:00 noon

PAPER 3

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***all four*** *questions from Section I and* ***at most five*** *questions from Section II. In Section II,* ***no more than three*** *questions on each course may be attempted.*

Complete answers are preferred to fragments.

Write on ***one*** *side of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in separate bundles, marked ***A***, ***B***, ***C***, ***D***, ***E*** *and* ***F*** *according to the code letter affixed to each question. Include in the same bundle all questions from Section I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheets
Green master cover sheet

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1D Groups

Say that a group is *dihedral* if it has two generators x and y, such that x has order n (greater than or equal to 2 and possibly infinite), y has order 2, and $yxy^{-1} = x^{-1}$. In particular the groups C_2 and $C_2 \times C_2$ are regarded as dihedral groups. Prove that:

(i) any dihedral group can be generated by two elements of order 2;

(ii) any group generated by two elements of order 2 is dihedral; and

(iii) any non-trivial quotient group of a dihedral group is dihedral.

2D Groups

How many cyclic subgroups (including the trivial subgroup) does S_5 contain? Exhibit two isomorphic subgroups of S_5 which are not conjugate.

3A Vector Calculus

(i) For $r = |\mathbf{x}|$ with $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, show that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \quad (i = 1, 2, 3).$$

(ii) Consider the vector fields $\mathbf{F}(\mathbf{x}) = r^2\mathbf{x}$, $\mathbf{G}(\mathbf{x}) = (\mathbf{a}\cdot\mathbf{x})\mathbf{x}$ and $\mathbf{H}(\mathbf{x}) = \mathbf{a}\times\hat{\mathbf{x}}$, where $\mathbf{a}$ is a constant vector in $\mathbb{R}^3$ and $\hat{\mathbf{x}}$ is the unit vector in the direction of $\mathbf{x}$. Using suffix notation, or otherwise, find the divergence and the curl of each of $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$.

4A Vector Calculus

The smooth curve $\mathcal{C}$ in $\mathbb{R}^3$ is given in parametrised form by the function $\mathbf{x}(u)$. Let s denote arc length measured along the curve.

(a) Express the tangent $\mathbf{t}$ in terms of the derivative $\mathbf{x}' = d\mathbf{x}/du$, and show that $du/ds = |\mathbf{x}'|^{-1}$.

(b) Find an expression for $d\mathbf{t}/ds$ in terms of derivatives of $\mathbf{x}$ with respect to u, and show that the curvature κ is given by

$$\kappa = \frac{|\mathbf{x}' \times \mathbf{x}''|}{|\mathbf{x}'|^3}.$$

[*Hint: You may find the identity* $(\mathbf{x}' \cdot \mathbf{x}'')\mathbf{x}' - (\mathbf{x}' \cdot \mathbf{x}')\mathbf{x}'' = \mathbf{x}' \times (\mathbf{x}' \times \mathbf{x}'')$ *helpful.*]

(c) For the curve

$$\mathbf{x}(u) = \begin{pmatrix} u\cos u \\ u\sin u \\ 0 \end{pmatrix},$$

with $u \geqslant 0$, find the curvature as a function of u.

SECTION II

5D Groups

What does it mean for a group G to *act on* a set X? For $x \in X$, what is meant by the *orbit* $\mathrm{Orb}(x)$ to which x belongs, and by the *stabiliser* G_x of x? Show that G_x is a subgroup of G. Prove that, if G is finite, then $|G| = |G_x| \cdot |\mathrm{Orb}(x)|$.

(a) Prove that the symmetric group S_n acts on the set $P^{(n)}$ of all polynomials in n variables $x_1, \ldots, x_n$, if we define $\sigma \cdot f$ to be the polynomial given by

$$(\sigma \cdot f)(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

for $f \in P^{(n)}$ and $\sigma \in S_n$. Find the orbit of $f = x_1x_2 + x_3x_4 \in P^{(4)}$ under S_4. Find also the order of the stabiliser of f.

(b) Let r, n be fixed positive integers such that $r \leqslant n$. Let B_r be the set of all subsets of size r of the set $\{1, 2, \ldots, n\}$. Show that S_n acts on B_r by defining $\sigma \cdot U$ to be the set $\{\sigma(u) : u \in U\}$, for any $U \in B_r$ and $\sigma \in S_n$. Prove that S_n is transitive in its action on B_r. Find also the size of the stabiliser of $U \in B_r$.

6D Groups

Let G, H be groups and let $\varphi \colon G \to H$ be a function. What does it mean to say that φ is a *homomorphism* with *kernel* K? Show that if $K = \{e, \xi\}$ has order 2 then $x^{-1}\xi x = \xi$ for each $x \in G$. [If you use any general results about kernels of homomorphisms, then you should prove them.]

Which of the following four statements are true, and which are false? Justify your answers.

(a) There is a homomorphism from the orthogonal group $\mathrm{O}(3)$ to a group of order 2 with kernel the special orthogonal group $\mathrm{SO}(3)$.

(b) There is a homomorphism from the symmetry group S_3 of an equilateral triangle to a group of order 2 with kernel of order 3.

(c) There is a homomorphism from $\mathrm{O}(3)$ to $\mathrm{SO}(3)$ with kernel of order 2.

(d) There is a homomorphism from S_3 to a group of order 3 with kernel of order 2.

7D Groups

(a) State and prove Lagrange's theorem.

(b) Let G be a group and let H, K be fixed subgroups of G. For each $g \in G$, any set of the form $HgK = \{hgk : h \in H, k \in K\}$ is called an (H, K) *double coset*, or simply a double coset if H and K are understood. Prove that every element of G lies in some (H, K) double coset, and that any two (H, K) double cosets either coincide or are disjoint.

Let G be a finite group. Which of the following three statements are true, and which are false? Justify your answers.

(i) The size of a double coset divides the order of G.

(ii) Different double cosets for the same pair of subgroups have the same size.

(iii) The number of double cosets divides the order of G.

8D Groups

(a) Let G be a non-trivial group and let $Z(G) = \{h \in G : gh = hg \text{ for all } g \in G\}$. Show that $Z(G)$ is a normal subgroup of G. If the order of G is a power of a prime, show that $Z(G)$ is non-trivial.

(b) The *Heisenberg group* H is the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

with $x, y, z \in \mathbb{R}$. Show that H is a subgroup of the group of non-singular real matrices under matrix multiplication.

Find $Z(H)$ and show that $H/Z(H)$ is isomorphic to $\mathbb{R}^2$ under vector addition.

(c) For p prime, the *modular Heisenberg group* H_p is defined as in (b), except that x, y and z now lie in the field of p elements. Write down $|H_p|$. Find both $Z(H_p)$ and $H_p/Z(H_p)$ in terms of generators and relations.

9A Vector Calculus

The vector field $\mathbf{F}(\mathbf{x})$ is given in terms of cylindrical polar coordinates (ρ, ϕ, z) by

$$\mathbf{F}(\mathbf{x}) = f(\rho)\mathbf{e}_\rho,$$

where f is a differentiable function of ρ, and $\mathbf{e}_\rho = \cos\phi\,\mathbf{e}_x + \sin\phi\,\mathbf{e}_y$ is the unit basis vector with respect to the coordinate ρ. Compute the partial derivatives $\partial F_1/\partial x$, $\partial F_2/\partial y$, $\partial F_3/\partial z$ and hence find the divergence $\nabla \cdot \mathbf{F}$ in terms of ρ and ϕ.

The domain V is bounded by the surface $z = (x^2+y^2)^{-1}$, by the cylinder $x^2+y^2 = 1$, and by the planes $z = \frac{1}{4}$ and $z = 1$. Sketch V and compute its volume.

Find the most general function $f(\rho)$ such that $\nabla \cdot \mathbf{F} = 0$, and verify the divergence theorem for the corresponding vector field $\mathbf{F}(\mathbf{x})$ in V.

10A Vector Calculus

State Stokes' theorem.

Let S be the surface in $\mathbb{R}^3$ given by $z^2 = x^2 + y^2 + 1 - \lambda$, where $0 \leqslant z \leqslant 1$ and λ is a positive constant. Sketch the surface S for representative values of λ and find the surface element $\mathbf{dS}$ with respect to the Cartesian coordinates x and y.

Compute $\nabla \times \mathbf{F}$ for the vector field

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} -y \\ x \\ z \end{pmatrix}$$

and verify Stokes' theorem for $\mathbf{F}$ on the surface S for every value of λ.

Now compute $\nabla \times \mathbf{G}$ for the vector field

$$\mathbf{G}(\mathbf{x}) = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

and find the line integral $\int_{\partial S} \mathbf{G} \cdot \mathbf{dx}$ for the boundary ∂S of the surface S. Is it possible to obtain this result using Stokes' theorem? Justify your answer.

11A Vector Calculus

(i) Starting with the divergence theorem, derive Green's first theorem

$$\int_V (\psi \nabla^2 \phi + \nabla \psi \cdot \nabla \phi)\, dV = \int_{\partial V} \psi \frac{\partial \phi}{\partial n}\, dS.$$

(ii) The function $\phi(\mathbf{x})$ satisfies Laplace's equation $\nabla^2 \phi = 0$ in the volume V with given boundary conditions $\phi(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in \partial V$. Show that $\phi(\mathbf{x})$ is the only such function. Deduce that if $\phi(\mathbf{x})$ is constant on ∂V then it is constant in the whole volume V.

(iii) Suppose that $\phi(\mathbf{x})$ satisfies Laplace's equation in the volume V. Let V_r be the sphere of radius r centred at the origin and contained in V. The function $f(r)$ is defined by

$$f(r) = \frac{1}{4\pi r^2} \int_{\partial V_r} \phi(\mathbf{x})\, dS.$$

By considering the derivative df/dr, and by introducing the Jacobian in spherical polar coordinates and using the divergence theorem, or otherwise, show that $f(r)$ is constant and that $f(r) = \phi(\mathbf{0})$.

(iv) Let M denote the maximum of ϕ on ∂V_r and m the minimum of ϕ on ∂V_r. By using the result from (iii), or otherwise, show that $m \leqslant \phi(\mathbf{0}) \leqslant M$.

12A Vector Calculus

(a) Let t_{ij} be a rank 2 tensor whose components are invariant under rotations through an angle π about each of the three coordinate axes. Show that t_{ij} is diagonal.

(b) An array of numbers a_{ij} is given in one orthonormal basis as $\delta_{ij} + \epsilon_{1ij}$ and in another rotated basis as δ_{ij}. By using the invariance of the determinant of any rank 2 tensor, or otherwise, prove that a_{ij} is not a tensor.

(c) Let a_{ij} be an array of numbers and b_{ij} a tensor. Determine whether the following statements are true or false. Justify your answers.

(i) If $a_{ij}b_{ij}$ is a scalar for any rank 2 tensor b_{ij}, then a_{ij} is a rank 2 tensor.

(ii) If $a_{ij}b_{ij}$ is a scalar for any symmetric rank 2 tensor b_{ij}, then a_{ij} is a rank 2 tensor.

(iii) If a_{ij} is antisymmetric and $a_{ij}b_{ij}$ is a scalar for any symmetric rank 2 tensor b_{ij}, then a_{ij} is an antisymmetric rank 2 tensor.

(iv) If a_{ij} is antisymmetric and $a_{ij}b_{ij}$ is a scalar for any antisymmetric rank 2 tensor b_{ij}, then a_{ij} is an antisymmetric rank 2 tensor.

END OF PAPER