UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part II

Friday, 6 June, 2014 9:00 am to 12:00 noon

PAPER 4

Before you begin read these instructions carefully.

The examination paper is divided into two sections. Each question in Section II carries twice the number of marks of each question in Section I. Candidates may attempt ***at most six*** *questions from Section I and any number of questions from Section II.*

Complete answers are preferred to fragments.

Write on **one side** *of the paper only and begin each answer on a separate sheet.*

Write legibly; otherwise you place yourself at a grave disadvantage.

At the end of the examination:

Tie up your answers in bundles, marked ***A***, ***B***, ***C***, ..., ***K*** *according to the code letter affixed to each question. Include in the same bundle all questions from Sections I and II with the same code letter.*

Attach a completed gold cover sheet to each bundle.

You must also complete a green master cover sheet listing all the questions you have attempted.

Every cover sheet must bear your examination number and desk number.

STATIONERY REQUIREMENTS
Gold cover sheet
Green master cover sheet

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1F Number Theory

State the Chinese Remainder Theorem.

Find all solutions to the simultaneous congruences

$$\begin{aligned} x &\equiv 2 \pmod 3 \\ x &\equiv 3 \pmod 5 \\ x &\equiv 5 \pmod 7). \end{aligned}$$

A positive integer is said to be *square-free* if it is the product of distinct primes. Show that there are 100 consecutive numbers that are not square-free.

2G Topics in Analysis

State Liouville's theorem on approximation of algebraic numbers by rationals.

Prove that the number $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{n^n}}$ is transcendental.

3F Geometry and Groups

Define the limit set $\Lambda(G)$ of a Kleinian group G. Assuming that G has no finite orbit in $\mathbb{H}^3 \cup S^2_\infty$, and that $\Lambda(G) \neq \emptyset$, prove that if $E \subset \mathbb{C} \cup \{\infty\}$ is any non-empty closed set which is invariant under G, then $\Lambda(G) \subset E$.

4I Coding and Cryptography

Explain what is meant by a *Bose–Ray Chaudhuri–Hocquenghem (BCH) code with design distance* δ. Prove that, for such a code, the minimum distance between code words is at least δ. How many errors will the code detect? How many errors will it correct?

5K Statistical Modelling

Consider the normal linear model where the n-vector of responses Y satisfies $Y = X\beta + \varepsilon$ with $\varepsilon \sim N_n(0, \sigma^2 I)$ and X is an $n \times p$ design matrix with full column rank. Write down a $(1-\alpha)$-level confidence set for β.

Define the *Cook's distance* for the observation (Y_i, x_i) where x_i^T is the ith row of X, and give its interpretation in terms of confidence sets for β.

In the model above with $n = 100$ and $p = 4$, you observe that one observation has Cook's distance 3.1. Would you be concerned about the influence of this observation? Justify your answer.

[*Hint: You may find some of the following facts useful:*

1. *If* $Z \sim \chi_4^2$, *then* $\mathbb{P}(Z \leqslant 1.06) = 0.1$, $\mathbb{P}(Z \leqslant 7.78) = 0.9$.
2. *If* $Z \sim F_{4,96}$, *then* $\mathbb{P}(Z \leqslant 0.26) = 0.1$, $\mathbb{P}(Z \leqslant 2.00) = 0.9$.
3. *If* $Z \sim F_{96,4}$, *then* $\mathbb{P}(Z \leqslant 0.50) = 0.1$, $\mathbb{P}(Z \leqslant 3.78) = 0.9$.]

6B Mathematical Biology

The concentration $c(x,t)$ of a chemical in one dimension obeys the equations

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(c^2 \frac{\partial c}{\partial x}\right), \qquad \int_{-\infty}^{\infty} c(x,t)\, dx = 1\,.$$

State the physical interpretation of each equation.

Seek a similarity solution of the form $c = t^\alpha f(\xi)$, where $\xi = t^\beta x$. Find equations involving α and β from the differential equation and the integral. Show that these are satisfied by $\alpha = \beta = -1/4$.

Find the solution for $f(\xi)$. Find and sketch the solution for $c(x,t)$.

7D Dynamical Systems

Consider the map $x_{n+1} = \lambda x_n(1 - x_n^2)$ for $-1 \leqslant x_n \leqslant 1$. What is the maximum value, λ_{max}, for which the interval $[-1,1]$ is mapped into itself?

Analyse the first two bifurcations that occur as λ increases from 0 towards λ_{max}, including an identification of the values of λ at which the bifurcation occurs and the type of bifurcation.

What type of bifurcation do you expect as the third bifurcation? Briefly give your reasoning.

8B Further Complex Methods

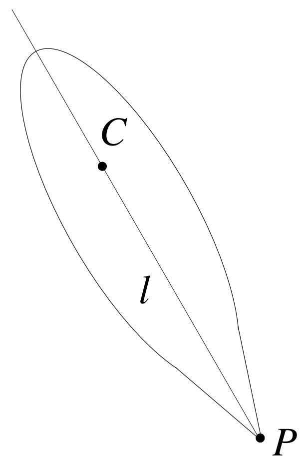
Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that

$$f(z+\omega_1) = f(z)\,, \quad f(z+\omega_2) = f(z)\,, \tag{1}$$

where $\omega_1, \omega_2 \in \mathbb{C}\backslash\{0\}$ and ω_1/ω_2 is not real. Show that if f is analytic on $\mathbb{C}$ then it is a constant. [Liouville's theorem may be used if stated.] Give an example of a non-constant meromorphic function which satisfies (1).

9A Classical Dynamics

Consider a heavy symmetric top of mass M with principal moments of inertia I_1, I_2 and I_3, where $I_1 = I_2 \neq I_3$. The top is pinned at point P, which is at a distance l from the centre of mass, C, as shown in the figure.



Its angular velocity in a body frame $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ is given by

$$\boldsymbol{\omega} = [\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi]\,\mathbf{e}_1 + [\dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi]\,\mathbf{e}_2 + [\dot\psi + \dot\phi \cos\theta]\,\mathbf{e}_3\,,$$

where ϕ, θ and ψ are the Euler angles.

(a) Assuming that $\{\mathbf{e}_a\}$, $a = 1, 2, 3$, are chosen to be the principal axes, write down the Lagrangian of the top in terms of ω_a and the principal moments of inertia. Hence find the Lagrangian in terms of the Euler angles.

(b) Find all conserved quantities. Show that ω_3, the spin of the top, is constant.

(c) By eliminating $\dot\phi$ and $\dot\psi$, derive a second-order differential equation for θ.

10E Cosmology

A homogeneous and isotropic universe, with cosmological constant Λ, has expansion scale factor $a(t)$ and Hubble expansion rate $H = \dot{a}/a$. The universe contains matter with density ρ and pressure P which satisfy the positive-energy condition $\rho + 3P/c^2 \geqslant 0$. The acceleration equation is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P/c^2) + \frac{1}{3}\Lambda c^2.$$

If $\Lambda \leqslant 0$, show that

$$\frac{d}{dt}(H^{-1}) \geqslant 1.$$

Deduce that $H \to \infty$ and $a \to 0$ at a finite time in the past or the future. What property of H distinguishes the two cases?

Give a simple counterexample with $\rho = P = 0$ to show that this deduction fails to hold when $\Lambda > 0$.

SECTION II

11F Number Theory

Define the *Legendre* and *Jacobi symbols*.

State the law of quadratic reciprocity for the Legendre symbol.

State the law of quadratic reciprocity for the Jacobi symbol, and deduce it from the corresponding result for the Legendre symbol.

Let p be a prime with $p \equiv 1 \pmod 4$. Prove that the sum of the quadratic residues in the set $\{1, 2, \ldots, p-1\}$ is equal to the sum of the quadratic non-residues in this set.

For which primes p is 7 a quadratic residue?

12F Geometry and Groups

Define the s-dimensional *Hausdorff measure* $\mathcal{H}^s(F)$ of a set $F \subset \mathbb{R}^N$. Explain briefly how properties of this measure may be used to define the *Hausdorff dimension* $\dim_H(F)$ of such a set.

Prove that the limit sets of conjugate Kleinian groups have equal Hausdorff dimension. Hence, or otherwise, prove that there is no subgroup of $\mathbb{P}SL(2,\mathbb{R})$ which is conjugate in $\mathbb{P}SL(2,\mathbb{C})$ to $\mathbb{P}SL(2,\mathbb{Z} \oplus \mathbb{Z}i)$.

13K Statistical Modelling

In a study on infant respiratory disease, data are collected on a sample of 2074 infants. The information collected includes whether or not each infant developed a respiratory disease in the first year of their life; the gender of each infant; and details on how they were fed as one of three categories (breast-fed, bottle-fed and supplement). The data are tabulated in `R` as follows:

```
  disease nondisease gender       food
1      77        381    Boy Bottle-fed
2      19        128    Boy Supplement
3      47        447    Boy Breast-fed
4      48        336   Girl Bottle-fed
5      16        111   Girl Supplement
6      31        433   Girl Breast-fed
```

Write down the model being fit by the `R` commands on the following page:

```
> total <- disease + nondisease
> fit <- glm(disease/total ~ gender + food, family = binomial,
+ weights = total)
```

The following (slightly abbreviated) output from R is obtained.

```
> summary(fit)

Coefficients:
               Estimate Std. Error z value Pr(>|z|)
(Intercept)     -1.6127     0.1124 -14.347  < 2e-16 ***
genderGirl      -0.3126     0.1410  -2.216   0.0267 *
foodBreast-fed  -0.6693     0.1530  -4.374 1.22e-05 ***
foodSupplement  -0.1725     0.2056  -0.839   0.4013
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 26.37529  on 5  degrees of freedom
Residual deviance:  0.72192  on 2  degrees of freedom
```

Briefly explain the justification for the standard errors presented in the output above.

Explain the relevance of the output of the following R code to the data being studied, justifying your answer:

```
 > exp(c(-0.6693 - 1.96*0.153, -0.6693 + 1.96*0.153))
[1] 0.3793940 0.6911351
```

[*Hint: It may help to recall that if* $Z \sim N(0,1)$ *then* $\mathbb{P}(Z \geqslant 1.96) = 0.025$.]

Let D_1 be the deviance of the model fitted by the following R command.

```
> fit1 <- glm(disease/total ~ gender + food + gender:food,
+ family = binomial, weights = total)
```

What is the numerical value of D_1? Which of the two models that have been fitted should you prefer, and why?

14D Dynamical Systems

A dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has a fixed point at the origin. Define the terms *Lyapunov stability*, *asymptotic stability* and *Lyapunov function* with respect to this fixed point. State and prove Lyapunov's first theorem and state (without proof) La Salle's invariance principle.

(a) Consider the system

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -y - x^3 + x^5 . \end{aligned}$$

Construct a Lyapunov function of the form $V = f(x) + g(y)$. Deduce that the origin is asymptotically stable, explaining your reasoning carefully. Find the greatest value of y_0 such that use of this Lyapunov function guarantees that the trajectory through $(0, y_0)$ approaches the origin as $t \to \infty$.

(b) Consider the system

$$\begin{aligned} \dot{x} &= x + 4y + x^2 + 2y^2 , \\ \dot{y} &= -3x - 3y\,. \end{aligned}$$

Show that the origin is asymptotically stable and that the basin of attraction of the origin includes the region $x^2 + xy + y^2 < \frac{1}{4}$.

15A Classical Dynamics

(a) Consider a system with one degree of freedom, which undergoes periodic motion in the potential $V(q)$. The system's Hamiltonian is

$$H(p,q) = \frac{p^2}{2m} + V(q)\,.$$

(i) Explain what is meant by the *angle* and *action variables*, θ and I, of the system and write down the integral expression for the action variable I. Is I conserved? Is θ conserved?

(ii) Consider $V(q) = \lambda q^6$, where λ is a positive constant. Find I in terms of λ, the total energy E, the mass M, and a dimensionless constant factor (which you need not compute explicitly).

(iii) Hence describe how E changes with λ if λ varies slowly with time. Justify your answer.

(b) Consider now a particle which moves in a plane subject to a central force-field $\mathbf{F} = -kr^{-2}\hat{\mathbf{r}}$.

(i) Working in plane polar coordinates (r,ϕ), write down the Hamiltonian of the system. Hence deduce two conserved quantities. Prove that the system is integrable and state the number of action variables.

(ii) For a particle which moves on an elliptic orbit find the action variables associated with radial and tangential motions. Can the relationship between the frequencies of the two motions be deduced from this result? Justify your answer.

(iii) Describe how E changes with m and k if one or both of them vary slowly with time.

[You may use

$$\int_{r_1}^{r_2} \left\{ \left(1 - \frac{r_1}{r}\right)\left(\frac{r_2}{r} - 1\right) \right\}^{\frac{1}{2}} dr \;=\; \frac{\pi}{2}\,(r_1 + r_2) - \pi\,\sqrt{r_1\, r_2}\;\;,$$

where $0 < r_1 < r_2$.]

16I Logic and Set Theory

Explain what is meant by a *chain-complete* poset. State the Bourbaki–Witt fixed-point theorem.

We call a poset $(P, \leqslant)$ *Bourbakian* if every order-preserving map $f : P \to P$ has a least fixed point $\mu(f)$. Suppose P is Bourbakian, and let $f, g \colon P \rightrightarrows P$ be order-preserving maps with $f(x) \leqslant g(x)$ for all $x \in P$; show that $\mu(f) \leqslant \mu(g)$. [*Hint: Consider the function* $h \colon P \to P$ *defined by* $h(x) = f(x)$ *if* $x \leqslant \mu(g)$, $h(x) = \mu(g)$ *otherwise.*]

Suppose P is Bourbakian and $f \colon \alpha \to P$ is an order-preserving map from an ordinal to P. Show that there is an order-preserving map $g \colon P \to P$ whose fixed points are exactly the upper bounds of the set $\{f(\beta) \mid \beta < \alpha\}$, and deduce that this set has a least upper bound.

Let C be a chain with no greatest member. Using the Axiom of Choice and Hartogs' Lemma, show that there is an order-preserving map $f \colon \alpha \to C$, for some ordinal α, whose image has no upper bound in C. Deduce that any Bourbakian poset is chain-complete.

17I Graph Theory

Define the *Ramsey number* $R^{(r)}(s,t)$. What is the value of $R^{(1)}(s,t)$? Prove that $R^{(r)}(s,t) \leqslant 1 + R^{(r-1)}(R^{(r)}(s-1,t), R^{(r)}(s,t-1))$ holds for $r \geqslant 2$ and deduce that $R^{(r)}(s,t)$ exists.

Show that $R^{(2)}(3,3) = 6$ and that $R^{(2)}(3,4)=9$.

Show that $7 \leqslant R^{(3)}(4,4) \leqslant 19$. [*Hint: For the lower bound, choose a suitable subset* U *and colour* e *red if* $|U \cap e|$ *is odd.*]

18H Galois Theory

(i) Let G be a finite subgroup of the multiplicative group of a field. Show that G is cyclic.

(ii) Let $\Phi_n(X)$ be the nth cyclotomic polynomial. Let p be a prime not dividing n, and let L be a splitting field for Φ_n over $\mathbb{F}_p$. Show that L has p^m elements, where m is the least positive integer such that $p^m \equiv 1 \pmod{n}$.

(iii) Find the degrees of the irreducible factors of $X^{35} - 1$ over $\mathbb{F}_2$, and the number of factors of each degree.

19H Representation Theory

Let G =SU(2).

(i) Sketch a proof that there is an isomorphism of topological groups $G/\{\pm I\} \cong$ SO(3).

(ii) Let V_2 be the irreducible complex representation of G of dimension 3. Compute the character of the (symmetric power) representation $S^n(V_2)$ of G for any $n \geqslant 0$. Show that the dimension of the space of invariants $(S^n(V_2))^G$, meaning the subspace of $S^n(V_2)$ where G acts trivially, is 1 for n even and 0 for n odd. [*Hint: You may find it helpful to restrict to the unit circle subgroup $S^1 \leqslant G$. The irreducible characters of G may be quoted without proof.*]

Using the fact that V_2 yields the standard 3-dimensional representation of SO(3), show that $\bigoplus_{n\geqslant 0} S^n V_2 \cong \mathbb{C}[x,y,z]$. Deduce that the ring of complex polynomials in three variables x, y, z which are invariant under the action of SO(3) is a polynomial ring in one generator. Find a generator for this polynomial ring.

20F Number Fields

Explain what is meant by an *integral basis* for a number field. Splitting into the cases $d \equiv 1 \pmod 4$ and $d \equiv 2, 3 \pmod 4$, find an integral basis for $K = \mathbb{Q}(\sqrt{d})$ where $d \neq 0, 1$ is a square-free integer. Justify your answer.

Find the fundamental unit in $\mathbb{Q}(\sqrt{13})$. Determine all integer solutions to the equation $x^2 + xy - 3y^2 = 17$.

21F Algebraic Topology

State the Lefschetz fixed point theorem.

Let X be an orientable surface of genus g (which you may suppose has a triangulation), and let $f : X \to X$ be a continuous map such that

1. $f^3 = \mathrm{Id}_X$,
2. f has no fixed points.

By considering the eigenvalues of the linear map $f_* : H_1(X;\mathbb{Q}) \to H_1(X;\mathbb{Q})$, and their multiplicities, show that g must be congruent to 1 modulo 3.

22G Linear Analysis

Define the *spectrum* $\sigma(T)$ and the *approximate point spectrum* $\sigma_{\text{ap}}(T)$ of a bounded linear operator T on a Banach space. Prove that $\sigma_{\text{ap}}(T) \subset \sigma(T)$ and that $\sigma(T)$ is a closed and bounded subset of $\mathbb{C}$. [You may assume without proof that the set of invertible operators is open.]

Let T be a hermitian operator on a non-zero Hilbert space. Prove that $\sigma(T)$ is not empty.

Let K be a non-empty, compact subset of $\mathbb{C}$. Show that there is a bounded linear operator $T\colon \ell_2 \to \ell_2$ with $\sigma(T) = K$. [You may assume without proof that a compact metric space is separable.]

23H Algebraic Geometry

Let X be a smooth projective curve of genus $g > 0$ over an algebraically closed field of characteristic $\neq 2$, and suppose there is a degree 2 morphism $\pi : X \to \mathbf{P}^1$. How many ramification points of π are there?

Suppose Q and R are distinct ramification points of π. Show that $Q \not\sim R$, but $2Q \sim 2R$.

Now suppose $g = 2$. Show that every divisor of degree 2 on X is linearly equivalent to $P + P'$ for some $P, P' \in X$, and deduce that every divisor of degree 0 is linearly equivalent to $P_1 - P_2$ for some $P_1, P_2 \in X$.

Show that the subgroup $\{[D] \in Cl^0(X) \mid 2[D] = 0\}$ of the divisor class group of X has order 16.

24G Differential Geometry

Let $I = [0,l]$ be a closed interval, $k(s), \tau(s)$ smooth real valued functions on I with k strictly positive at all points, and $\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0$ a positively oriented orthonormal triad of vectors in $\mathbf{R}^3$. An application of the fundamental theorem on the existence of solutions to ODEs implies that there exists a unique smooth family of triples of vectors $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ for $s \in I$ satisfying the differential equations

$$\mathbf{t}' = k\mathbf{n}, \quad \mathbf{n}' = -k\mathbf{t} - \tau\mathbf{b}, \quad \mathbf{b}' = \tau\mathbf{n},$$

with initial conditions $\mathbf{t}(0) = \mathbf{t}_0$, $\mathbf{n}(0) = \mathbf{n}_0$ and $\mathbf{b}(0) = \mathbf{b}_0$, and that $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ forms a positively oriented orthonormal triad for all $s \in I$. Assuming this fact, consider $\alpha : I \to \mathbf{R}^3$ defined by $\alpha(s) = \int_0^s \mathbf{t}(t)dt$; show that α defines a smooth immersed curve parametrized by arc-length, which has curvature and torsion given by $k(s)$ and $\tau(s)$, and that α is uniquely determined by this property up to rigid motions of $\mathbf{R}^3$. Prove that α is a plane curve if and only if τ is identically zero.

If $a > 0$, calculate the curvature and torsion of the smooth curve given by

$$\alpha(s) = (a\cos(s/c), a\sin(s/c), bs/c), \quad \text{where } c = \sqrt{a^2 + b^2}.$$

Suppose now that $\alpha : [0, 2\pi] \to \mathbf{R}^3$ is a smooth simple closed curve parametrized by arc-length with curvature everywhere positive. If both k and τ are constant, show that $k = 1$ and $\tau = 0$. If k is constant and τ is not identically zero, show that $k > 1$. Explain what it means for α to be *knotted*; if α is knotted and τ is constant, show that $k(s) > 2$ for some $s \in [0, 2\pi]$. [You may use standard results from the course if you state them precisely.]

25K Probability and Measure

Let $(X_n : n \in \mathbb{N})$ be a sequence of independent identically distributed random variables. Set $S_n = X_1 + \cdots + X_n$.

(i) State the strong law of large numbers in terms of the random variables X_n.

(ii) Assume now that the X_n are non-negative and that their expectation is infinite. Let $R \in (0, \infty)$. What does the strong law of large numbers say about the limiting behaviour of S_n^R/n, where $S_n^R = (X_1 \wedge R) + \cdots + (X_n \wedge R)$?

Deduce that $S_n/n \to \infty$ almost surely.

Show that

$$\sum_{n=0}^{\infty} \mathbb{P}(X_n \geqslant n) = \infty.$$

Show that $X_n \geqslant Rn$ infinitely often almost surely.

(iii) Now drop the assumption that the X_n are non-negative but continue to assume that $\mathbb{E}(|X_1|) = \infty$. Show that, almost surely,

$$\limsup_{n\to\infty} |S_n|/n = \infty.$$

26J Applied Probability

(i) Define the $M/M/1$ *queue* with arrival rate λ and service rate μ. Find conditions on the parameters λ and μ for the queue to be transient, null recurrent, and positive recurrent, briefly justifying your answers. In the last case give with justification the invariant distribution explicitly. Answer the same questions for an $M/M/\infty$ queue.

(ii) At a taxi station, customers arrive at a rate of 3 per minute, and taxis at a rate of 2 per minute. Suppose that a taxi will wait no matter how many other taxis are present. However, if a person arriving does not find a taxi waiting he or she leaves to find alternative transportation.

Find the long-run proportion of arriving customers who get taxis, and find the average number of taxis waiting in the long run.

An agent helps to assign customers to taxis, and so long as there are taxis waiting he is unable to have his coffee. Once a taxi arrives, how long will it take on average before he can have another sip of his coffee?

27J Principles of Statistics

Suppose you have at hand a pseudo-random number generator that can simulate an i.i.d. sequence of uniform $U[0,1]$ distributed random variables $U_1^*, \dots, U_N^*$ for any $N \in \mathbb{N}$. Construct an algorithm to simulate an i.i.d. sequence $X_1^*, \dots, X_N^*$ of standard normal $N(0,1)$ random variables. [Should your algorithm depend on the inverse of any cumulative probability distribution function, you are required to provide an explicit expression for this inverse function.]

Suppose as a matter of urgency you need to approximately evaluate the integral

$$I = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{(\pi + |x|)^{1/4}} e^{-x^2/2} dx.$$

Find an approximation I_N of this integral that requires N simulation steps from your pseudo-random number generator, and which has stochastic accuracy

$$\Pr(|I_N - I| > N^{-1/4}) \leqslant N^{-1/2},$$

where Pr denotes the joint law of the simulated random variables. Justify your answer.

28J Optimization and Control

A girl begins swimming from a point $(0,0)$ on the bank of a straight river. She swims at a constant speed v relative to the water. The speed of the downstream current at a distance y from the shore is $c(y)$. Hence her trajectory is described by

$$\dot{x} = v\cos\theta + c(y)\,, \quad \dot{y} = v\sin\theta\,,$$

where θ is the angle at which she swims relative to the direction of the current.

She desires to reach a downstream point $(1,0)$ on the same bank as she starts, as quickly as possible. Construct the Hamiltonian for this problem, and describe how Pontryagin's maximum principle can be used to give necessary conditions that must hold on an optimal trajectory. Given that $c(y)$ is positive, increasing and differentiable in y, show that on an optimal trajectory

$$\frac{d}{dt}\tan\big(\theta(t)\big) = -c'\big(y(t)\big)\,.$$

29K Stochastic Financial Models

Write down the Black–Scholes partial differential equation (PDE), and explain briefly its relevance to option pricing.

Show how a change of variables reduces the Black–Scholes PDE to the heat equation:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0 \text{ for all } (t,x) \in [0,T) \times \mathbb{R},$$
$$f(T,x) = \varphi(x) \text{ for all } x \in \mathbb{R},$$

where φ is a given boundary function.

Consider the following numerical scheme for solving the heat equation on the equally spaced grid $(t_n, x_k) \in [0,T] \times \mathbb{R}$ where $t_n = n\Delta t$ and $x_k = k\Delta x$, $n = 0,1,\ldots,N$ and $k \in \mathbb{Z}$, and $\Delta t = T/N$. We approximate $f(t_n, x_k)$ by f_k^n where

$$0 = \frac{f^{n+1} - f^n}{\Delta t} + \theta L f^{n+1} + (1-\theta) L f^n, \qquad f_k^N = \varphi(x_k), \qquad (*)$$

and $\theta \in [0,1]$ is a constant and the operator L is the matrix with non-zero entries $L_{kk} = -\dfrac{1}{(\Delta x)^2}$ and $L_{k,k+1} = L_{k,k-1} = \dfrac{1}{2(\Delta x)^2}$. By considering what happens when $\varphi(x) = \exp(i\omega x)$, show that the finite-difference scheme $(*)$ is stable if and only if

$$1 \geqslant \lambda(2\theta - 1),$$

where $\lambda \equiv \Delta t/(\Delta x)^2$. For what values of θ is the scheme $(*)$ unconditionally stable?

30D Partial Differential Equations

(a) Derive the solution of the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0\,, \qquad u(0,x) = u_0(x)\,, \quad u_t(0,x) = u_1(x)\,, \tag{1}$$

with Cauchy data given by C^2 functions $u_j = u_j(x)\,, j = 0,1$, and where $x \in \mathbb{R}$ and $u_{tt} = \partial_t^2 u$ etc. Explain what is meant by the property of *finite propagation speed* for the wave equation. Verify that the solution to (1) satisfies this property.

(b) Consider the Cauchy problem

$$u_{tt} - u_{xx} + x^2 u = 0\,, \qquad u(0,x) = u_0(x)\,, \quad u_t(0,x) = u_1(x)\,. \tag{2}$$

By considering the quantities

$$e = \tfrac{1}{2}\big(u_t^2 + u_x^2 + x^2u^2\big) \qquad \text{and} \qquad p = -u_t u_x\,,$$

prove that solutions of (2) also satisfy the property of finite propagation speed.

(c) Define what is meant by a strongly continuous one-parameter group of unitary operators on a Hilbert space. Consider the Cauchy problem for the Schrödinger equation for $\psi(x,t) \in \mathbb{C}$:

$$i\psi_t \;=\; -\psi_{xx} + x^2\psi\,, \qquad \psi(x,0) = \psi_0(x)\,, \qquad -\infty < x < \infty\,. \tag{3}$$

[In the following you may use without proof the fact that there is an orthonormal set of (real-valued) Schwartz functions $\{f_j(x)\}_{j=1}^{\infty}$ which are eigenfunctions of the differential operator $P = -\partial_x^2 + x^2$ with eigenvalues $2j+1$, i.e.

$$Pf_j = (2j+1)f_j\,, \quad f_j \in \mathcal{S}(\mathbb{R})\,, \quad (f_j, f_k)_{L^2} = \int_{\mathbb{R}} f_j(x)f_k(x)dx = \delta_{jk}\,,$$

and which have the property that any function $u \in L^2$ can be written uniquely as a sum $u(x) = \sum_j (f_j, u)_{L^2} f_j(x)$ which converges in the metric defined by the L^2 norm.]

Write down the solution to (3) in the case that ψ_0 is given by a finite sum $\psi_0 = \sum_{j=1}^{N}(f_j, \psi_0)_{L^2} f_j$ and show that your formula extends to define a strongly continuous one-parameter group of unitary operators on the Hilbert space L^2 of square-integrable (complex-valued) functions, with inner product $(f,g)_{L^2} = \int_{\mathbb{R}} \overline{f(x)}g(x)dx$.

31C Asymptotic Methods

Derive the leading-order Liouville–Green (or WKBJ) solution for $\epsilon \ll 1$ to the ordinary differential equation

$$\epsilon^2 \frac{d^2 f}{dy^2} + \Phi(y) f = 0\,,$$

where $\Phi(y) > 0$.

The function $f(y;\epsilon)$ satisfies the ordinary differential equation

$$\epsilon^2 \frac{d^2 f}{dy^2} + \left(1 + \frac{1}{y} - \frac{2\epsilon^2}{y^2}\right) f = 0\,, \tag{1}$$

subject to the boundary condition $f''(0) = 2$. Show that the Liouville–Green solution of (1) for $\epsilon \ll 1$ takes the asymptotic forms

$$f \sim \alpha_1 y^{\frac{1}{4}} \exp(2i\sqrt{y}/\epsilon) + \alpha_2 y^{\frac{1}{4}} \exp(-2i\sqrt{y}/\epsilon) \qquad \text{for} \quad \epsilon^2 \ll y \ll 1$$

and

$$f \sim B \cos\left[\theta_2 + (y + \log\sqrt{y})/\epsilon\right] \qquad \text{for} \quad y \gg 1\,,$$

where α_1, α_2, B and θ_2 are constants.

[*Hint : You may assume that* $\displaystyle\int_0^y \sqrt{1 + u^{-1}}\, du = \sqrt{y(1+y)} + \sinh^{-1}\sqrt{y}\,.$]

Explain, showing the relevant change of variables, why the leading-order asymptotic behaviour for $0 \leqslant y \ll 1$ can be obtained from the reduced equation

$$\frac{d^2 f}{dx^2} + \left(\frac{1}{x} - \frac{2}{x^2}\right) f = 0\,. \tag{2}$$

The unique solution to (2) with $f''(0) = 2$ is $f = x^{1/2} J_3(2x^{1/2})$, where the Bessel function $J_3(z)$ is known to have the asymptotic form

$$J_3(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{7\pi}{4}\right) \text{ as } z \to \infty\,.$$

Hence find the values of α_1 and α_2.

32A Principles of Quantum Mechanics

Define the *interaction picture* for a quantum mechanical system with Schrödinger picture Hamiltonian $H_0 + V(t)$ and explain why the interaction and Schrödinger pictures give the same physical predictions for transition rates between eigenstates of H_0. Derive the equation of motion for the interaction picture states $|\overline{\psi(t)}\rangle$.

A system consists of just two states $|1\rangle$ and $|2\rangle$, with respect to which

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}, \qquad V(t) = \hbar\lambda \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}.$$

Writing the interaction picture state as $|\overline{\psi(t)}\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle$, show that the interaction picture equation of motion can be written as

$$i\dot{a}_1(t) = \lambda e^{i\mu t} a_2(t)\,, \qquad i\dot{a}_2(t) = \lambda e^{-i\mu t} a_1(t)\,, \qquad (*)$$

where $\mu = \omega - \omega_{21}$ and $\omega_{21} = (E_2 - E_1)/\hbar$. Hence show that $a_2(t)$ satisfies

$$\ddot{a}_2 + i\mu\,\dot{a}_2 + \lambda^2 a_2 = 0\,.$$

Given that $a_2(0) = 0$, show that the solution takes the form

$$a_2(t) = \alpha\, e^{-i\mu t/2} \sin \Omega t\,,$$

where Ω is a frequency to be determined and α is a complex constant of integration.

Substitute this solution for $a_2(t)$ into $(*)$ to determine $a_1(t)$ and, by imposing the normalization condition $\||\overline{\psi(t)}\rangle\|^2 = 1$ at $t = 0$, show that $|\alpha|^2 = \lambda^2/\Omega^2$.

At time $t = 0$ the system is in the state $|1\rangle$. Write down the probability of finding the system in the state $|2\rangle$ at time t.

33A Applications of Quantum Mechanics

Let Λ be a Bravais lattice in three dimensions. Define the *reciprocal lattice* Λ^*.

State and prove *Bloch's theorem* for a particle moving in a potential $V(\mathbf{x})$ obeying

$$V(\mathbf{x}+\boldsymbol{\ell}) = V(\mathbf{x}) \qquad \forall\, \boldsymbol{\ell} \in \Lambda,\ \mathbf{x} \in \mathbb{R}^3 .$$

Explain what is meant by a *Brillouin zone* for this potential and how it is related to the reciprocal lattice.

A simple cubic lattice Λ_1 is given by the set of points

$$\Lambda_1 \quad = \quad \left\{ \boldsymbol{\ell} \in \mathbb{R}^3 : \boldsymbol{\ell} = n_1 \hat{\mathbf{i}} + n_2 \hat{\mathbf{j}} + n_3 \hat{\mathbf{k}},\ n_1, n_2, n_3 \in \mathbb{Z} \right\} ,$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors parallel to the Cartesian coordinate axes in $\mathbb{R}^3$. A body-centred cubic (BCC) lattice Λ_{BCC} is obtained by adding to Λ_1 the points at the centre of each cube, i.e. all points of the form

$$\boldsymbol{\ell} + \frac{1}{2}\left(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}\right), \qquad \boldsymbol{\ell} \in \Lambda_1 .$$

Show that Λ_{BCC} is Bravais with primitive vectors

$$\begin{aligned} \mathbf{a}_1 &= \frac{1}{2}\left(\hat{\mathbf{j}} + \hat{\mathbf{k}} - \hat{\mathbf{i}}\right), \\ \mathbf{a}_2 &= \frac{1}{2}\left(\hat{\mathbf{k}} + \hat{\mathbf{i}} - \hat{\mathbf{j}}\right), \\ \mathbf{a}_3 &= \frac{1}{2}\left(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}\right). \end{aligned}$$

Find the reciprocal lattice Λ^*_{BCC}. Hence find a consistent choice for the first Brillouin zone of a potential $V(\mathbf{x})$ obeying

$$V(\mathbf{x}+\boldsymbol{\ell}) \quad = \quad V(\mathbf{x}) \qquad \forall\, \boldsymbol{\ell} \in \Lambda_{BCC},\ \mathbf{x} \in \mathbb{R}^3 .$$

[*Hint: The matrix* $M = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ *has inverse* $M^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.]

34E Statistical Physics

The Dieterici equation of state of a gas is

$$P = \frac{k_B T}{v-b} \exp\left(-\frac{a}{k_B T v}\right),$$

where P is the pressure, $v = V/N$ is the volume divided by the number of particles, T is the temperature, and k_B is the Boltzmann constant. Provide a physical interpretation for the constants a and b.

Briefly explain how the Dieterici equation captures the liquid–gas phase transition. What is the maximum temperature at which such a phase transition can occur?

The Gibbs free energy is given by

$$G = E + PV - TS,$$

where E is the energy and S is the entropy. Explain why the Gibbs free energy is proportional to the number of particles in the system.

On either side of a first-order phase transition the Gibbs free energies are equal. Use this fact to derive the Clausius–Clapeyron equation for a line along which there is a first-order liquid–gas phase transition,

$$\frac{dP}{dT} = \frac{L}{T(V_{\text{gas}} - V_{\text{liquid}})}, \qquad (*)$$

where L is the latent heat which you should define.

Assume that the volume of liquid is negligible compared to the volume of gas and that the latent heat is constant. Further assume that the gas can be well approximated by the ideal gas law. Solve $(*)$ to obtain an equation for the phase-transition line in the (P, T) plane.

35C Electrodynamics

(i) The action S for a point particle of rest mass m and charge q moving along a trajectory $x^\mu(\lambda)$ in the presence of an electromagnetic 4-vector potential A^μ is

$$S = -mc \int \left(-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda + q \int A_\mu \frac{dx^\mu}{d\lambda} \, d\lambda \, ,$$

where λ is an arbitrary parametrization of the path and $\eta_{\mu\nu}$ is the Minkowski metric. By varying the action with respect to $x^\mu(\lambda)$, derive the equation of motion $m\ddot{x}^\mu = qF^\mu{}_\nu \dot{x}^\nu$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and overdots denote differentiation with respect to proper time for the particle.

(ii) The particle moves in constant electric and magnetic fields with non-zero Cartesian components $E_z = E$ and $B_y = B$, with $B > E/c > 0$ in some inertial frame. Verify that a suitable 4-vector potential has components

$$A^\mu = (0, 0, 0, -Bx - Et)$$

in that frame.

Find the equations of motion for x, y, z and t in terms of proper time τ. For the case of a particle that starts at rest at the spacetime origin at $\tau = 0$, show that

$$\ddot{z} + \frac{q^2}{m^2} \left(B^2 - \frac{E^2}{c^2} \right) z = \frac{qE}{m} \, .$$

Find the trajectory $x^\mu(\tau)$ and sketch its projection onto the (x, z) plane.

36E General Relativity

A plane-wave spacetime has line element

$$ds^2 = H du^2 + 2du\, dv + dx^2 + dy^2,$$

where $H = x^2 - y^2$. Show that the line element is unchanged by the coordinate transformation

$$u = \bar{u}, \quad v = \bar{v} + \bar{x}e^{\bar{u}} - \tfrac{1}{2}e^{2\bar{u}}, \quad x = \bar{x} - e^{\bar{u}}, \quad y = \bar{y}. \qquad (*)$$

Show more generally that the line element is unchanged by coordinate transformations of the form

$$u = \bar{u} + a, \quad v = \bar{v} + b\bar{x} + c, \quad x = \bar{x} + p, \quad y = \bar{y},$$

where a, b, c and p are functions of $\bar{u}$, which you should determine and which depend in total on four parameters (arbitrary constants of integration).

Deduce (without further calculation) that the line element is unchanged by a 6-parameter family of coordinate transformations, of which a 5-parameter family leave invariant the surfaces $u =$ constant.

For a general coordinate transformation $x^a = x^a(\bar{x}^b)$, give an expression for the transformed Ricci tensor $\bar{R}_{cd}$ in terms of the Ricci tensor R_{ab} and the transformation matrices $\dfrac{\partial x^a}{\partial \bar{x}^c}$. Calculate $\bar{R}_{\bar{x}\bar{x}}$ when the transformation is given by $(*)$ and deduce that $R_{vv} = R_{vx}$.

37B Fluid Dynamics II

An incompressible fluid of density ρ and kinematic viscosity ν is confined in a channel with rigid stationary walls at $y = \pm h$. A spatially uniform pressure gradient $-G\cos\omega t$ is applied in the x-direction. What is the physical significance of the dimensionless number $S = \omega h^2/\nu$?

Assuming that the flow is unidirectional and time-harmonic, obtain expressions for the velocity profile and the total flux. [You may leave your answers as the real parts of complex functions.]

In each of the limits $S \to 0$ and $S \to \infty$, find and sketch the flow profiles, find leading-order asymptotic expressions for the total flux, and give a physical interpretation.

Suppose now that $G = 0$ and that the channel walls oscillate in their own plane with velocity $U\cos\omega t$ in the x-direction. Without explicit calculation of the solution, sketch the flow profile in each of the limits $S \to 0$ and $S \to \infty$.

38C Waves

A one-dimensional shock wave propagates at a constant speed along a tube aligned with the x-axis and containing a perfect gas. In the reference frame where the shock is at rest at $x = 0$, the gas has speed U_0, density ρ_0 and pressure p_0 in the region $x < 0$ and speed U_1, density ρ_1 and pressure p_1 in the region $x > 0$.

Write down equations of conservation of mass, momentum and energy across the shock. Show that

$$\frac{\gamma}{\gamma - 1}\left(\frac{p_1}{\rho_1} - \frac{p_0}{\rho_0}\right) = \frac{p_1 - p_0}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_0}\right) ,$$

where γ is the ratio of specific heats.

From now on, assume $\gamma = 2$ and let $P = p_1/p_0$. Show that $\frac{1}{3} < \rho_1/\rho_0 < 3$.

The increase in entropy from $x < 0$ to $x > 0$ is given by $\Delta S = C_V \log(p_1\rho_0^2/p_0\rho_1^2)$, where C_V is a positive constant. Show that ΔS is a monotonic function of P.

If $\Delta S > 0$, deduce that $P > 1$, $\rho_1/\rho_0 > 1$, $(U_0/c_0)^2 > 1$ and $(U_1/c_1)^2 < 1$, where c_0 and c_1 are the sound speeds in $x < 0$ and $x > 0$, respectively. Given that ΔS must have the same sign as U_0 and U_1, interpret these inequalities physically in terms of the properties of the flow upstream and downstream of the shock.

39D Numerical Analysis

Let A be a real symmetric $n \times n$ matrix with n distinct real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ and a corresponding orthogonal basis of normalized real eigenvectors $\{\mathbf{w}_i\}_{i=1}^n$.

(i) Let $s \in \mathbb{R}$ satisfy $s < \lambda_1$. Given a unit vector $\mathbf{x}^{(0)} \in \mathbb{R}^n$, the iteration scheme

$$(A - sI)\mathbf{y} = \mathbf{x}^{(k)},$$
$$\mathbf{x}^{(k+1)} = \mathbf{y}/\|\mathbf{y}\|,$$

generates a sequence of vectors $\mathbf{x}^{(k+1)}$ for $k = 0, 1, 2, \ldots$. Assuming that $\mathbf{x}^{(0)} = \sum c_i \mathbf{w}_i$ with $c_1 \neq 0$, prove that $\mathbf{x}^{(k)}$ tends to $\pm\, \mathbf{w}_1$ as $k \to \infty$. What happens to $\mathbf{x}^{(k)}$ if $s > \lambda_1$? [Consider all cases.]

(ii) Describe how to implement an inverse-iteration algorithm to compute the eigenvalues and eigenvectors of A, given some initial estimates for the eigenvalues.

(iii) Let $n = 2$. For iterates $\mathbf{x}^{(k)}$ of an inverse-iteration algorithm with a fixed value of $s \neq \lambda_1, \lambda_2$, show that if

$$\mathbf{x}^{(k)} = (\mathbf{w}_1 + \epsilon_k \mathbf{w}_2)/(1 + \epsilon_k^2)^{1/2},$$

where $|\epsilon_k|$ is small, then $|\epsilon_{k+1}|$ is of the same order of magnitude as $|\epsilon_k|$.

(iv) Let $n = 2$ still. Consider the iteration scheme

$$s_k = \left(\mathbf{x}^{(k)}\,,\, A\mathbf{x}^{(k)}\right), \qquad (A - s_k I)\mathbf{y} = \mathbf{x}^{(k)}, \qquad \mathbf{x}^{(k+1)} = \mathbf{y}/\|\mathbf{y}\|$$

for $k = 0, 1, 2, \ldots$, where (,) denotes the inner product. Show that with this scheme $|\epsilon_{k+1}| = |\epsilon_k|^3$.

END OF PAPER